GLOBAL LIPSCHITZ CONTINUITY FOR ELLIPTIC TRANSMISSION PROBLEMS WITH A BOUNDARY INTERSECTING INTERFACE

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(Received February 6, 2012)

Abstract. We investigate the regularity of the weak solution to elliptic transmission problems that involve two layered anisotropic materials separated by a boundary intersecting interface. Under a pair of compatibility conditions for the angle of the two surfaces and the boundary data at the contact line, we prove the existence of up to the boundary square-integrable second derivatives, and the global Lipschitz continuity of the solution. If only the weakest, necessary condition is satisfied, we show that the second weak derivatives remain integrable to a certain power less than two.

Keywords: elliptic transmission problem, regularity theory, Lipschitz continuity

MSC 2010: 35B65, 35J25

1. Introduction

The paper is concerned with the Lipschitz continuity and the regularity of the second derivatives of weak solutions to a class of elliptic equations with transmission conditions that occurs in manifold areas of mathematical physics. We consider a bounded domain $\Omega \subset \mathbb{R}^3$ partitioned by a 2-dimensional interface S into two disjoint subdomains Ω_i $(i = 1, 2)$ that represent two materials, or two different phases of the same material. The interface S is a free surface,[1] whose intersection with the outer boundary Γ of the domain Ω is a *closed* curve. We study the regularity of the

This research is supported by DFG Research Center 'Mathematics for Key Technologies' Matheon in Berlin.

[1] By *free surface* we mean a bounded, connected surface with boundary as opposed to a *closed* surface. This is not to be understood in the context of free boundary problems, the surface S being given troughout the paper.

function $u\colon \Omega \to \mathbb{R}$ that solves the problem

$$(1.1) \qquad -\operatorname{div}(\kappa \nabla u) = f \quad \text{in } \Omega \setminus S,$$

$$(1.2) \qquad [u]_S = 0, \ [-\kappa \nabla u \cdot n_S]_S = 0 \quad \text{on } S,$$

in connection with one of the following boundary conditions on the surface $\Gamma := \partial\Omega$:

$$(1.3) \qquad -\kappa \nabla u \cdot n_\Gamma = Q \quad \text{on } \Gamma \ [= \text{ Problem } (P_N)],$$

$$(1.4) \qquad u = u_e \quad \text{on } \Gamma \ [= \text{ Problem } (P_D)].$$

In the equation (1.1), the function f is a given source, and the diffusion coefficient κ is assumed to be phase-dependent, that means

$$(1.5) \qquad \kappa = \kappa_i(x) \quad \text{if } \ x \in \Omega_i,$$

with mappings $\kappa_i\colon \overline{\Omega_i} \to \mathbb{R}^{3\times 3}$ $(i = 1, 2)$. The conditions (1.2) are the transmission conditions. The symbol $[\cdot]_S$ denotes the jump of a quantity across S, and n_S is the unit normal to the surface S that points into Ω_2. In the boundary condition (1.3), the function Q is given, and n_Γ denotes the outward unit normal. In (1.4), the function u_e is given.

We address the problem (1.1), (1.2), with either (1.3) or (1.4) as (P). The problem (P) is a second order elliptic transmission problem. For transmission conditions near surfaces of class $\mathcal{C}^2$ that do not intersect the boundary of the domain, it has been known for a relatively long time that the solution is globally in $W^{2,2}$ up to the interface ([15], [8] and Chapter 3, paragraph 16 of [6]). Lipschitz continuity, for the case that the jumps of the coefficients occur at not intersecting surfaces of class $\mathcal{C}^2$, was proved in the book [6] Chapter 3, paragraph 16, and for surfaces of class $W^{2,q}$, $q > 3$ in the paper [7]. These results were recently generalized in [9] for interfaces of class $\mathcal{C}^{1,\alpha}$ $(\alpha > 0)$.

For the case under study of one single smooth interface intersecting the outer boundary in three-dimensions, one can expect the integrability of ∇u to a certain power $q_0 > 3$, as shown in [4]. In the latest paper, the boundary Γ is polyhedral, and mixed conditions thereon are also allowed. But it seems that *sufficient conditions* for the existence of square-integrable (or p-integrable) second derivatives, and for the boundedness of ∇u, have not yet been investigated. In this paper, we present a set of complex compatibility conditions (the conditions (2.19), (2.20), (2.22) below) involving the geometry, the anisotropic diffusion coefficient κ, and the type of the boundary condition on Γ, that ensures the Lipschitz continuity and the $W^{2,2}$ regularity of the solution to (P) in each subdomain. To our best knowledge, these compatibility conditions have not yet been known. We also prove a weaker regularity result for the second derivatives if only (2.20), (2.22) are satisfied.

