ON THE UNIQUENESS OF POSITIVE SOLUTIONS FOR TWO-POINT BOUNDARY VALUE PROBLEMS OF EMDEN-FOWLER DIFFERENTIAL EQUATIONS

Satoshi Tanaka, Okayama

(Received October 15, 2009)

Dedicated to Professor Manabu Naito on the occasion of his sixtieth birthday

Abstract. The two-point boundary value problem

$$u'' + h(x)u^p = 0, \quad a < x < b, \qquad u(a) = u(b) = 0$$

is considered, where $p > 1$, $h \in C^1[0,1]$ and $h(x) > 0$ for $a \leqslant x \leqslant b$. The existence of positive solutions is well-known. Several sufficient conditions have been obtained for the uniqueness of positive solutions. On the other hand, a non-uniqueness example was given by Moore and Nehari in 1959. In this paper, new uniqueness results are presented.

Keywords: uniqueness, positive solution, two-point boundary value problem, Emden-Fowler equation

MSC 2010: 34B15

1. Introduction and main results

We consider the two-point boundary value problem

$$\begin{cases} u'' + h(x)u^p = 0, \quad a < x < b, \\ u(a) = u(b) = 0, \end{cases} \tag{1.1}$$

where $p > 1$, $h \in C^1[a,b]$ and $h(x) > 0$ for $x \in [a,b]$.

The Emden-Fowler equation

$$u'' + h(x)u^p = 0 \tag{1.2}$$

is classical. It is the origin of various equations such as the equation with one-dimensional p-Laplacian $(|u'|^{p-2}u')' + h(x)f(u) = 0$ and elliptic partial differential equations of the form $\Delta u + K(|x|)u^p = 0$. It is well-known that if $p > 0$ and $p \neq 1$, then problem (1.1) has at least one positive solution. See, for example, [6], [8] and [14]. It is also well-known that if $0 < p < 1$, then the positive solution is unique. See, for example, [9]. A number of studies have been made on the uniqueness of positive solutions in the case $p > 1$. However, there are still cases for which it is not known whether the positive solution is unique or not. Moore and Nehari [6] presented a function $h(x)$ such that (1.1) has at least three positive solutions. See also [12]. Sufficient conditions for the uniqueness of positive solutions were obtained in [1], [2], [4], [5], [7], [10], [11], [12] and [15]. If one of the following conditions (1.3)–(1.10) is satisfied, then the positive solution of problem (1.1) is unique:

$$h(c-x) = h(c+x), \quad h'(x) \leqslant 0 \quad \text{on } [c,b]; \tag{1.3}$$

$$a > 0, \quad h(x) = x^l, \quad l \in \mathbb{R}; \tag{1.4}$$

$$a > 0, \quad -4m \geqslant m(p-1) + xg(x) \geqslant -2 \text{ on } [a,b] \quad \text{for some } m \leqslant 0; \tag{1.5}$$

$$a > 0, \quad -4m \leqslant m(p-1) + xg(x) \leqslant -2 \text{ on } [a,b] \quad \text{for some } m \geqslant 1; \tag{1.6}$$

$$-2/(x-a) \leqslant g(x) \leqslant 2/(b-x) \text{ on } (a,b); \tag{1.7}$$

$$g(x) \text{ is nonincreasing on } [a,b]; \tag{1.8}$$

$$-2/(x-a) \leqslant g(x) \text{ on } (a,c], \ h'(x) \leqslant 0 \text{ on } [a,b]; \tag{1.9}$$

$$h \in C^2[a,b], \quad ([h(x)]^{-1/2})'' = 4^{-1}(h(x))^{-1/2}\left[(g(x))^2 - 2(g(x))'\right] \leqslant 0, \tag{1.10}$$

where $c = (a+b)/2$ and $g(x) = h'(x)/h(x)$. By the result of Moroney [7], we can obtain (1.3). See also Dalmasso [2]. Conditions (1.4)–(1.6) were established by Coffman [1]. For condition (1.4), see also Ni and Nussbaum [10, Theorem 3.8]. Kwong [4] obtained condition (1.7), and later, in [5], he generalized it as follows: there exist concave functions φ, ψ: $(a,b) \to (0,\infty)$ such that $[\varphi(t)]^2h(t)$ is nonincreasing and $[\psi(t)]^2h(t)$ is nondecreasing. Using the result of Yanagida [15], we have condition (1.8). Condition (1.9) was established by Korman [3]. Condition (1.10) was obtained in [12].

