REMARKS ON STAR COVERING PROPERTIES IN PSEUDOCOMPACT SPACES

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Abstract. Let P be a topological property. A space X is said to be star P if whenever $\mathcal{U}$ is an open cover of X, there exists a subspace $A \subseteq X$ with property P such that $X = \mathrm{St}(A,\mathcal{U})$, where $\mathrm{St}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U}\colon\ U \cap A \neq \emptyset\}$. In this paper, we study the relationships of star P properties for $P \in$ {Lindelöf, compact, countably compact} in pseudocompact spaces by giving some examples.

Keywords: Lindelöf, star Lindelöf, compact, star compact, countably compact, star countably compact space

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1. Introduction

By a space we mean a topological space. In this section, we give definitions of terms which are used in this paper. Let X be a space and $\mathcal{U}$ a collection of subsets of X. For $A \subseteq X$, let $\mathrm{St}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U}\colon\ U \cap A \neq \emptyset\}$.

Definition ([1], [2]). Let P be a topological property. A space X is said to be *star P* if whenever $\mathcal{U}$ is an open cover of X, there exists a subspace $A \subseteq X$ with property P such that $X = \mathrm{St}(A,\mathcal{U})$. The set A will be called a *star kernel* of the cover $\mathcal{U}$.

The term star P was coined in [1], [2] but certain star properties, specifically those corresponding to "$\mathcal{P}$=compact" were first studied by Ikenaga and Tani in [6], "$\mathcal{P}$ = Lindelöf" was first studied by Hiremath in [5] and the author [12], and "$\mathcal{P}$ = countably compact" was first studied by the author in [10]. A survey of star covering

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properties with a comprehensive bibliography can be found in [3], [8]. Here, we use the terminology from [1], [2]. In [12] and earlier [5], a star Lindelöf space is called $\mathcal{L}$-starcompact and sLc property, respectively. In [11], a star compact space is called $\mathcal{K}$-starcompact, and in [10], a star countably compact space is called $\mathcal{C}$-starcompact. From the above definitions, it is not difficult to see that every star compact space is star countably compact and every star compact space is star Lindelöf. In [10], the author studied the relationships of star P properties for $P \in$ {Lindelöf, compact, countably compact} by giving some examples.

The purpose of this note is to study the relationships of star P properties for $P \in$ {Lindelöf, compact, countably compact} in pseudocompact spaces by giving some examples.

Throughout this paper, the cardinality of a set A is denoted by $|A|$. For a cardinal κ, $\mathrm{cf}(\kappa)$ denotes the cofinality of κ. Let ω denote the first infinite cardinal and $\mathfrak{c}$ the cardinality of the continuum. As usual, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. When viewed as a space, every cardinal has the usual order topology. For each ordinal α, β with $\alpha < \beta$, we write $(\alpha, \beta) = \{\gamma \colon \alpha < \gamma < \beta\}$ and $(\alpha, \beta] = \{\gamma \colon \alpha < \gamma \leqslant \beta\}$. Other terms and symbols that we do not define will be used as in [4].

2. Some examples on star covering properties in pseudocompact spaces

In this section we study the relationships of star P properties for $P \in$ {Lindelöf, compact, countably compact} in pseudocompact spaces by giving some examples. For a Tychonoff space X, let βX denote the Čech-Stone compactification of X.

Example 2.1. There exists a star countably compact, pseudocompact Tychonoff space which is not star Lindelöf.

Proof. Let D be a discrete space of cardinality $\mathfrak{c}$, and let

$$X = (\beta D \times (\mathfrak{c} + 1)) \setminus ((\beta D \setminus D) \times \{\mathfrak{c}\})$$

be the subspace of the product of βD and $\mathfrak{c} + 1$. Then X is star countably compact pseudocompact Tychonoff, since it has a countably compact dense subspace $\beta D \times \mathfrak{c}$.

