A NONEXISTENCE RESULT FOR THE KURZWEIL INTEGRAL

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Abstract. It is shown that there exist a continuous function f and a regulated function g defined on the interval $[0,1]$ such that g vanishes everywhere except for a countable set, and the K^*-integral of f with respect to g does not exist. The problem was motivated by extensions of evolution variational inequalities to the space of regulated functions.

Keywords: Kurzweil integral, regulated functions

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Introduction

The present paper has been motivated by an auxiliary problem which arose in connection with the investigation of generalized evolution variational inequalities in the space of regulated functions in [4]. It can be stated as follows: which notions of the integral have the property that

$$\int_a^b f(t)\,\mathrm{d}g(t) = 0 \tag{0.1}$$

for every pair of regulated functions $f, g\colon [a,b] \to \mathbb{R}$ such that $g(t) = 0$ everywhere except for a countable subset $N \subset \,]a,b[$?

Recall that a function $f\colon [a,b] \to \mathbb{R}$ is said to be *regulated* (cf. [1]) if finite one-sided limits $f(t-)$, $f(t+)$ exist for every $t \in [a,b]$ with the convention $f(a-) = f(a)$, $f(b+) = f(b)$. A more systematic information about regulated functions can be found e.g. in [2].

The identity (0.1) is obviously fulfilled if it is interpreted as the Young integral in the form presented in [3]. The problem is more delicate for both versions of

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the Kurzweil integral, the original one introduced in [5] (the K-integral), and its generalization proposed in [7] (the so-called K^*-integral). In general, the above statement holds only conditionally, that is,

$$\text{if } \int_a^b f(t)\,\mathrm{d}g(t) \text{ exists, then } \int_a^b f(t)\,\mathrm{d}g(t) = 0, \tag{0.2}$$

see Proposition 2.4 of [6]. On the other hand, (0.1) is true if at least one of the functions f, g is of bounded variation, see Proposition 2.13 and Corollary 2.14 of [9].

The main result of this paper consists in giving a negative answer to the above problem for the K^*-integral (and, a fortiori, for the K-integral). We construct explicitly a regulated function $g\colon [0,1] \to \mathbb{R}$ which vanishes everywhere except for a countable set, and a continuous function $f\colon [0,1] \to \mathbb{R}$ such that $(K^*)\int_0^1 f(t)\,\mathrm{d}g(t)$ does not exist, see Theorem 1.4 below. This means in particular that the Young integral is not included in Kurzweil's theory. An interested reader can find more information about the Kurzweil integral and its relation to other types of integrals e.g. in [6], [7], [8] or [9].

1. Statement of the main result

We first recall the definition of the Kurzweil integral. Consider a compact interval $[a,b] \subset \mathbb{R}$. The basic concept in the whole theory is that of a *δ-fine partition*. We define the set

$$\Gamma(a,b) := \{\delta\colon [a,b] \to \mathbb{R};\ \delta(t) > 0 \quad \text{for every } t \in [a,b]\}. \tag{1.1}$$

An element $\delta \in \Gamma(a,b)$ is called a *gauge*.

Let $a = t_0 < t_1 < \ldots < t_m = b$ be a division of $[a,b]$ and let $\tau = \{\tau_1, \ldots, \tau_m\}$, $a \leqslant \tau_1 \leqslant \tau_2 \leqslant \ldots \leqslant \tau_m \leqslant b$ be a sequence such that $\tau_j \in [t_{j-1}, t_j]$ for $j = 1, \ldots, m$. Then the system $D = \{(\tau_j, [t_{j-1}, t_j]);\ j = 1, \ldots, m\}$ is called a *partition*. For $t \in [a,b]$ and $\delta \in \Gamma(a,b)$ we denote

$$I_\delta(t) := \,]t - \delta(t), t + \delta(t)[. \tag{1.2}$$

Definition 1.1. Let $\delta \in \Gamma(a,b)$ be a gauge. A partition $D = \{(\tau_j, [t_{j-1}, t_j]);\ j = 1, \ldots, m\}$ is said to be *δ-fine* if for every $j = 1, \ldots, m$ we have

$$\tau_j \in [t_{j-1}, t_j] \subset I_\delta(\tau_j). \tag{1.3}$$

If moreover a δ-fine partition D satisfies the implications

$$\tau_j = t_{j-1} \Rightarrow j = 1, \quad \tau_j = t_j \Rightarrow j = m, \tag{1.3*}$$

then it is called a *δ-fine** partition.