It should be noted that (1.3) and (1.5)–(1.10) are conditions for more general equations such as $u'' + h(x)f(u) = 0$ or $u'' + f(x,u) = 0$. On the other hand, there are only a few uniqueness results for the special problem (1.1). In this paper we study only the special problem (1.1), and then we can obtain new sufficient conditions for the uniqueness of positive solutions.

Theorem 1.1. *Assume that*

$$-\frac{2}{x-a} - d \leqslant \frac{h'(x)}{h(x)}, \quad a < x \leqslant \frac{a+b}{2}, \tag{1.11}$$

$$\frac{h'(x)}{h(x)} \leqslant -d, \quad a \leqslant x \leqslant b \tag{1.12}$$

for some $d \geqslant 0$. *Then the positive solution of* (1.1) *is unique.*

In the case $d = 0$, (1.11) and (1.12) become (1.9).

Let λ_k be the k-th eigenvalue of

$$\varphi'' + \lambda h(x)\varphi = 0, \quad a < x < b, \quad \varphi(a) = \varphi(b) = 0,$$

and let φ_k be an eigenfunction corresponding to λ_k. Then

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_k < \lambda_{k+1} < \ldots, \quad \lim_{k\to\infty} \lambda_k = \infty,$$

and φ_k has exactly $k-1$ zeros in (a, b). (See, for example, [13, Chap. VI, Sec. 27].) Define the constant M_h by

$$M_h = \max_{a\leqslant x\leqslant b} \min\left\{ \frac{(x-a)^p}{\int_a^x (s-a)^{p+1} h(s)\,\mathrm{d}s}, \frac{(b-x)^p}{\int_x^b (b-s)^{p+1} h(s)\,\mathrm{d}s} \right\}.$$

Theorem 1.2. *If* $pM_h \leqslant \lambda_2$, *then the positive solution of* (1.1) *is unique.*

Now set

$$h_* = \min_{a\leqslant x\leqslant b} h(x), \quad h^* = \max_{a\leqslant x\leqslant b} h(x).$$

We can estimate λ_2 and M_h by h_* and h^*. It is easy to see that

$$\lambda_2 \geqslant \frac{1}{h^*}\left(\frac{2\pi}{b-a}\right)^2. \tag{1.13}$$

Assume to the contrary that $\lambda_2 h^* < (2\pi)^2/(b-a)^2$. Since the eigenfunction φ_2 has three zeros in $[a, b]$, the Sturm comparison theorem shows that every solution of

$$\psi'' + \left(\frac{2\pi}{b-a}\right)^2 \psi = 0 \tag{1.14}$$

has at least two zeros in (a, b). This contradicts the fact that $\sin(2\pi(x-a)/(b-a))$ is a solution of (1.14) which has exactly one zero in (a, b). Hence we obtain (1.13).

Using $h(x) \geqslant h_*$, we can see that

$$M_h \leqslant \frac{p+2}{h_*}\Big(\frac{2}{b-a}\Big)^2.$$

Therefore, by Theorem 1.2, we find that if

$$\frac{h^*}{h_*} \leqslant \frac{\pi^2}{p(p+2)}, \tag{1.15}$$

then the positive solution of (1.1) is unique. However, we have the following better result.

Theorem 1.3. *If*

$$\frac{h^*}{h_*} \leqslant \frac{2\pi^2}{p(p+1)[T(p)]^2},$$

then the positive solution of (1.1) *is unique, where* $T(p) = \int_0^1 (1-t^{p+1})^{-1/2}\,\mathrm{d}t$.

We can express the function $T(p)$ by the beta function or the gamma function. Since $T(p)$ is decreasing for $p \geqslant 1$, we see that $1 < T(p) < T(1) = \pi/2$ for $p > 1$. Therefore we obtain the following corollary of Theorem 1.3.