It should be mentioned that second order elliptic transmission problems are a field of intense research. The main line of investigation in the last years concerns transmission conditions at piecewise smooth (or polyhedral) interfaces, in connection with intersection of the outer boundary of Lipschitz class. This is obviously a much more general situation than the one we consider. The methods of edge and corner asymptotic are here well suited (see among others [13] and references), or more generally a regularity theory in weighted spaces (see [1] for a recent study with references). In the language of standard Sobolev spaces, the $W^{s,2}$ regularity ([12], Section 5 of [14], [13]) for $s < 3/2$ is at most expected for similar problems without additional compatibility conditions.

The structure of the paper is as follows. In Section 2, we introduce the basic notation, and then formulate the compatibility condition and the main theorem. In Subsection 2.3, the compatibility conditions are explicitly interpreted for the simpler case of isotropic diffusion. In Section 3, we design a regularization procedure to approximate the problem (P) by problems with continuous coefficients. The most fundamental section is Section 4, in which we derive appropriate integral relations and boundary conditions satisfied by the gradient of the approximate solutions. With these results at hand, we can derive in Sections 5, 6 and 7 the higher-order estimates from well-known arguments of the PDE theory. In Section 8, we give some conclusions of our investigation.

2. Notation and statement of the main result

2.1. Notation. Throughout the paper, Ω denotes a bounded domain with boundary Γ of class $\mathcal{C}^2$. There are a free hypersurface $S \subset \Omega$ of class $\mathcal{C}^2$ such that $\overline{S} \cap \Gamma$ is a closed curve, and two disjoint open sets $\Omega_i \subset \Omega$ $(i = 1, 2)$ such that the partition $\Omega \setminus S = \Omega_1 \cup \Omega_2$ is valid.

The outward unit normal to Γ is denoted by n_Γ, and n_S denotes the unit normal to S that points into Ω_2. We set $\Gamma_i := \partial\Omega_i \cap \Gamma$ for $i = 1, 2$. The angle of contact $\alpha \in \,]0, \pi[$ of the surfaces Γ and S is defined on the curve $\Gamma \cap \overline{S}$ via

$$\cos\alpha := n_S \cdot n_\Gamma, \ \sin\alpha := \sqrt{1 - \cos^2\alpha} \quad \text{on } \ \Gamma \cap \overline{S}. \tag{2.1}$$

Remark 2.1 (Data extension). Since S is of class $\mathcal{C}^2$, we lose no generality by assuming that S is also defined outside of Ω. Otherwise, we always will find an extension surface $S' \in \mathcal{C}^2$ such that S is compactly included in the interior of the surface S'. For $\varrho > 0$, define $\Omega_\varrho := \{x \in \Omega \colon \operatorname{dist}(x, S) < \varrho\}$. Choosing $\varrho \leqslant \varrho_0(S)$

sufficiently small, there is for all $x \in \Omega_\varrho$ a unique projection $y(x) \in S$ such that $|x-y| = \operatorname{dist}(x,S)$. Moreover, since S is defined in a neighborhood of Ω, the point y belongs to the interior of S. We introduce the signed distance function

$$d_S(x) := \begin{cases} -\operatorname{dist}(x,S) & \text{for } x \in \Omega_1, \\ \operatorname{dist}(x,S) & \text{for } x \in \Omega_2. \end{cases} \tag{2.2}$$

In Ω_{ϱ_0}, set $n_S := \nabla d_S$ so that $n_S \in [C^1(\overline{\Omega_{\varrho_0}})]^3$ ([3], Lemma 14.16). From the neighborhood Ω_{ϱ_0}, it is then possible to extend n_S to the rest of Ω in order to obtain $n_S \in [C^1(\overline{\Omega})]^3$. The normal n_Γ has by similar arguments a continuously differentiable extension into Ω. Due to (2.1), the functions $\cos\alpha$ and $\sin\alpha$ also possess natural extensions into Ω. In order to track the dependence on the surfaces Γ and S in the regularity estimate, we introduce

$$g_0 := \|\nabla n_\Gamma\|_{[L^\infty(\Omega)]^3} + \|\nabla n_S\|_{[L^\infty(\Omega)]^3}. \tag{2.3}$$