Next, we show that X is not star Lindelöf. Since $|D| = \mathfrak{c}$, we can enumerate D as $\{d_\alpha \colon \alpha < \mathfrak{c}\}$. For each $\alpha < \mathfrak{c}$, let $U_\alpha = \{d_\alpha\} \times (\alpha, \mathfrak{c}]$. Then U_α is open in X and $U_\alpha \cap U_{\alpha'} = \emptyset$ for $\alpha \neq \alpha'$. Let us consider the open cover

$$\mathcal{U} = \{U_\alpha \colon \alpha < \mathfrak{c}\} \cup \{\beta D \times [0, \mathfrak{c})\}$$

of X. Let L be a Lindelöf subset of X. Then $\Lambda = \{\alpha\colon \langle d_\alpha, \mathfrak{c}\rangle \in L\}$ is countable, since $\{\langle d_\alpha, \mathfrak{c}\rangle\colon \alpha < \mathfrak{c}\}$ is discrete and closed in X. Let $L' = L \setminus \bigcup\{U_\alpha\colon \alpha \in \Lambda\}$. If $L' = \emptyset$, then there exists an $\alpha_0 < \mathfrak{c}$ such that $L \cap U_{\alpha_0} = \emptyset$, hence $\langle d_{\alpha_0}\mathfrak{c}, \rangle \notin \mathrm{St}(L, \mathcal{U})$, since U_{α_0} is the only element of $\mathcal{U}$ containing the point $\langle d_{\alpha_0}, \mathfrak{c}\rangle$. On the other hand, if $L' \neq \emptyset$, since L' is closed in L, we have L' is Lindelöf and $L' \subseteq \beta D \times \mathfrak{c}$, hence $\pi(L')$ is a Lindelöf subset of the countably compact space $\mathfrak{c}$, where $\pi\colon \beta D \times \mathfrak{c} \to \mathfrak{c}$ is the projection, thus there exists an $\alpha'' < \mathfrak{c}$ such that $\pi(L') \cap (\alpha'', \mathfrak{c}) = \emptyset$. Choose $\alpha < \mathfrak{c}$ such that $\alpha > \alpha''$ and $\alpha \notin \Lambda$. Then $\langle d_\alpha, \mathfrak{c}\rangle \notin \mathrm{St}(L, \mathcal{U})$, since U_α is the only element of $\mathcal{U}$ containing the point $\langle d_\alpha, \mathfrak{c}\rangle$ and $U_\alpha \cap L = \emptyset$, which shows that X is not star Lindelöf. □

Recall from [8] that a space X is called $1\frac{1}{2}$-*starcompact* if for every open cover $\mathcal{U}$ of X, there exists a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $\mathrm{St}(\bigcup\mathcal{V}, \mathcal{U}) = X$. In [3], a $1\frac{1}{2}$-starcompact space is called 1-starcompact. To demonstrate the following example, we need the following lemma from [8, Theorem 28].

Lemma 2.2. *If a regular space X contains a discrete closed subspace Y such that $|X| = |Y| \geqslant \omega$, then X is not $1\frac{1}{2}$-starcompact.*

Example 2.3. There exists a star Lindelöf, pseudocompact Tychonoff space which is not star countably compact.

Proof. Let $X = \omega \cup \mathcal{R}$ be the Isbell-Mrówka space ([9]), where $\mathcal{R}$ is a maximal almost disjoint family of infinite subsets of ω with $|\mathcal{R}| = \mathfrak{c}$. We topologize X as follows: every subset with only one point of ω is open in X; a basic neighborhood of a point $r \in \mathcal{R}$ takes the form

$$O_F(r) = \{r\} \cup (r \setminus F) \text{ where } F \text{ is a finite subset of } r.$$

It is well known that X is pseudocompact Tychonoff. Since ω is a countable dense subset of X, the space X is star countable, hence it is star Lindelöf.

Next, we show that X is not star countably compact. First, we show that every countably compact subset of X is compact. To this end, let C be a countably compact subset of X and $\mathcal{U}$ an open cover of C. Then $C \cap \mathcal{R}$ is finite, since $\mathcal{R}$ is a discrete closed subset of X. For each $r \in C \cap \mathcal{R}$, there exists a finite subset $F_r \subseteq r$ such that $O_{F_r}(r) \subseteq U_r$ for some $U_r \in \mathcal{U}$. Then

$$C \setminus \bigcup\{U_r\colon r \in C \cap \mathcal{R}\} \subseteq C \setminus \bigcup\{O_{F_r}(r)\colon r \in C \cap \mathcal{R}\}$$

is finite by the construction of the topology of X and countable compactness of X, which shows that C is compact. From this fact, it is easy to show that X is star countably compact if and only if X is star compact. By Lemma 2.2, X is not $1\frac{1}{2}$-starcompact, so X is not star compact, since every star compact space is $1\frac{1}{2}$-starcompact. Thus X is not star countably compact. □

It is well known that every countably compact Lindelöf space is compact. The following example shows that the result cannot be generalized to star compact even in the class of pseudocompact spaces.