Corollary 1.1. *If*

$$\frac{h^*}{h_*} \leqslant \frac{8}{p(p+1)}, \tag{1.16}$$

then the positive solution of (1.1) *is unique.*

Unfortunately, since $h^*/h_* \geqslant 1$, there is a restriction $1 < p \leqslant (\sqrt{33}-1)/2 = 2.37\ldots$ in Corollary 1.1. We also obtain $1 < p \leqslant \sqrt{\pi^2+1} - 1 = 2.29\ldots$ when (1.15) is satisfied. We easily see that $\pi^2/[p(p+2)] < 8/[p(p+1)]$ for $1 < p < 3$, so that (1.16) is a better condition than (1.15).

Roughly speaking, if $p > 1$ is close to 1 and the function $h(x)$ changes slowly, then the positive solution is unique. On the other hand, by [6] or [12], problem (1.1) has at least three positive solutions for some function $h(x)$ such that $h(x) > 0$ on $[a,b]$ and h^*/h_* is sufficiently large. It is emphasized here that the condition concerning $h'(x)/h(x)$ is not needed in Corollary 1.1. Therefore, the uniqueness of positive solutions does not depend on only the behavior of the function $h'(x)/h(x)$. In fact, we have the next example.

E x a m p l e 1.1. We consider the case where $p = 3/2$ and $h(x) = 3 + \sin(nx)$, $n \geqslant 1$. Then

$$\frac{h^*}{h_*} \leqslant \frac{3+1}{3-1} = 2 < \frac{32}{15} = \frac{8}{p(p+1)}.$$

Hence Corollary 1.1 implies that the positive solution of (1.1) is unique. On the other hand, we observe that

$$\frac{h'(x)}{h(x)} = \frac{n\cos(nx)}{3+\sin(nx)},$$

so that

$$\lim_{n\to\infty} \max_{x\in[a,b]} \frac{h'(x)}{h(x)} = \infty, \quad \lim_{n\to\infty} \min_{x\in[a,b]} \frac{h'(x)}{h(x)} = -\infty.$$

Hence we cannot apply conditions (1.3)–(1.10) if n is sufficiently large.

We can apply the technique of this paper to the study of radially symmetric solutions of the Dirichlet problem

$$\Delta u + K(|x|)u^p = 0 \text{ in } B, \qquad u = 0 \text{ on } \partial B,$$

where $B = \{x \in \mathbb{R}^N \colon |x| < 1\}$, $N \geqslant 3$, $p > 1$, $K \in C^1[0,1]$ and $K(r) > 0$ for $0 \leqslant r \leqslant 1$. However, as own space is limited, it cannot be discussed here. We leave the details to another paper.

2. Proof of theorem 1.1

Let u be a positive solution of problem (1.1) and let w be the solution of the linearized problem

$$w'' + ph(x)u^{p-1}w = 0, \quad w(a) = 0, \quad w'(a) = 1. \tag{2.1}$$

The next proposition follows by the standard argument of the Kolodner-Coffman method. See, for example, Kwong [4].

Proposition 2.1. *For each positive solution u of* (1.1), *if the solution w of problem* (2.1) *satisfies $w(b) < 0$, then problem* (1.1) *has at most one positive solution.*

Lemma 2.1. *Let u be a positive solution of problem* (1.1). *Then the solution w of problem* (2.1) *has at least one zero in* (a, b).

P r o o f. We note that u is a solution of

$$u'' + h(x)u^{p-1}u = 0, \quad u(a) = u(b) = 0.$$

Since $ph(x)u^{p-1} > h(x)u^{p-1}$ on (a, b), the Sturm comparison theorem implies that w has at least one zero in (a, b). □

Lemma 2.2. *Let u be a positive solution of problem* (1.1), *and let $d \geqslant 0$ be a constant. Then $y := u' - [d/(p-1)]u$ satisfies*

$$y'' + ph(x)u^{p-1}y = -h(x)\Big(d + \frac{h'(x)}{h(x)}\Big)u^p, \quad x \in [a, b].$$

P r o o f. A direct calculation shows that Lemma 2.2 follows immediately, by noting that $u''' = -(h(x)u^p)' = -h'(x)u^p - ph(x)u^{p-1}u'$. □

Lemma 2.3. *Let u be a positive solution of problem* (1.1), *and let $d \geqslant 0$ be a constant. Then $Y := (x-a)(u' - [d/(p-1)]u)$ satisfies*

$$Y'' + ph(x)u^{p-1}Y = -\frac{2d}{p-1}u' - h(x)\Big(2 + d(x-a) + (x-a)\frac{h'(x)}{h(x)}\Big)u^p \tag{2.2}$$

for $x \in [a, b]$.