Particular systems of tangential vectors arise naturally to derive estimates near the curve $\Gamma \cap S$. Those are

$$\tau^{(1)} := \frac{n_S \times n_\Gamma}{|n_S \times n_\Gamma|}, \quad \tau^{(2)} := \frac{(n_S \times n_\Gamma) \times n_\Gamma}{|(n_S \times n_\Gamma) \times n_\Gamma|} \quad \text{on } \Gamma, \tag{2.4}$$

$$T^{(1)} := \frac{n_S \times n_\Gamma}{|n_S \times n_\Gamma|}, \quad T^{(2)} := \frac{(n_S \times n_\Gamma) \times n_S}{|(n_S \times n_\Gamma) \times n_S|} \quad \text{on } S. \tag{2.5}$$

Lemma C.3 in the appendix states the elementary relationships of these vectors. The orthogonal matrix that transforms the standard Euclidean basis of $\mathbb{R}^3$ into the orthonormal system $\{T^{(1)}, T^{(2)}, n_S\}$ is denoted by O. Further relevant matrices are, first, the matrix $A := O^T \kappa O$, the entries of which are given by

$$A = \begin{pmatrix} \kappa T^{(1)} \cdot T^{(1)} & \kappa T^{(1)} \cdot T^{(2)} & \kappa T^{(1)} \cdot n_S \\ \kappa T^{(2)} \cdot T^{(1)} & \kappa T^{(2)} \cdot T^{(2)} & \kappa T^{(2)} \cdot n_S \\ \kappa n_S \cdot T^{(1)} & \kappa n_S \cdot T^{(2)} & \kappa n_S \cdot n_S \end{pmatrix}, \tag{2.6}$$

and, second, the perturbation $\tilde{\kappa}$ of the matrix κ defined by

$$\tilde{\kappa} := O \begin{pmatrix} a^{1,1} & 2a^{1,2} & 2a^{1,3} \\ 0 & a^{2,2} & a^{2,3} \\ 0 & a^{3,2} & a^{3,3} \end{pmatrix} O^T. \tag{2.7}$$

For $B \in \mathbb{R}^{3\times 3}$, the *minors* $m^{i,j}(B)$ $(i,j = 1,2,3)$ are the numbers

$$m^{i,j}(B) := \det(B^{i,j}), \ B^{i,j} := \{b^{k,l}\}_{k\neq i, l\neq j} \quad \text{for } k,l,i,j = 1,2,3, \tag{2.8}$$

Since almost exclusively the minors of the matrix A (cf. (2.6)) are needed in the paper, we define

$$m^{i,j} := m^{i,j}(A) \quad \text{for } i,j = 1,2,3. \tag{2.9}$$

A function $\nu \in L^\infty(\Omega)$ is called *piecewise Lipschitz continuous* if there are $\nu_i \in W^{1,\infty}(\Omega_i)$ such that $\nu = \nu_i$ in Ω_i $(i = 1,2)$. For piecewise Lipschitz continuous ν, the jump across S is the quantity

$$[\nu]_S(x) = \nu_2(x) - \nu_1(x) \quad \text{for } x \in S. \tag{2.10}$$

Since the functions ν_i have Lipschitz continuous extensions to $\overline{\Omega}$, the symbol $[\nu]_S$ still makes sense outside of S and

$$[\nu]_S \in C^{0,1}(\overline{\Omega}). \tag{2.11}$$

For a symmetric and positive definite $B \in \mathbb{R}^{3\times 3}$ and for $\theta \in]0,\pi[$ define the *compatibility function*

$$f_d(\theta, B) := \begin{cases} \cot\theta b^{3,3}/m^{1,1}(B) + b^{2,3}/m^{1,1}(B) & \text{for } (P_N), \\ \cot\theta b^{3,3} + b^{2,3} & \text{for } (P_D) \end{cases} \tag{2.12}$$

that plays the fundamental role with respect to compatibility conditions near $\Gamma \cap \overline{S}$.

We finally introduce some functional spaces. We denote by q' the number conjugated to $q \in]1,+\infty[$ in the sense that $1/q + 1/q' = 1$. The usual Lebesgue spaces $L^q(\Omega)$, the Sobolev spaces $W^{1,q}(\Omega)$, and their trace spaces $W^{1/q',q}(\partial\Omega)$, are needed. The definition and relevant properties of these spaces are to be found in standard monographes (for instance [5]). Also well-known in the context of regularity theory are the subspaces of $W^{1/q',q}(\Gamma)$ associated with the linear operators of extension by zero. Define

$$\gamma^-(u) := \begin{cases} u & \text{on } \Gamma_1, \\ 0 & \text{on } \Gamma_2, \end{cases} \qquad \gamma^+(u) := \begin{cases} 0 & \text{on } \Gamma_1, \\ u & \text{on } \Gamma_2, \end{cases} \tag{2.13}$$

$$V^q(\Gamma) := \{u \in W^{1/q',q}(\Gamma) \colon \gamma^-(u) \in W^{1/q',q}(\Gamma)\}, \tag{2.14}$$

$$\|u\|_{V^q(\Gamma)} := \|u\|_{W^{1/q',q}(\Gamma)} + \|\gamma^-(u)\|_{W^{1/q',q}(\Gamma)}.$$

Relevant properties of the V^q spaces are recalled in the appendix, Lemma B.2.