Example 2.4. There exists a star countably compact and star Lindelöf, pseudocompact Tychonoff space which is not star compact.

Proof. Let $S_1 = (\beta D \times (\mathfrak{c}+1)) \setminus ((\beta D \setminus D) \times \{\mathfrak{c}\})$ be the same space X as in Example 2.1. Then S_1 is star countably compact pseudocompact Tychonoff. But, S_1 is not star compact.

Let $S_2 = \omega \cup \mathcal{R}$ be the same Isbell-Mrówka space X as in Example 2.3, where $\mathcal{R}$ is a maximal almost disjoint family of infinite subsets of ω with $|\mathcal{R}| = \mathfrak{c}$. Then S_2 is star Lindelöf pseudocompact Tychonoff. But, S_2 is not star compact.

Let $\varphi\colon D \times \{\mathfrak{c}\} \to \mathcal{R}$ be a bijection. Let X be the quotient space obtained from the sum $S_1 \oplus S_2$ by identifying $\langle d_\alpha, \mathfrak{c}\rangle$ and $\varphi(\langle d_\alpha, \mathfrak{c}\rangle)$ for each $\alpha < \mathfrak{c}$. Let $\pi\colon S_1 \oplus S_2 \to X$ be the quotient map. Then X is pseudocompact Tychonoff, but X is not star compact by the definition of the topology of X.

Now, we show that X is star countably compact. Let $\mathcal{U}$ be an open cover of X. Since $\pi(\beta D \times \mathfrak{c})$ is a countably compact dense subset of $\pi(S_1)$, we have

$$\pi(S_1) \subseteq \mathrm{St}(\pi(\beta D \times \mathfrak{c}), \mathcal{U}).$$

On the other hand, since $\pi(S_2)$ is homeomorphic to S_2, every infinite subset of $\pi(\omega)$ has an accumulation point in $\pi(S_2)$. Hence, there exists a finite subset F_1 of $\pi(\omega)$ such that $\pi(\omega) \subseteq \mathrm{St}(F_1, \mathcal{U})$. For, if $\pi(\omega) \nsubseteq \mathrm{St}(B, \mathcal{U})$ for any finite subset $B \subseteq \pi(\omega)$, then, by induction, we can define a sequence $\{x_n\colon n \in \omega\}$ in $\pi(\omega)$ such that $x_n \notin \mathrm{St}(\{x_i\colon i < n\}, \mathcal{U})$ for each $n \in \omega$. By the above mentioned property of $\pi(\omega)$, the sequence $\{x_n\colon n \in \omega\}$ has an accumulation point x^* in $\pi(S_2)$. Pick $U \in \mathcal{U}$ such that $x^* \in U$. Choose $n < m < \omega$ such that $x_n \in U$ and $x_m \in U$. Then $x_m \in \mathrm{St}(\{x_i\colon i < n\}, \mathcal{U})$, which contradicts the definition of the sequence $\{x_n\colon n \in \omega\}$. Let $F = F_1 \cup \pi(\beta D \times \mathfrak{c})$. Then F is countably compact and $X = \mathrm{St}(F, \mathcal{U})$. Hence, X is star countably compact.

Next, we show that X is star Lindelöf. Since $\pi(\omega)$ is a countable dense subset of $\pi(S_2)$, we have $\pi(S_2) \subseteq \mathrm{St}(\pi(\omega), \mathcal{U})$. On the other hand, since $\pi(\beta D \times \mathfrak{c})$ is countably compact, there exists a finite subset F_1 of $\pi(\beta D \times \mathfrak{c})$ such that $\pi(\beta D \times \mathfrak{c}) \subseteq \mathrm{St}(F_1, \mathcal{U})$. If we put $L = \pi(\omega) \cup F_1$, then L is a countable subset of X and $X = \mathrm{St}(L, \mathcal{U})$, which shows that X is star Lindelöf. □

For normal spaces, it is well known that countable compactness is equivalent to pseudocompactness, and every countably compact space is star finite. Thus we have the following result.

Theorem 2.5. *Every pseudocompact normal space X is star compact.*

Remark 2.1. The author does not know if there exists an example of a star countably compact and star Lindelöf normal space that is not star compact.

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