P r o o f. Set $y = u' - [d/(p-1)]u$. Then $Y = (x-a)y$ and

$$Y'' + ph(x)u^{p-1}Y = 2y' + (x-a)(y'' + ph(x)u^{p-1}y), \quad x \in [a, b].$$

By Lemma 2.2, we see that

$$Y'' + ph(x)u^{p-1}Y = 2y' - (x-a)h(x)\Big(d + \frac{h'(x)}{h(x)}\Big)u^p, \quad x \in [a, b].$$

This implies (2.2). □

Every positive solution u of (1.1) admits only one point of maximum, since $u'' = -h(x)u^p < 0$ for $x \in (a, b)$. We denote by m the point of maximum of the positive solution u. Then we see that $u'(m) = 0$, $u'(x) > 0$ on $[a, m)$, and $u'(x) < 0$ on $(m, b]$.

The following result was obtained by Korman [3, Lemma 2.2].

Lemma 2.4. *Assume that* $h'(x) \leqslant 0$ *for* $x \in [a, b]$*. Then* $a < m \leqslant (a + b)/2$.

P r o o f of Theorem 1.1. Let u be a positive solution of (1.1). Set $y = u' - [d/(p-1)]u$. Then $y' = u'' - [d/(p-1)]u' = -h(x)u^p - [d/(p-1)]u' < 0$ on $[a, m]$. Since $y(a) = u'(a) > 0$ and $y(x) < 0$ on $[m, b]$, there exists $z \in (a, m)$ such that $y(z) = 0$, $y(x) > 0$ on $[a, z)$ and $y(x) < 0$ on $(z, b]$.

We claim that $w(x) > 0$ for $x \in (a, z)$. Assume to the contrary that there exists a number $x_0 \in (a, z)$ such that $w(x_0) = 0$ and $w(x) > 0$ on (a, x_0). We note that $w'(x_0) < 0$. From (1.11) and Lemmas 2.3 and 2.4, it follows that $Y := (x - a)y$ satisfies

$$Y'' + ph(x)u^{p-1}Y \leqslant -\frac{2d}{p-1}u', \quad x \in [a, z].$$

Hence we see that

$$wY'' - w''Y \leqslant -\frac{2d}{p-1}u'w, \quad x \in [a, x_0],$$

so that

$$\int_a^{x_0} (wY'' - w''Y)\,\mathrm{d}x \leqslant 0, \quad x \in [a, x_0].$$

On the other hand, since $w'(x_0) < 0$ and $Y(x_0) > 0$, we find that

$$\int_a^{x_0} (wY'' - w''Y)\,\mathrm{d}x = wY' - w'Y \,\Big|_a^{x_0} = -w'(x_0)Y(x_0) > 0.$$

This is a contradiction. Therefore $w(x) > 0$ for $x \in (a, z)$ as claimed.

By Lemma 2.1, there exists $x_1 \in [z, b)$ such that $w(x_1) = 0$ and $w(x) > 0$ on (a, x_1). We show that $w(x) < 0$ for $(x_1, b]$. Suppose that there exists $x_2 \in (x_1, b]$ such that $w(x_2) = 0$ and $w(x) < 0$ on (x_1, x_2). Lemma 2.2 and (1.12) imply that

$$\int_{x_1}^{x_2} (wy'' - w''y)\,\mathrm{d}x = -\int_{x_1}^{x_2} h(x)\Big(d + \frac{h'(x)}{h(x)}\Big)u^p w\,\mathrm{d}x \leqslant 0.$$

Since $w'(x_1) < 0$, $w'(x_2) > 0$ and $y(x) < 0$ on $(z, b]$, we see that

$$\int_{x_1}^{x_2} (wy'' - w''y)\,\mathrm{d}x = -w'(x_2)y(x_2) + w'(x_1)y(x_1) > 0.$$

This is a contradiction. Hence $w(x) < 0$ for $(x_1, b]$. Proposition 2.1 shows that the positive solution of (1.1) is unique. □