The set of all δ-fine (δ-fine*) partitions is denoted by $\mathcal{F}_\delta(a,b)$ ($\mathcal{F}^*_\delta(a,b)$, respectively).

We have indeed $\mathcal{F}^*_\delta(a,b) \subset \mathcal{F}_\delta(a,b)$. The next lemma implies in particular that these sets are nonempty for every $\delta \in \Gamma(a,b)$.

Lemma 1.2. *Let $\delta \in \Gamma(a,b)$ and a dense subset $\Omega \subset \,]a,b[$ be given. Then there exists $D = \{(\tau_j, [t_{j-1}, t_j]);\ j = 1,\dots,m\} \in \mathcal{F}^*_\delta(a,b)$ such that $t_j \in \Omega$ for every $j = 1,\dots,m-1$.*

P r o o f. We have $[a,b] \subset \bigcup\limits_{t\in[a,b]} I_\delta(t)$, hence there exists a finite covering

$$[a,b] \subset \bigcup_{j=1}^{m} I_\delta(\tau_j), \quad a \leqslant \tau_1 \leqslant \ldots \leqslant \tau_m \leqslant b. \tag{1.4}$$

The inclusion remains valid if we eliminate all intervals $I_\delta(\tau_j)$ for which there exists $k \neq j$, $I_\delta(\tau_j) \subset I_\delta(\tau_k)$. We claim that then we have

$$\min\{\tau_{j+1}, \tau_j + \delta(\tau_j)\} > \max\{\tau_j, \tau_{j+1} - \delta(\tau_{j+1})\} \tag{1.5}$$

for every $j = 1,\dots,m-1$. Indeed, we obviously have $\tau_{j+1} > \tau_j$, since otherwise $I_\delta(\tau_{j+1}) \subset I_\delta(\tau_j)$ or $I_\delta(\tau_j) \subset I_\delta(\tau_{j+1})$ according to whether $\delta(\tau_{j+1}) \leqslant \delta(\tau_j)$ or $\delta(\tau_{j+1}) \geqslant \delta(\tau_j)$. Assume now that for some j we have

$$\min\{\tau_{j+1}, \tau_j + \delta(\tau_j)\} \leqslant \max\{\tau_j, \tau_{j+1} - \delta(\tau_{j+1})\}.$$

Then $\tau_{j+1} > \tau_{j+1} - \delta(\tau_{j+1}) \geqslant \tau_j + \delta(\tau_j) > \tau_j$, hence the points $\tau_j + \delta(\tau_j)$, $\tau_{j+1} - \delta(\tau_{j+1})$ do not belong to $I_\delta(\tau_j) \cup I_\delta(\tau_{j+1})$. Then there exists necessarily either $k < j$ such that $\tau_j + \delta(\tau_j) \in I_\delta(\tau_k)$, hence $I_\delta(\tau_j) \subset I_\delta(\tau_k)$, or $k > j+1$ such that $\tau_{j+1} - \delta(\tau_{j+1}) \in I_\delta(\tau_k)$, hence $I_\delta(\tau_{j+1}) \subset I_\delta(\tau_k)$, which is a contradiction. Inequality (1.5) is thus verified and we may choose arbitrarily

$$t_j \in \big]\max\{\tau_j, \tau_{j+1} - \delta(\tau_{j+1})\},\ \min\{\tau_{j+1}, \tau_j + \delta(\tau_j)\}\big[\cap \Omega, \quad j = 1,\dots,m-1,$$

