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LINEAR DIFFERENTIAL LAPPO-DANILEVSKII SYSTEMS

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Abstract. The class of linear differential systems with coefficient matrices which are commutative with their integrals is considered. The results on asymptotic equivalence of these systems and their distribution among linear systems are given.

Keywords: linear differential systems, Lyapunov transformations, functional commutative matrices, Lappo-Danilevskii systems

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Consider the linear system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \in I = [t_0, +\infty[, \tag{1}$$

where $A(t)$ is an $n \times n$ matrix of a real-valued continuous and bounded functions of a real variable t on the non-negative half-line I. Usually [1, p. 117], (1) is called the Lappo-Danilevskii system if the matrix A is commutative with its integral, i.e.

$$A(t) \int_s^t A(u)\,\mathrm{d}u = \int_s^t A(u)\,\mathrm{d}u A(t), \tag{2}$$

for some s, $t \in I$.

We define three types of the Lappo-Danilevskii systems.

Definition. We say that

i) $A(t)$ is a right Lappo-Danilevskii matrix with the initial point s ($A \in LD_r(s)$) if there exists an s, $s \in I$, such that (2) is fulfilled for all $t \in I_r(s) = [s, +\infty[$;

ii) $A(t)$ is a left Lappo-Danilevskii matrix with the initial point s ($A \in LD_l(s)$) if there exists an $s \in I$, $s > t_0$, such that (2) is fulfilled for all $t \in I_l(s) = [t_0, s]$;

iii) $A(t)$ is a bilateral Lappo-Danilevskii matrix with the initial point s ($A \in LD_b(s)$) if there exists an s, $s \geqslant t_0$, such that (2) is fulfilled for all $t \in I$.

The corresponding systems (1) are called right, left or bilateral Lappo-Danilevskii systems. Note that a special case of the bilateral Lappo-Danilevskii system is the system (1) with a functional commutative matrix A, where for all $s, t \in I$

$$A(t)A(s) - A(s)A(t) = 0. \tag{3}$$

It is well known that if A is a right, left or bilateral Lappo-Danilevskii matrix, then the fundamental solution matrix $X_s(t)$ of (1) ($X_s(s) = E$, E is the identity matrix) can be represented as

$$X_s(t) = \exp \int_s^t A(u)\,\mathrm{d}u \tag{4}$$

for $t \in I_r(s)$, $t \in I_l(s)$, $t \in I$, respectively. This simple representation (4) of the fundamental solution matrix explains the fact that the class of Lappo-Danilevskii systems is one of the main and interesting classes of linear systems. For example, in some cases it is possible to calculate asymptotic characteristics, in particular, Lyapunov exponents of the solutions of (1) directly using the coefficients of (1) (see for instance [2]). In this connection we consider the problem of reducibility of an arbitrary linear system with bounded coefficients to the Lappo-Danilevskii system and to the system with a functional commutative matrix of coefficients.

It is well known [3, p. 274] that any linear system is almost reducible to some diagonal system. It is a trivial fact that any diagonal matrix is a functional commutative matrix. However, the case of linear systems under Lyapunov's transformations is quite different.

A linear transformation

$$x = L(t)y \tag{5}$$

is a Lyapunov transformation if $L(t)$ is a Lyapunov matrix, i.e.

$$\max\left\{\sup_{t\geqslant t_0}\|L(t)\|, \sup_{t\geqslant t_0}\|L^{-1}(t)\|, \sup_{t\geqslant t_0}\left\|\frac{\mathrm{d}}{\mathrm{d}t}L(t)\right\|\right\} < +\infty. \tag{6}$$

It is easy to see that if (5) reduces (1) to the system

$$\frac{\mathrm{d}y}{\mathrm{d}t} = B(t)y, \quad y \in \mathbb{R}^n, \quad t \in I, \tag{7}$$

then

$$B(t) = L^{-1}(t)A(t)L(t) - L^{-1}(t)\frac{\mathrm{d}}{\mathrm{d}t}L(t). \tag{8}$$

We follow Yu. Bogdanov [4] and say that two linear systems are asymptotically equivalent if there exists a Lyapunov transformation reducing one of them to the other. Note that the Lyapunov transformations do not change the asymptotic properties of the solutions, in particular, their stability.