3. Proofs of theorems 1.2 and 1.3

Lemma 3.1. *Assume that there exists $M > 0$ such that $u(x) \leqslant M^{1/(p-1)}$ on $[a, b]$ for each positive solution u of* (1.1). *If $pM \leqslant \lambda_2$, then the positive solution of* (1.1) *is unique.*

Proof. Let u be a positive solution of (1.1). (Recall that the existence of positive solutions is well-known.) We see that

$$ph(x)[u(x)]^{p-1} \leqslant ph(x)M \leqslant \lambda_2 h(x), \quad x \in [a, b],$$

and $ph(x)[u(x)]^{p-1} \not\equiv \lambda_2 h(x)$ for $x \in [a, b]$. Since φ_2, which is an eigenfunction corresponding to λ_2, has exactly one zero in (a, b) and satisfies $\varphi_2(a) = \varphi_2(b) = 0$, the Sturm comparison theorem implies that w has at most one zero in $(a, b]$. By Lemma 2.1, we see that w has at lest one zero in (a, b), hence w has exactly one zero in (a, b) and $w(b) < 0$. From Proposition 2.1 it follows that the positive solution of (1.1) is unique. □

Lemma 3.2. *Let u be a positive solution of problem* (1.1). *Then $[u(x)]^{p-1} < M_h$ for $x \in [a, b]$.*

Proof. Since u is concave, we see that

$$u(x) > \frac{u(m)}{m-a}(x-a), \quad x \in (a, m) \tag{3.1}$$

and

$$u(x) > \frac{u(m)}{b-m}(b-x), \quad x \in (m, b), \tag{3.2}$$

where $m \in (a, b)$ is the point of maximum of u. Integrating (1.2) over $[t, m]$ and integrating it again over $[a, m]$, we obtain

$$u(m) = \int_a^m \int_t^m h(x)[u(x)]^p \, \mathrm{d}x \, \mathrm{d}t = \int_a^m (x-a)h(x)[u(x)]^p \, \mathrm{d}x. \tag{3.3}$$

From (3.1) and (3.3) it follows that

$$u(m) > \int_a^m (x-a)h(x)\Big(\frac{u(m)}{m-a}(x-a)\Big)^p \, \mathrm{d}x = \frac{[u(m)]^p}{(m-a)^p}\int_a^m (x-a)^{p+1}h(x)\, \mathrm{d}x,$$

that is,

$$[u(m)]^{p-1} < \frac{(m-a)^p}{\int_a^m (x-a)^{p+1}h(x)\, \mathrm{d}x}. \tag{3.4}$$

In the same way, integrating (1.2) over $[m,t]$ and integrating it again over $[m,b]$, and using (3.2), we have

$$[u(m)]^{p-1} < \frac{(b-m)^p}{\int_m^b (b-x)^{p+1} h(x)\,\mathrm{d}x}. \tag{3.5}$$

Combining (3.4) and (3.5), we see that $[u(x)]^{p-1} \leqslant [u(m)]^{p-1} < M_h$ on $[a,b]$. □

Lemma 3.3. *Let u be a positive solution of problem* (1.1)*. Then*

$$[u(x)]^{p-1} \leqslant \frac{2(p+1)[T(p)]^2}{h_*(b-a)^2}, \quad x \in [a,b].$$

Proof. We denote by m the point of maximum of the positive solution u. Then we see that $u'(m) = 0$, $u'(x) > 0$ on $[a,m)$, and $u'(x) < 0$ on $(m,b]$. Multiplying $u'' + h_* u^p \leqslant 0$ by $2u'$, we find that

$$[(u')^2]' + C(u^{p+1})' \leqslant 0, \quad x \in [a,m], \tag{3.6}$$

where $C = 2h_*/(p+1)$. Integrating (3.6) over $[x,m]$, we have

$$-(u')^2 + C\left([u(m)]^{p+1} - u^{p+1}\right) \leqslant 0, \quad x \in [a,m],$$

which implies that

$$C^{1/2} \leqslant ([u(m)]^{p+1} - u^{p+1})^{-1/2} u', \quad x \in [a,m). \tag{3.7}$$

Integrating (3.7) over $[a,m)$ and substituting $t = u(x)/u(m)$, we see that

$$C^{1/2}(m-a) \leqslant \int_a^m ([u(m)]^{p+1} - u^{p+1})^{-1/2} u'\,\mathrm{d}x = [u(m)]^{-(p-1)/2} T(p). \tag{3.8}$$