$t_0 := a$, $t_m := b$, and the assertion immediately follows. □

We are now ready to give a formal definition of both types of the Kurzweil integral. For given functions $f, g\colon [a,b] \to \mathbb{R}$ and a partition $D = \{(\tau_j, [t_{j-1}, t_j]);\ j = 1,\dots,m\}$ we define the integral sum $S_D(f\Delta g)$ by the formula

$$S_D(f\Delta g) = \sum_{j=1}^{m} f(\tau_j)(g(t_j) - g(t_{j-1})). \tag{1.6}$$

Definition 1.3. We say that $J \in \mathbb{R}$ ($J^* \in \mathbb{R}$) is the *K-integral* (*K^*-integral*, respectively) over $[a,b]$ of f with respect to g and denote

$$(1.7) \qquad J = (K)\int_a^b f(t)\,\mathrm{d}g(t), \qquad \Bigl(J^* = (K^*)\int_a^b f(t)\,\mathrm{d}g(t), \quad \text{respectively}\Bigr),$$

if for every $\varepsilon > 0$ there exists $\delta \in \Gamma(a,b)$ such that for every $D \in \mathcal{F}_\delta(a,b)$ ($D^* \in \mathcal{F}^*_\delta(a,b)$, respectively) we have

$$(1.8) \qquad |J - S_D(f\Delta g)| \leqslant \varepsilon, \qquad (|J^* - S_{D^*}(f\Delta g)| \leqslant \varepsilon, \quad \text{respectively}).$$

Using the fact that the implication

$$(1.9) \qquad \delta \leqslant \min\{\delta_1, \delta_2\} \Rightarrow \begin{cases} \mathcal{F}^*_\delta(a,b) \subset \mathcal{F}^*_{\delta_1}(a,b) \cap \mathcal{F}^*_{\delta_2}(a,b), \\ \mathcal{F}_\delta(a,b) \subset \mathcal{F}_{\delta_1}(a,b) \cap \mathcal{F}_{\delta_2}(a,b) \end{cases}$$

holds for every $\delta, \delta_1, \delta_2 \in \Gamma(a,b)$, we easily check that the values J, J^* in Definition 1.3 are uniquely determined. Since $\mathcal{F}^*_\delta(a,b) \subset \mathcal{F}_\delta(a,b)$ for every gauge δ, we also see that if $(K)\int_a^b f(t)\,\mathrm{d}g(t)$ exists, then $(K^*)\int_a^b f(t)\,\mathrm{d}g(t)$ exists and both are equal.

Let I be the interval $[0,1]$. Consider the function $Q\colon I \to I$ given by the formula

$$(1.10) \qquad Q(t) := \begin{cases} 2^{-n} & \text{if } t = (2j-1)2^{-n},\ j = 1,\dots,2^{n-1},\ n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

The main result of this paper reads as follows.

Theorem 1.4. *Let $\alpha \in \,]0,1[$ be given and for $t \in I$ put $g(t) := Q^\alpha(t)$. Then there exists a continuous function $f\colon I \to \mathbb{R}$ such that the integral $(K^*)\int_0^1 f(t)\,\mathrm{d}g(t)$ does not exist.*

A proof of Theorem 1.4 will be given in the next section. It makes substantial use of the concept of *outer measure* $\mu(E) \in [0,\infty]$ of an arbitrary set $E \subset \mathbb{R}$. For the reader's convenience, we briefly recall here its basic properties that are used in the sequel. By 'meas' we denote the Lebesgue measure in $\mathbb{R}$.

Proposition 1.5.
(i) *For every $E \subset \mathbb{R}$ we have $\mu(E) = \inf\{\operatorname{meas}(V);\ V \text{ open subset of } \mathbb{R},\ E \subset V\}$.*
(ii) *If E is measurable, then $\mu(E) = \operatorname{meas}(E)$.*
(iii) *For every $E_1 \subset E_2 \subset \mathbb{R}$ we have $\mu(E_1) \leqslant \mu(E_2)$.*
(iv) *For every $E_1, E_2 \subset \mathbb{R}$ we have $\mu(E_1 \cup E_2) \leqslant \mu(E_1) + \mu(E_2)$.*

(v) *Let* $E_1 \subset E_2 \subset \ldots \subset I$ *be a sequence of sets such that* $I = \bigcup_{n=1}^{\infty} E_n$. *Then we have* $\lim_{n\to\infty} \mu(E_n) = 1$.