2.2. Statement of the main result. To investigate the regularity of the solution to (P_N) or (P_D), we first formulate some assumptions on the data. We assume that the surfaces Γ and S, and their angle of contact α, satisfy

$$\Gamma, S \in \mathcal{C}^2, \quad \alpha \in W^{1,\infty}(\Gamma \cap S), \quad \alpha \in]0,\pi[\text{ on } \Gamma \cap S. \tag{2.15}$$

Let, moreover, the matrix $\kappa(x)$ be symmetric for all $x \in \overline{\Omega}$ and satisfy

$$k_0|\eta|^2 \leqslant \kappa(x)\eta \cdot \eta \leqslant k_1|\eta|^2 \quad \text{for all } x \in \overline{\Omega},\ \eta \in \mathbb{R}^3, \tag{2.16}$$

with two constants $0 < k_0 \leqslant k_1 < \infty$. The matrix A (cf. (2.6)) is then also symmetric, and the matrices A and $\tilde{\kappa}$ (cf. (2.7)) uniformly satisfy the inequality (2.16) with the same constants k_0, k_1. For the matrix κ, we furthermore assume that

$$k_1' := \|\nabla\kappa_1\|_{L^\infty(\Omega_1)} + \|\nabla\kappa_2\|_{L^\infty(\Omega_2)} < \infty. \tag{2.17}$$

The right-hand side f of equation (1.1) is supposed to have the regularity

$$f \in L^q(\Omega). \tag{2.18}$$

In (2.18), and also in the integrability conditions formulated hereafter, we focus on the cases $q = 2$ (for the $W^{2,p}$ analysis), and $q = q_0 > 3$ (for the $W^{1,\infty}$ analysis).

We come now to the *sufficient compatibility conditions* for higher regularity. The angle of contact α and the matrix κ (or the matrix A) are assumed to satisfy the compatibility condition

$$[f_d(\alpha, A)]_S := f_d(\alpha, A_2) - f_d(\alpha, A_1) \geqslant 0 \quad \text{on} \ \ \Gamma \cap S, \tag{2.19}$$

where $A_i := A_{|\Omega_i}$. For the problem (P_N), we additionally require on the surface Γ that

$$\exists Q_1 \in W^{1/q',q}(\Gamma),\ Q_2 \in V^q(\Gamma)\colon\ \left[\frac{a^{3,3}}{m^{1,1}}\right]_S \frac{Q}{\sin\alpha} = [f_d(\alpha, A)]_S Q_1 + Q_2, \tag{2.20}$$

$$\exists g_1 \in W^{1,\infty}(\Gamma)\colon\ \left[\frac{m^{2,1}}{m^{1,1}}\right]_S = g_1[f_d(\alpha, A)]_S. \tag{2.21}$$

For the problem (P_D), we require that

$$\begin{aligned} &\nabla u_e \in W^{1/q',q}(\Gamma) \ \text{ and } \ \exists\, U_1 \in W^{1/q',q}(\Gamma),\\ &U_2 \in V^q(\Gamma)\colon\ [a^{1,3}]_S(\tau^{(1)} \cdot \nabla u_e) - [a^{3,3}]_S \frac{\tau^{(2)} \cdot \nabla u_e}{\sin\alpha} = [f_d(\alpha, A)]_S U_1 + U_2. \end{aligned} \tag{2.22}$$

The condition (2.19) ensures the compatibility of the geometry, of the coefficient κ and of the type of the boundary condition: If it is satisfied the function $[f_d(\alpha, A)]_S$ is a regularizing factor in the problem. The representation conditions (2.20) and (2.22) ensure a suitable decay of the boundary data if the function $[f_d(\alpha, A)]_S$ approaches its critical value (see Section 2.3 below for an alternative characterization). The main results of the paper are contained in the following two theorems.