Theorem 1. *The linear system* (7) *is asymptotically equivalent to the system* (1) *with a functional commutative matrix of coefficients if and only if the Cauchy matrix* $K_B(t,s)$ *of* (7) *can be presented in the form*

$$K_B(t,s) = L(t)\exp\int_s^t A(u)\,\mathrm{d}u L^{-1}(s) \quad \forall t,s \geqslant t_0, \tag{9}$$

where $L(t)$ *is Lyapunov's matrix.*

Proof. 1. Let (7) be asymptotically equivalent to (1) with the functional commutative matrix A satisfying (3). Then there exist $s_0 \geqslant t_0$ and a non-singular constant matrix C such that

$$Y_{s_0}(t)CX_{s_0}^{-1}(t) = L(t), \tag{10}$$

where $L(t)$ is Lyapunov's matrix, X_{s_0} and Y_{s_0} are fundamental matrices of the solutions of (1) and (7), respectively ($X_{s_0}(s_0) = Y_{s_0}(s_0) = E$). Since A is a functional commutative matrix, we have

$$K_A(t,s) = \exp\int_s^t A(u)\,\mathrm{d}u, \tag{11}$$

where $K_A(t,s)$ is the Cauchy matrix of (1). It follows from (10) that

$$K_B(t,s) = Y_{s_0}(t)Y_{s_0}^{-1}(s) = L(t)X_{s_0}(t)C^{-1}CX_{s_0}^{-1}(s)L^{-1}(s) = L(t)K_A(t,s)L^{-1}(s).$$

Using (11), we obtain the required relation (9).

2. Let a transformation

$$y = L(t)x \tag{12}$$

with the Lyapunov matrix $L(t)$ satisfying (6) reduce (7) to a linear system $\frac{\mathrm{d}}{\mathrm{d}t}x = P(t)x$. Then P satisfies (see (8)) the equality

$$P(t) = L^{-1}(t)B(t)L(t) - L^{-1}(t)\frac{\mathrm{d}}{\mathrm{d}t}L(t).$$

Since $L(t) = K_B(t,s)L(s)\exp(-\int_s^t A(u)\,\mathrm{d}u)$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}L(t) &= B(t)K_B(t,s)L(s)\exp\Big(-\int_s^t A(u)\,\mathrm{d}u\Big)\\ &\quad - K_B(t,s)L(s)\exp\Big(-\int_s^t A(u)\,\mathrm{d}u\Big)\frac{\mathrm{d}}{\mathrm{d}t}\Big(\exp\int_s^t A(u)\,\mathrm{d}u\Big)\exp\Big(-\int_s^t A(u)\,\mathrm{d}u\Big)\\ &= B(t)L(t) - L(t)\frac{\mathrm{d}}{\mathrm{d}t}\Big(\exp\int_s^t A(u)\,\mathrm{d}u\Big)\exp\Big(-\int_s^t A(u)\,\mathrm{d}u\Big).\end{aligned}$$

Therefore, $P(t) = \frac{\mathrm{d}}{\mathrm{d}t}(\exp\int_s^t A(u)\,\mathrm{d}u)\exp(-\int_s^t A(u)\,\mathrm{d}u)$, hence

$$P(t)\exp\int_s^t A(u)\,\mathrm{d}u = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\exp\int_s^t A(u)\,\mathrm{d}u\Big) \quad \forall t, s \geqslant t_0.$$

Thus,

$$\begin{aligned}P(t)\Big(E + \sum_{m=1}^{\infty}\frac{1}{m!}\Big(\int_s^t A(u)\,\mathrm{d}u\Big)^m\Big) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(E + \sum_{m=1}^{\infty}\frac{1}{m!}\Big(\int_s^t A(u)\,\mathrm{d}u\Big)^m\Big)\\ &= A(t) + \sum_{m=2}^{\infty}\frac{1}{m!}\sum_{k=0}^{m-1}\Big(\int_s^t A(u)\,\mathrm{d}u\Big)^k A(t)\Big(\int_s^t A(u)\,\mathrm{d}u\Big)^{m-1-k} \quad \forall t, s \geqslant t_0.\end{aligned}$$

Substituting t for s, we get $P(t) = A(t)$ for all $t \geqslant t_0$. Therefore, (12) reduces (7) to (1). It suffices to show that A is a functional commutative matrix.

Consider the transformation (12); if Y is any fundamental matrix of solutions of (7), then $X(t) = L^{-1}(t)Y(t)$ is a fundamental matrix of (1). Therefore, from (9) it follows that

$$\begin{aligned}K_A(t,s) &= X(t)X^{-1}(s) = L^{-1}(t)Y(t)Y^{-1}(s)L(s)\\ &= L^{-1}(t)K_B(t,s)L(s) = \exp\int_s^t A(u)\,\mathrm{d}u \qquad \forall t, s \geqslant t_0.\end{aligned}$$

From [5] it follows that A is a functional commutative matrix. The theorem is proved.