Sketch of the proof. Statements (i)–(iv) belong to the standard course on Lebesgue measure. To prove (v), we denote $m_n := \mu(E_n)$ for $n \in \mathbb{N}$. The sequence $\{m_n\}$ is nondecreasing, we can therefore put $m := \lim_{n\to\infty} m_n \leqslant 1$. Assume that $m < 1$ and put $\eta := (1-m)/2$. By definition, for every $n \in \mathbb{N}$ there exists an open set $V_n \supset E_n$ such that $\operatorname{meas}(V_n) \leqslant 1 - \eta$. The sets $A_n := I \setminus (\bigcap_{k=n}^{\infty} V_k)$ are measurable, $\operatorname{meas}(A_n) \geqslant \eta$, $A_n \cap E_n = \emptyset$ for every $n \in \mathbb{N}$, $I \supset A_1 \supset A_2 \supset \ldots$. Put $A_\infty := \bigcap_{n=1}^{\infty} A_n$, $D_n := A_n \setminus A_{n+1}$ for $n \in \mathbb{N}$. Then $A_n = A_\infty \cup (\bigcup_{k=n}^{\infty} D_k)$ and the union is disjoint, hence $\operatorname{meas}(A_n) = \operatorname{meas}(A_\infty) + \sum_{k=n}^{\infty} \operatorname{meas}(D_k)$. Letting n tend to infinity we conclude that $\operatorname{meas}(A_\infty) \geqslant \eta$, $A_\infty \cap E_n = \emptyset$ for every $n \in \mathbb{N}$, which is a contradiction. $\square$

2. Proof of Theorem 1.4

In order to construct the function f satisfying the conditions of Theorem 1.4, we choose a decreasing sequence $\{s_n;\ n \in \mathbb{N}\}$ of positive numbers such that

$$\sigma := \sum_{n=1}^{\infty} s_n < \frac{1}{4}. \tag{2.1}$$

We introduce the intervals

$$\begin{cases} K_i^n := \,](i-1+s_n)2^{-n}, (i-s_n)2^{-n}[\,, \quad i = 1, \ldots, 2^n,\ n \in \mathbb{N}, \\ J_i^n := [(i-s_n)2^{-n}, (i+s_n)2^{-n}]\,, \quad i = 1, \ldots, 2^n - 1,\ n \in \mathbb{N} \end{cases} \tag{2.2}$$

completed by $J_0^n := [0, s_n 2^{-n}]$, $J_{2^n}^n := [1 - s_n 2^{-n}, 1]$, see Fig. 1. We further denote

$$J^n := \bigcup_{i=0}^{2^n} J_i^n, \quad K^n := \bigcup_{i=1}^{2^n} K_i^n, \quad K := \bigcap_{n=1}^{\infty} K^n. \tag{2.3}$$

Then $J^n \cup K^n = I$ and $\operatorname{meas}(J^n) = 2s_n$ for every $n \in \mathbb{N}$, hence

$$\operatorname{meas}(I \setminus K) = \operatorname{meas}\Big(\bigcup_{n=1}^{\infty} J^n\Big) \leqslant 2\sum_{n=1}^{\infty} s_n = 2\sigma. \tag{2.4}$$

The function f will be constructed in the following way. We fix some $h > 0$ such that

$$2^{\alpha-1} < h < 1 \tag{2.5}$$

and for $n \in \mathbb{N}$ and $t \in I$ put

$$f_n(t) := \begin{cases} h^n & \text{if } t \in K_i^n,\ i \text{ odd}, \\ 0 & \text{if } t \in K_i^n,\ i \text{ even}, \\ h^n \dfrac{s_n + (-1)^i(2^n t - i)}{2s_n} & \text{if } t \in J_i^n,\ i = 0, \ldots, 2^n. \end{cases} \tag{2.6}$$