Theorem 2.2. *Let $u \in W^{1,2}(\Omega)$ denote the unique weak solution to (P_D) or to (P_N). Assume that the conditions* (2.15), (2.16) *and* (2.17) *are satisfied, and that* (2.18) *is valid with $q = 2$. Assume that the condition* (2.19)*, and either* (2.20), (2.21) *for the problem (P_N), or* (2.22) *for the problem (P_D), are valid with $q = 2$. Then $u \in W^{2,2}(\Omega_i)$ $(i = 1, 2)$.*

Assume moreover that (2.18)*, and either* (2.20), (2.21) *for the problem (P_N), or* (2.22) *for the problem (P_D), are satisfied for $q = q_0 > 3$. Then $u \in W^{1,\infty}(\Omega)$.*

In the case that the principal hypothesis (2.19) of Theorem 2.2 is violated, we can still prove that the weak solution to (P) has second derivatives at least integrable to the power $6/5$.

Theorem 2.3. *Except of* (2.19)*, the same assumptions as in Theorem* 2.2 *with $q = 2$. Let $u \in W^{1,2}(\Omega)$ denote the unique weak solution to (P_D) or to (P_N). Then there is $q_0 > 3$ such that $\nabla u \in L^{q_0}(\Omega)$. Define $s_0 := \min\{q_0, 6\}$. Then, for $1 \leqslant p < 2s_0/(s_0 + 2)$ arbitrary, $\nabla u \in W^{1,p}(\Omega_i)$ $(i = 1, 2)$.*

2.3. Interpretation of the compatibility conditions. A few remarks can help better understand the conditions (2.19), (2.20) and (2.22).

In the case that κ is a scalar, one can verify that $f_d := \cot\alpha\kappa^{-1}$ for (P_N), while $f_d := \cot\alpha\kappa$ for (P_D). The condition (2.19) reduces to

$$\cot\alpha[\kappa]_S \leqslant 0 \quad \text{for } (P_N), \quad \cot\alpha[\kappa]_S \geqslant 0 \quad \text{for } (P_D) \text{ on } \Gamma \cap S. \tag{2.23}$$

Elementary consequences of (2.23) for the result of Theorem 2.2 in the case of the isotropic diffusion are the following:

(1) For given data (κ, α), the result does not apply to both the Neumann problem *and* the Dirichlet problem, unless $\alpha \equiv \pi/2$ on $\Gamma \cap S$ (the two surfaces meet at right angle). Otherwise, the choice which quantity to prescribe on the outer boundary Γ has to follow the condition (2.23).

(2) If κ is, moreover, piecewise constant (that is, if $\kappa_1, \kappa_2 \in \mathbb{R}$), the changes in sign of $\cos\alpha$ along $\Gamma \cap S$ are critical for the applicability of the result.

We also briefly comment on the representation conditions (2.20) and (2.22). In the scalar case, the condition (2.20) reduces to

$$[\kappa]_S Q = \cos\alpha[\kappa^{-1}]_S Q_1 + \sin\alpha Q_2 \quad \text{on } \Gamma, \tag{2.24}$$

and (2.22) reduces to

$$-[\kappa]_S \tau^{(2)} \cdot \nabla u_e = \cos\alpha[\kappa]_S U_1 + \sin\alpha U_2 \quad \text{on } \Gamma. \tag{2.25}$$

Thus, if the contact angle is such that $|\cos\alpha| \geqslant \delta_0 > 0$ on $\Gamma \cap S$, the condition (2.20) is trivially satisfied for every $Q \in W^{1/q',q}(\Gamma)$: set $Q_1 = -\kappa_2\kappa_1 Q/\cos\alpha$ and $Q_2 = 0$. Similarly, set $U_1 := -\tau^{(2)} \cdot \nabla u_e/\cos\alpha$, $U_2 = 0$ to obtain (2.22). The compatibility conditions (2.20) and (2.22) are therefore only needed for the limiting case that the function $[f_d(\alpha, A)]_S$ tends to zero on some part of $\Gamma \cap S$. It is obvious and easy to motivate that representation conditions of the type (2.24) and (2.25) are *necessary* for every higher regularity of u that implies the existence of traces for ∇u on manifolds.

Note that the function $[f_d(\alpha, A)]_S$ is intrinsically given only on $\Gamma \cap S$. Therefore, the representations (2.20) and (2.22) depend on the choice of its extension to Γ. However, assuming additional regularity of the data Q, u_e, we show in the next Lemma that (2.20) and (2.22) more intrinsically amount to require a certain decay along the curve $\Gamma \cap S$. To this aim denote

$$K := \Gamma \cap S, \quad K_0 := \{x \in K\colon\ |[f_d(\alpha, A)]_S| > 0\},$$
$$d_{K_0}(x) := \operatorname{dist}(x, K \setminus K_0) \quad \text{ for } x \in K.$$

For $s \in \mathbb{R}$, the properties of the spaces $W^{s,2}(U)$, $U \in \mathbb{R}^n$ have been studied in [11]. It is impossible to expose in a few lines the localization arguments that justify to extend these properties to the case that U is a $\mathcal{C}^2$-submanifold. We recall that our aim here is only to throw some light on the compatibility conditions. Define $W^{s,2}_K(\Gamma)$ as the space $W^{s,2}_0(\Gamma_1) \oplus W^{s,2}_0(\Gamma_2)$. If $s > 1/2$, every function $g \in W^{s,2}(\Gamma)$ has a trace $\operatorname{tr}(g) \in W^{s-1/2,2}(K)$ (see [11], Theorem 9.4).

