A SIMPLE PROOF OF WHITNEY'S THEOREM ON CONNECTIVITY IN GRAPHS

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Abstract. In 1932 Whitney showed that a graph G with order $n \geqslant 3$ is 2-connected if and only if any two vertices of G are connected by at least two internally-disjoint paths. The above result and its proof have been used in some Graph Theory books, such as in Bondy and Murty's well-known Graph Theory with Applications. In this note we give a much simple proof of Whitney's Theorem.

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We consider a finite undirected simple graph G with the vertex set $V(G)$. If $x, y \in V(G)$ then $d(x, y)$ denotes the distance between x and y, a path in G with end-vertices x and y will be denoted by (x, y).

In 1932 Whitney [2], [3] showed the following well-known result.

Theorem. *A graph G with order $n \geqslant 3$ is 2-connected if and only if any two vertices of G are connected by at least two internally-disjoint paths.*

Whitney's Theorem is Theorem 3.2 in [1]. However, the proof in [1] (pp. 44–45) used Theorem 2.3 [1] (pp. 27–28), so the proof is more complex than the one given here.

Simple Proof of Theorem. If any two vertices of G are connected by at least two internally-disjoint paths, then, clearly, G is connected and has no 1-vertex cut. Hence G is 2-connected.

Conversely, let G be 2-connected graph and assume there exist two vertices u and v without two internally-disjoint (u, v)-paths. Let P and Q be two (u, v)-paths with the common vertex set S as small as possible. Let $w \in S \setminus \{u, v\}$ and P_1, P_2

denote the sections of P from u to w and w to v and Q_1, Q_2 denote the sections of Q from u to w and w to v, respectively. Since G is 2-connected, let R denote a shortest path from some vertex x of $(V(P_1) \cup V(Q_1)) \setminus \{w\}$ to some vertex y of $(V(P_2) \cup V(Q_2)) \setminus \{w\}$ without passing through $\{w\}$. We may assume, without loss of generality, that x is in P_1 and y in Q_2. Let T denote the (u, v)-path composed of the section of P_1 from u to x and the section of Q_2 from y to v together with R. Clearly the common vertices of T and the (u, v)-path composed of Q_1 and P_2 are all in $S \setminus \{w\}$. This contradicts the choice of both P and Q as having the smallest number of vertices. □

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References

[1] *J. A. Bondy, U. S. R. Murty*: Graph Theory with Applications. Elsevier, New York, 1976.

[2] *H. Whitney*: Congruent graphs and the connectivity of graphs. Amer. J. Math. *54* (1932), 150–168. zbl

[3] *H. Whitney*: Non-separable and planar graphs. Trans. Amer. Math. Soc. *34* (1932), 339–362. zbl

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