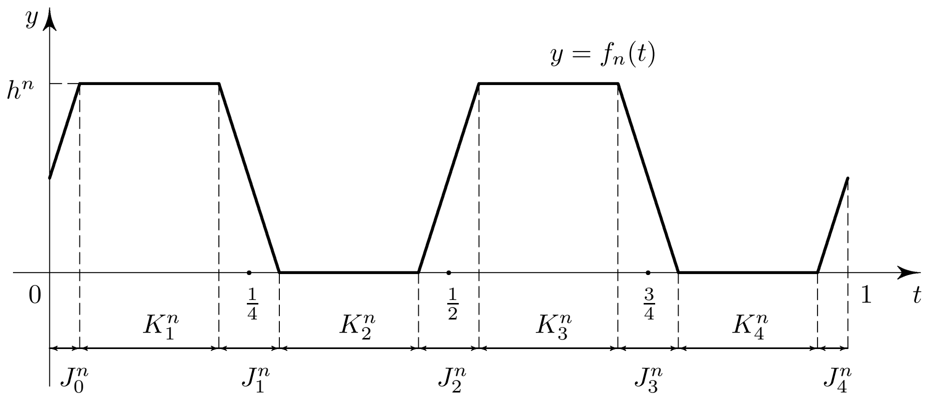

Figure 1. Graph of the function $y = f_n(t)$ for $n = 2$.

Then f_n are continuous, $|f_n|_\infty = h^n$ for every $n \in \mathbb{N}$, see Figure 1. We next fix an integer $r \in \mathbb{N}$ such that

$$h^r < \frac{1}{2}, \tag{2.7}$$

and for $t \in I$ put

$$f(t) := \sum_{k=1}^{\infty} f_{rk}(t). \tag{2.8}$$

The series in (2.8) is uniformly convergent, hence the definition is meaningful and f is continuous.

It remains to check that the integral $(K^*) \int_0^1 f(t)\,\mathrm{d}g(t)$ does not exist. To this end, we consider an arbitrary gauge $\delta \in \Gamma(0,1)$, and for $n \in \mathbb{N}$ we define the sets

$$E_n := \{t \in I;\ \delta(t) \geqslant 2^{-n}\}. \tag{2.9}$$

By Proposition 1.5 we have $\lim\limits_{n\to\infty}\mu(E_n)=1$. We fix $\nu>0$ and $\ell\in\mathbb{N}$ such that

$$2\sigma<\nu<\tfrac{1}{2},\quad \mu(E_\ell)\geqslant 1-\nu+2\sigma. \tag{2.10}$$

For every $n\geqslant\ell$ we then have by (2.4) and Proposition 1.5 that

$$\mu(E_n\cap K)\geqslant\mu(E_\ell)-\operatorname{meas}(I\setminus K)\geqslant 1-\nu. \tag{2.11}$$

Let us define the sets of indices

$$\begin{cases} B_n:=\{i\in\{1,\dots,2^n\};\ E_n\cap K\cap K_i^n\neq\emptyset\},\\ C_n:=\{i\in\{1,\dots,2^n\};\ E_n\cap K\cap K_i^n=\emptyset\}.\end{cases} \tag{2.12}$$

For every $n\in\mathbb{N}$ we obviously have

$$E_n\cap K=\bigcup_{i\in B_n}(E_n\cap K\cap K_i^n), \tag{2.13}$$

hence

$$\mu(E_n\cap K)\leqslant(\#B_n)\max_i\{\operatorname{meas}(K_i^n)\}\leqslant 2^{-n}(\#B_n), \tag{2.14}$$

where the symbol '#' means 'number of elements'. This and (2.11) yields for $n\geqslant\ell$ that

$$\#B_n\geqslant(1-\nu)2^n,\quad \#C_n\leqslant\nu 2^n. \tag{2.15}$$

We continue by introducing the sets

$$\begin{cases} X_n:=\{j\in\{1,\dots,2^{n-1}\};\ 2j-1\in C_n\ \text{ or }\ 2j\in C_n\},\\ Y_n:=\{j\in\{1,\dots,2^{n-1}\};\ 2j-1\in B_n\ \text{ and }\ 2j\in B_n\}.\end{cases} \tag{2.16}$$