Lemma 2.4. *Assume that there are $\beta \in\,]0, 1]$ and constants $0 < c_1 \leqslant c_2$ such that $c_1 d^{\beta}_{K_0} \leqslant [f_d(\alpha, A)]_S \leqslant c_2 d^{\beta}_{K_0}$ on K_0. Assume that $g \in W^{s,2}(\Gamma)$, $s > 1/2$ is such that*

$$\operatorname{tr}(g) \in \begin{cases} W^{s-1/2+\beta,2}_{00}(K_0) & \text{if } s - 1/2 + \beta = j + 1/2 \quad \text{for a } j \in \mathbb{N},\\ W^{s-1/2+\beta,2}_{0}(K_0) & \text{otherwise.}\end{cases} \tag{2.26}$$

Then, for each extension of the function $[f_d(\alpha, A)]_S$ to Γ, there are $g_1 \in W^{s,2}(\Gamma)$ and $g_2 \in W^{s,2}_K(\Gamma)$ such that $g = [f_d(\alpha, A)]_S g_1 + g_2$.

P r o o f. Define $\tilde g_1 := g/[f_d(\alpha, A)]_S$ on K_0 and $\tilde g_1 := 0$ on $K \setminus K_0$. Then, from [11], Theorem 11.7, it follows that

$$\tilde g_1 \in W^{s-1/2}(K). \tag{2.27}$$

Due to (2.26) and to the trace theorem for $W^{s,2}$, there exists $g_1 \in W^{s,2}(\Gamma)$ such that $\operatorname{tr}(g_1) = \tilde g_1$ on K. Choosing an arbitrary extension of $[f_d(\alpha, A)]_S$ to Γ, we easily obtain that $g_2 := g - [f_d(\alpha, A)]_S g_1$ belongs to $W^{s,2}_K(\Gamma)$. □

3. Method of the proof

To prove Theorem 2.2, we investigate a regularization of the problems (P_N) and (P_D). For $\varrho > 0$, $t \in \mathbb{R}$, define

$$I_\varrho(t) := \begin{cases} 0 & \text{if } t \leqslant 0, \\ t/\varrho & \text{for } t \in \,]0, \varrho], \\ 1 & \text{for } t > \varrho. \end{cases} \tag{3.1}$$

For $\nu_1, \nu_2 \in L^\infty(\Omega)$, define $\nu := \nu_i$ in Ω_i. Recalling (2.2), we introduce

$$L_\varrho(\nu)(x) := \nu_1(x) + I_\varrho(d_S(x))(\nu_2(x) - \nu_1(x)) \in L^\infty(\Omega). \tag{3.2}$$

Note that

$$L_\varrho(\nu) \longrightarrow \nu \text{ everywhere in } \Omega \setminus S, \tag{3.3}$$

and, also taking Remark 2.1 into account, we obtain for piecewise Lipschitz continuous ν that

$$\nabla L_\varrho(\nu) = I'_\varrho(d_S(x))\nabla d_S(x) + L_\varrho(\nabla\nu)(x) = \frac{[\nu]_S(x)}{\varrho} b_\varrho(x) n_S(x) + L_\varrho(\nabla\nu)(x). \tag{3.4}$$

In (3.4) we have abbreviated $b_\varrho := \chi_{\{0\leqslant \mathrm{dist}(x,S)\leqslant\varrho\}}(x)$, and L_ϱ applies componentwise to vector fields. We now introduce a regularization of κ via the matrix A. For the problem (P_D), we apply the regularization (3.1), (3.2) to introduce the coefficients

$$a_\varrho^{i,j} = L_\varrho(a^{i,j}) \in C^{0,1}(\overline{\Omega}) \quad \text{for } i, j = 1, 2, 3, \tag{3.5}$$

where $a^{i,j}$ are taken from the matrix (2.6). For the problem (P_N), we introduce

$$a_\varrho^{1,1} := L_\varrho(a^{1,1}), \quad a_\varrho^{3,1} := L_\varrho(a^{3,1}), \quad m_\varrho^{1,1} := L_\varrho(m^{1,1}), \tag{3.6}$$