A similar result is valid for the right and bilateral Lappo-Danilevskii systems.

Theorem 2. *The linear system* (7) *is asymptotically equivalent to the right (bilateral) Lappo-Danilevskii system* (1) *if and only if there exists a fundamental matrix* $Y(t)$ *of* (7) *which can be presented in the form*

$$Y(t) = L(t)\exp\int_s^t A(u)\,\mathrm{d}u \quad \forall t \geqslant s \geqslant t_0 \ (\forall t \geqslant t_0),$$

where $L(t)$ *is Lyapunov's matrix and* $A \in LD_r(s)$ $(A \in LD_b(s))$.

There is no problem to reduce linear systems to left Lappo-Danilevskii systems, because it is easy to prove that any linear system is asymptotically equivalent to some left Lappo-Danilevskii system. On the other hand, the following result is valid [6], [7].

Theorem 3. *There exists a linear system which is asymptotically equivalent neither to a system with a functional commutative matrix of coefficients nor to a right (bilateral) Lappo-Danilevskii system.*

To prove this fact it is sufficient to consider the linear system with the matrix of coefficients (E_{n-2} is the $(n-2) \times (n-2)$ identity matrix)

$$\begin{pmatrix} 0 & 1 & 0 \dots 0 \\ 0 & (t-t_0+1)^{-1} & 0 \dots 0 \\ 0 & 0 & \\ \dots & \dots & E_{n-2} \\ 0 & 0 & \end{pmatrix}, \quad t \in [t_0, +\infty[, \tag{13}$$

and to use the specific structure and the distribution of zeros of the integrals of the Lappo-Danilevskii matrices.

However, system (13) is a regular system (in the Lyapunov sense) and can be reduced (Basov-Grobman-Bogdanov's criterion [8, p. 77]) to a system with a functional commutative matrix of coefficients by the generalized Lyapunov transformation (5) with a matrix L such that $\overline{\lim\limits_{t\to+\infty}} t^{-1} \ln \|L(t)\| = \overline{\lim\limits_{t\to+\infty}} t^{-1} \ln \|L^{-1}(t)\| = 0$.

But even if we expand the set of our transformations up to the set of generalized Lyapunov transformations there is a statement which is similar to Theorem 3 (see [9], [10]).

Theorem 4. *There exists a two-dimensional linear system which is generalized asymptotically equivalent neither to a system with a functional commutative matrix of coefficients nor to a right (bilateral) Lappo-Danilevskii system.*

We believe that this fact holds for linear systems of an arbitrary dimension n, but now we have the proof only for $n = 2$.

Note that condition (2) is sufficiently strong and small perturbations of the elements of A can output the matrix from the class of the Lappo-Danilevskii matrices. So we consider some problems on the behavior of the Lappo-Danilevskii matrices in the set of all matrices.

Let the distance between matrices A and B be defined by the formula $\varrho(A, B) = \sup\limits_{t \geqslant t_0} \|A(t) - B(t)\|$, where $\| \cdot \|$ is an arbitrary matrix norm.

The following results are valid [11].

Theorem 5. *Let $A_i \in LD_\alpha(s_i)$, $i \in \mathbb{N}$, $\alpha \in \{b, r\}$, and $\varrho(A, A_i) \to 0$ as $i \to +\infty$. If there exists M such that $s_i \leqslant M < +\infty$ for all $i \in \mathbb{N}$, then A is a bilateral (right) Lappo-Danilevskii matrix.*

Theorem 6. *Let $A_i \in LD_l(s_i)$, $i \in \mathbb{N}$, and $\varrho(A, A_i) \to 0$ as $i \to +\infty$. If there exist m, M such that $t_0 < m \leqslant s_i \leqslant M < +\infty$ for all $i \in \mathbb{N}$, then A is a left Lappo-Danilevskii matrix.*

However, if the sequences (s_i) for the sequences of the right and left Lappo-Danilevskii matrices are not bounded, then the previous results are not valid, namely, the following facts hold.