Multiplying $u'' + h_* u^p \leqslant 0$ by $2u'$, we have

$$[(u')^2]' + C(u^{p+1})' \geqslant 0, \quad x \in [m,b]. \tag{3.9}$$

Integrating (3.9) over $[m,x]$, we obtain

$$C^{1/2} \leqslant -([u(m)]^{p+1} - u^{p+1})^{-1/2} u', \quad x \in (m,b]. \tag{3.10}$$

Integrating (3.10) over $(m,b]$, we have

$$C^{1/2}(b-m) \leqslant [u(m)]^{-(p-1)/2} T(p). \tag{3.11}$$

From (3.8) and (3.11) it follows that

$$C^{1/2}(b-a)=C^{1/2}(b-m)+C^{1/2}(m-a)\leqslant 2[u(m)]^{-(p-1)/2}T(p),$$

which means

$$[u(m)]^{p-1}\leqslant \frac{2(p+1)[T(p)]^2}{h_*(b-a)^2}.$$

This completes the proof. □

P r o o f of Theorems 1.2 and 1.3. Theorem 1.2 follows from Lemmas 3.1 and 3.2. Combining Lemmas 3.1, 3.3 and (1.13), we can obtain Theorem 1.3. □

References

[1] *C. V. Coffman*: On the positive solutions of boundary-value problems for a class of nonlinear differential equations. J. Differ. Equations *3* (1967), 92–111.

[2] *R. Dalmasso*: Uniqueness of positive solutions of nonlinear second-order equations. Proc. Amer. Math. Soc. *123* (1995), 3417–3424. zbl

[3] *P. Korman*: On the multiplicity of solutions of semilinear equations. Math. Nachr. *229* (2001), 119–127. zbl

[4] *M. K. Kwong*: On the Kolodner-Coffman method for the uniqueness problem of Emden-Fowler BVP. Z. Angew. Math. Phys. *41* (1990), 79–104. zbl

[5] *M. K. Kwong*: Uniqueness results for Emden-Fowler boundary value problems. Nonlinear Anal. *16* (1991), 435–454. zbl

[6] *R. Moore, Z. Nehari*: Nonoscillation theorems for a class of nonlinear differential equations. Trans. Amer. Math. Soc. *93* (1959), 30–52. zbl

[7] *R. M. Moroney*: Note on a theorem of Nehari. Proc. Amer. Math. Soc. *13* (1962), 407–410. zbl

[8] *M. Naito, Y. Naito*: Solutions with prescribed numbers of zeros for nonlinear second order differential equations. Funkcial. Ekvac. *37* (1994), 505–520. zbl

[9] *Y. Naito*: Uniqueness of positive solutions of quasilinear differential equations. Differ. Int. Equations *8* (1995), 1813–1822. zbl

[10] *W.-M. Ni, R. D. Nussbaum*: Uniqueness and nonuniqueness for positive radial solutions of $\Delta u+f(u,r)=0$. Comm. Pure Appl. Math. *38* (1985), 67–108. zbl

[11] *S. Tanaka*: On the uniqueness of solutions with prescribed numbers of zeros for a two-point boundary value problem. Differ. Int. Equations *20* (2007), 93–104.

[12] *S. Tanaka*: An identity for a quasilinear ODE and its applications to the uniqueness of solutions of BVP. J. Math. Anal. Appl. *351* (2009), 206–217. zbl

[13] *W. Walter*: Ordinary Differential Equations. Graduate Texts in Mathematics, 182, Springer, New York, 1998. zbl

[14] *H. Wang*: On the existence of positive solutions for semilinear elliptic equations in the annulus. J. Differ. Equations *109* (1994), 1–7. zbl

[15] *E. Yanagida*: Sturmian theory for a class of nonlinear second-order differential equations. J. Math. Anal. Appl. *187* (1994), 650–662. zbl

Author's address: *Satoshi Tanaka*, Department of Applied Mathematics, Faculty of Science, Okayama University of Science, Ridaichou 1-1, Okayama 700–0005, Japan, e-mail: `tanaka@xmath.ous.ac.jp`.