where the relevant $a^{i,j}$ are taken from the matrix (2.6), and $m^{1,1}$ is given by (2.9). The remaining entries are defined in the following way:

$$\begin{aligned} &a_\varrho^{3,3} := m_\varrho^{1,1} L_\varrho\Big(\frac{a^{3,3}}{m^{1,1}}\Big), \ a_\varrho^{2,3} := m_\varrho^{1,1} L_\varrho\Big(\frac{a^{2,3}}{m^{1,1}}\Big), \\ &a_\varrho^{2,2} := (a_\varrho^{3,3})^{-1}(m_\varrho^{1,1} + [a_\varrho^{2,3}]^2), \\ &a_\varrho^{2,1} := (a_\varrho^{3,3})^{-1}\Big(m_\varrho^{1,1} L_\varrho\Big(\frac{m^{2,1}}{m^{1,1}}\Big) + a_\varrho^{2,3} a_\varrho^{3,1}\Big). \end{aligned} \tag{3.7}$$

The construction (3.6), (3.7), has the properties

$$\frac{a_\varrho^{3,3}}{m_\varrho^{1,1}} = L_\varrho\Big(\frac{a^{3,3}}{m^{1,1}}\Big), \quad \frac{a_\varrho^{2,3}}{m_\varrho^{1,1}} = L_\varrho\Big(\frac{a^{2,3}}{m^{1,1}}\Big), \quad \frac{m_\varrho^{2,1}}{m_\varrho^{1,1}} = L_\varrho\Big(\frac{m^{2,1}}{m^{1,1}}\Big). \tag{3.8}$$

In view of (3.4), the regularized coefficients have, for (P_D), the important property

$$T^{(k)} \cdot \nabla a_\varrho^{i,j} = L_\varrho(T^{(k)} \cdot \nabla a^{i,j}) \quad \text{for } i,j = 1,2,3 \text{ and } k = 1,2, \tag{3.9}$$

and therefore, due to (2.17),

$$|T^{(k)} \cdot \nabla a_\varrho^{i,j}| \leqslant c(k_1', g_0) \quad \text{for } i,j = 1,2,3 \text{ and } k = 1,2. \tag{3.10}$$

For (P_N), the coefficients $a_\varrho^{i,j}$ are also constructed from the original $a^{i,j}$ by applying L_ϱ, so that the estimate (3.10) is also easy to verify.

In view of (3.5), or of (3.6), (3.7), the matrix $A_\varrho := \{a_\varrho^{i,j}\}_{i,j=1,2,3}$ satisfies (cf. (3.3))

$$A_\varrho \longrightarrow A \text{ everywhere in } \Omega \setminus S. \tag{3.11}$$

Define $\kappa_\varrho := O A_\varrho O^T$, and, similarly, $\tilde\kappa_\varrho$ using (2.7). Then $\kappa_\varrho, \tilde\kappa_\varrho$ belong to $C^{0,1}(\overline{\Omega}; \mathbb{R}^{3\times 3})$. Moreover, $\kappa_\varrho \to \kappa$ and $\tilde\kappa_\varrho \to \tilde\kappa$ everywhere in $\Omega \setminus S$. We define $u_\varrho \in W^{1,2}(\Omega)$ to be the unique weak solution to the problem (P_ϱ)

$$-\operatorname{div}(\kappa_\varrho \nabla u_\varrho) = f \quad \text{in } \Omega, \quad \Big[-\kappa_\varrho \frac{\partial u_\varrho}{\partial n_S}\Big]_S = 0 \quad \text{on } S, \tag{3.12}$$

together with one of the conditions

$$-\kappa_\varrho \frac{\partial u_\varrho}{\partial n_\Gamma} = Q \quad \text{on } \Gamma \ [=: (P_{N,\varrho})], \quad u_\varrho = u_e \quad \text{on } \Gamma \ [=: (P_{D,\varrho})]. \tag{3.13}$$

Lemma 3.1. *Assume that κ satisfies (2.16) and (2.17). Let $f \in L^2(\Omega)$, $Q \in W^{1,2}(\Omega)$ and $u_e \in W^{2,2}(\Omega)$. Denote by $u \in W^{1,2}(\Omega)$ and $u_\varrho \in W^{1,2}(\Omega)$ the weak solution to (P) and (P_ϱ), respectively. Then $u_\varrho \in W^{2,2}(\Omega)$, and*

$$u_\varrho \longrightarrow u \text{ in } W^{1,2}(\Omega). \tag{3.14}$$

Moreover, there is a constant c, depending only on Ω and on k_1/k_0, such that the function u_ϱ satisfies the uniform estimates

$$\|\nabla u_\varrho\|_{L^2(\Omega)} \leqslant c k_0^{-1}(\|f\|_{L^2(\Omega)} + \|Q\|_{L^2(\Gamma)}) \quad \textit{in case of } (1.3),$$
$$\|\nabla u_\varrho\|_{L^2(\Omega)} \leqslant c(k_0^{-1}\|f\|_{L^2(\Omega)} + \|\nabla u_e\|_{L^2(\Omega)}) \quad \textit{in case of } (1.4).$$