Then $\#X_n+\#Y_n=2^{n-1}$, $\#X_n\leqslant\#C_n$, hence

$$\#Y_n\geqslant\left(\tfrac{1}{2}-\nu\right)2^n\qquad\forall n\geqslant\ell. \tag{2.17}$$

We fix some $n\geqslant\ell$ of the form $n=rp$, $p\in\mathbb{N}$, and for every $j\in Y_{rp}$ we find $\tau_{2j-1}\in E_{rp}\cap K\cap K_{2j-1}^{rp}$, $\tau_{2j}\in E_{rp}\cap K\cap K_{2j}^{rp}$, and put $t_{2j-1}:=(2j-1)2^{-rp}$.

The next step consists in constructing a suitable δ-fine* partition D with an arbitrarily large integral sum $S_D(f\Delta g)$. We are able to control the contribution to $S_D(f\Delta g)$ on points τ_{2j-1}, τ_{2j} and t_{2j-1} for $j\in Y_{rp}$, while the gaps between the

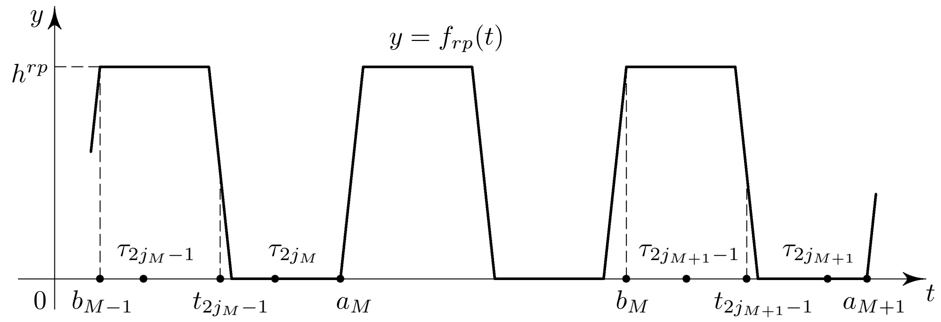

Figure 2. Illustration to the proof of Theorem 1.4.

points τ_{2j} and $\tau_{2j'-1}$ for any two consecutive elements $j, j' \in Y_{rp}$ will be filled in by δ-fine* partitions with zero contribution to $S_D(f\Delta g)$.

Let $1 \leqslant j_1 < j_2 < \ldots < j_N \leqslant 2^{rp-1}$ be all elements of Y_{rp}, $N = \#Y_{rp}$. We denote $a_0 := 0$, $b_N := 1$, and

$$\begin{aligned} a_M &:= (2j_M - s_{rp})2^{-rp} \quad \text{for} \quad M = 1, \ldots, N, \\ b_M &:= (2j_{M+1} - 2 + s_{rp})2^{-rp} \quad \text{for} \quad M = 0, \ldots, N-1, \end{aligned}$$

see Fig. 2. In each interval $[a_M, b_M]$ we use Lemma 1.2 for $\Omega = I \setminus \mathbb{Q}$ (by $\mathbb{Q}$ we denote the set of rational numbers) and find a partition $D^M = \{(\tau_i^M, [t_{i-1}^M, t_i^M]);\ i = 1, \ldots, m_M\} \in \mathcal{F}_\delta^*(a_M, b_M)$ such that $t_i^M \in \Omega$ for every $i = 1, \ldots, m_M - 1$, $t_0^M = a_M$, $t_{m_M}^M = b_M$. We now choose arbitrarily

$$\begin{aligned} \hat{t}_0^M &\in I_\delta(\tau_1^M) \cap]\tau_{2j_M}, a_M[\cap \Omega \quad \text{for } M = 1, \ldots, N, \\ \hat{t}_{m_M}^M &\in I_\delta(\tau_{m_M}^M) \cap]b_M, \tau_{2j_{M+1}-1}[\cap \Omega \quad \text{for } M = 0, \ldots, N-1, \end{aligned}$$