Theorem 7. *There exists a sequence A_i, $A_i \in LD_r(s_i)$, $(A_i \in LD_l(s_i))$, $i \in \mathbb{N}$, $\varrho(A, A_i) \to 0$ and $s_i \to +\infty$ $(s_i \to t_0+0)$ as $i \to +\infty$, such that $A \notin LD_r$ $(A \notin LD_l)$.*

To prove this statement it is sufficient to construct the following sequences of A_k $(k \in \mathbb{N},\ t_0 = 0)$:

$$A_k(t) = \begin{pmatrix} B_k(t) & O_1 \\ O_2 & C(t) \end{pmatrix}, \quad B_k(t) = \begin{pmatrix} g(t) & f_k(t) \\ \mathrm{e}^{-t} & g(t) \end{pmatrix}, \quad t \in [0, +\infty[,$$

where O_1, O_2 are the $2 \times (n-2)$, $(n-2) \times 2$, zero-matrices, respectively, $C(t)$ is an $(n-2) \times (n-2)$ functional commutative matrix, g is a continuous bounded function on $[0, +\infty[$. If

$$f_k = \begin{cases} (1 - \mathrm{e}^{-t})\mathrm{e}^{-t}, & 0 \leqslant t \leqslant k, \\ (1 - \mathrm{e}^{-k})\mathrm{e}^{-t}, & t > k, \end{cases}$$

then $A_k \in LD_r(k)$, but the limit matrix A does not belong to LD_r; if

$$f_k = \begin{cases} \mathrm{e}^{-\frac{1}{k}-t}, & 0 \leqslant t \leqslant \frac{1}{k}, \\ \mathrm{e}^{-2t}, & t > \frac{1}{k}, \end{cases}$$

then $A_k \in LD_l(\frac{1}{k})$, but the limit matrix A does not belong to LD_l.

The following result establishes the closedness of the set of two dimensional bilateral Lappo-Danilevskii matrices in the set of all matrices.

Theorem 8. *Let $A_i \in LD_b(s_i)$, $i \in \mathbb{N}$. If $\varrho(A, A_i) \to 0$ as $i \to +\infty$, then A is a bilateral Lappo-Danilevskii matrix.*

To complete our review of the Lappo-Danilevskii systems we say some words about the connection between the properties (2) and (4).

It is well known that condition (2) is sufficient for the representation (4). J. F. P. Martin proved (see [12]) that if the differences of the eigenvalues of the integral of A are not zero roots of the equation

$$e^z - z - 1 = 0, \tag{14}$$

then (4) implies (2). From the results of J. F. P. Martin [12] and V. N. Laptinskii [13] it follows that if the coefficients of (1) are analytic functions on I, then (4) also implies (2). However, there was an open question of the existence of a linear system with infinitely differentiable non-analytic coefficients such that this system was not a Lappo-Danilevskii system but its fundamental solution matrix had the form (4). We have proved that such system exists [14].

To verify this fact it is sufficient to consider the system (1) with the matrix

$$A(t) = \begin{pmatrix} -\mu a(t) & 0 & -\nu a(t) \\ b(t) & 0 & 0 \\ \nu a(t) & 0 & -\mu a(t) \end{pmatrix}, \quad t \in [0, +\infty[,$$

where $\mu \pm i\nu$ are roots of the equation (14), a and b are infinitely differentiable non-analytic functions such that

$$\int_0^t a(u)\,du > 0 \quad \forall\, t \in]0, s_0], \quad \int_0^{s_0} a(u)\,du = 1, \qquad a(t) = 0 \quad \forall\, t \geqslant s_0 > 0, \tag{15}$$

$$b(t) = \begin{cases} 0, & t \in [0, s_0[, \\ b_k(t) \not\equiv 0, & t \in [s_{2k}, s_{2k+1}[, \\ 0 & t \in [s_{2k+1}, s_{2k+2}[, k = 0, 1, \dots, \end{cases} \tag{16}$$

((s_k) is an arbitrary sequence of positive numbers such that $s_{k+1} > s_k$ and $s_k \to +\infty$ as $k \to +\infty$). In this case the fundamental solution matrix $X_0(t)$ of (1) may be represented as (4) with $s = 0$ but $A(t)$ is not a Lappo-Danilevskii matrix with the initial point $s = 0$.

Note that for a two dimensional real-valued matrix A condition (2) is necessary and sufficient for representation (4), which follows from the distribution of the roots of (14) and the eigenvalues of the integral of A. But for a two dimensional complex-valued matrix A condition (2) is not necessary for (4). For example, if γ is a root of (14) and the functions a, b satisfy (15) and (16), then the matrix

$$A(t) = \begin{pmatrix} -\frac{\gamma a(t)}{2} & 0 \\ b(t) & \frac{\gamma a(t)}{2} \end{pmatrix}, \quad t \in [0, +\infty[,$$

is not a Lappo-Danilevskii matrix with the initial point $s = 0$, however the fundamental solution matrix $X_0(t)$ of (1) may be represented as (4) with $s = 0$.

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