and put $\hat{t}_i^M = t_i^M$ otherwise. Then

$$\tag{2.18} \begin{aligned} D := &\bigcup_{M=0}^{N} \{(\tau_i^M, [\hat{t}_{i-1}^M, \hat{t}_i^M]);\ i = 1, \ldots, m_M\} \\ &\cup \bigcup_{M=1}^{N} \{(\tau_{2j_M-1}, [\hat{t}_{m_{M-1}}^{M-1}, t_{2j_M-1}]),\ (\tau_{2j_M}, [t_{2j_M-1}, \hat{t}_0^M])\} \end{aligned}$$

is a δ-fine* partition of I and using the fact that $g(\hat{t}_i^M) = 0$ for every $i = 1, \ldots, m_M$, $M = 0, \ldots, N$, we obtain

$$\tag{2.19} S_D(f\Delta g) = \sum_{j \in Y_{rp}} g(t_{2j-1})(f(\tau_{2j-1}) - f(\tau_{2j})).$$

Let us now evaluate separately the three terms on the right-hand side of the identity

$$f(\tau_{2j-1}) - f(\tau_{2j}) = \sum_{k=1}^{p-1}(f_{rk}(\tau_{2j-1}) - f_{rk}(\tau_{2j})) + f_{rp}(\tau_{2j-1}) - f_{rp}(\tau_{2j}) + \sum_{k=p+1}^{\infty}(f_{rk}(\tau_{2j-1}) - f_{rk}(\tau_{2j})). \tag{2.20}$$

By construction we have $|f_{rk}(\tau_{2j-1}) - f_{rk}(\tau_{2j})| \leqslant h^{rk}$ for every k, hence

$$\Big|\sum_{k=p+1}^{\infty}(f_{rk}(\tau_{2j-1}) - f_{rk}(\tau_{2j}))\Big| \leqslant h^{rp}\,\frac{h^r}{1-h^r}, \tag{2.21}$$

and, due to the choice of τ_{2j-1}, τ_{2j} (see Figure 2), we obtain that

$$f_{rp}(\tau_{2j-1}) - f_{rp}(\tau_{2j}) = h^{rp}. \tag{2.22}$$

From the inclusion $K^{rp}_{2j-1} \cup K^{rp}_{2j} \subset \,](j-1)2^{1-rp}, j2^{1-rp}[$ it follows for $k < p$ that

$$\begin{aligned} K \cap \left(K^{rp}_{2j-1} \cup K^{rp}_{2j}\right) &\subset K^{rk} \cap \,](j-1)2^{1-rp}, j\,2^{1-rp}[\\ &\subset K^{rk} \cap \,](m-1)2^{-rk}, m\,2^{-rk}[\, = K^{rk}_m, \end{aligned} \tag{2.23}$$

where m is the integer part of the rational number $1 + (j-1)2^{1-r(p-k)}$. Since f_{rk} is constant on K^{rk}_m, we obtain from (2.23) that

$$f_{rk}(\tau_{2j-1}) - f_{rk}(\tau_{2j}) = 0 \quad \text{for } k < p. \tag{2.24}$$

Combining (2.20) with (2.21), (2.22) and (2.24) yields

$$f(\tau_{2j-1}) - f(\tau_{2j}) \geqslant h^{rp}\frac{1-2h^r}{1-h^r} \tag{2.25}$$

for every $j \in Y_{rp}$. We moreover have $g(t_{2j-1}) \geqslant 2^{-\alpha rp}$ for every $j \in Y_{rp}$, and from (2.17), (2.19) we conclude that

$$S_D(f\Delta g) \geqslant (\#Y_{rp})(2^{-\alpha}h)^{rp}\frac{1-2h^r}{1-h^r} \geqslant (2^{1-\alpha}h)^{rp}\Big(\frac{1}{2} - \nu\Big)\frac{1-2h^r}{1-h^r}. \tag{2.26}$$

Since p can be chosen arbitrarily large and $2^{1-\alpha}h > 1$, we see that $(K^*)\int_0^1 f(t)\,\mathrm{d}g(t)$ does not exist and Theorem 1.4 is proved.

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