P r o o f. The matrix κ_ϱ is symmetric and uniformly positive definite. Since $\kappa_\varrho \in \mathcal{C}^{0,1}(\overline{\Omega}; \mathbb{R}^{3\times 3})$, the standard regularity theory for second order elliptic equations in divergence form ([6], Chapter 3, Paragraph 10, Theorem 10.1, or [17], Chapter 2, Section 2.5, Lemma 2.20 and Theorem 2.24 among others) proves the $W^{2,2}$ regularity claim for the solution to (P_ϱ). The strong convergence (3.14) for the entire sequence is obvious due to the uniqueness of the respective weak solutions to (P) and (P_ϱ). $\square$

Our method will consist in deriving uniform estimates for the main components of ∇u_ϱ with respect to the system $\{T^{(1)}, T^{(2)}, n_S\}$, that means, the functions

$$\xi_\varrho^{(1)} := T^{(1)} \cdot \nabla u_\varrho, \quad \xi_\varrho^{(2)} := T^{(2)} \cdot \nabla u_\varrho, \quad \xi_\varrho^{(3)} := \kappa_\varrho n_S \cdot \nabla u_\varrho. \tag{3.15}$$

In Section 4, we reformulate the problem of regularity in a more suitable coordinate system. Section 5 contains the core of the proof of the $W^{2,2}$ regularity, whereas Section 6 is dedicated to the boundedness of ∇u.

4. Preliminary propositions

This section mainly contains the technical rearrangements needed to, so to say, restate the problem in a more convenient coordinates. Throughout the remaining sections, the matrices $A_\varrho, \tilde{\kappa}_\varrho$ are as defined in Section 3. In the next Lemma, basic relationships satisfied by the functions $\xi_\varrho^{(i)}$ $(i = 1, 2, 3)$ are derived. We recall the notation (2.3) for the number g_0.

Lemma 4.1. *Let $u_\varrho \in W^{2,2}(\Omega)$ denote the weak solution to (P_ϱ). Then there are $G_\varrho^{(i)} \in [L^2(\Omega)]^3$ $(i = 1, 2, 3)$ and $M_\varrho^{(3)} \in [L^2(\Omega)]^9$ such that*

$$|G_\varrho^{(1)}| + |G_\varrho^{(2)}| + k_0^2 |G_\varrho^{(3)}| \leqslant c(|f| + g_0 k_1 |\nabla u_\varrho|), \ |M_\varrho^{(3)}| \leqslant c \tag{4.1}$$

with $c = c(\Omega, k_1/k_0, k_1'/k_0)$, and such that the following identities are valid almost everywhere in Ω:

(4.2)

$$\begin{aligned}
\kappa_\varrho \nabla \xi_\varrho^{(1)} &= G_\varrho^{(1)} + \Big(T^{(2)} - \frac{a_\varrho^{2,3}}{a_\varrho^{3,3}} n_S\Big) \times \nabla \xi_\varrho^{(3)} - \sum_{i=1}^{2} \Big(a_\varrho^{2,i} - \frac{a_\varrho^{i,3} a_\varrho^{3,2}}{a_\varrho^{3,3}}\Big)(n_S \times \nabla \xi_\varrho^{(i)}),\\
\tilde{\kappa}_\varrho \nabla \xi_\varrho^{(2)} &= G_\varrho^{(2)} - T^{(1)} \times \nabla \xi_\varrho^{(3)} + (a_\varrho^{1,1} n_S - a_\varrho^{1,3} T^{(1)}) \times \nabla \xi_\varrho^{(1)},\\
\tilde{\kappa}_\varrho \nabla \xi_\varrho^{(3)} &= m_\varrho^{1,1}(G_\varrho^{(3)} + M_\varrho^{(3)} \nabla \xi_\varrho^{(1)} + T^{(1)} \times \nabla \xi^{(2)}).
\end{aligned}$$

P r o o f. *First step.* In the proof, g_ϱ, $\overline{g}_\varrho$ denote generic functions, and G_ϱ, $\overline{G}_\varrho$ generic vector fields, that may change from line to line, but that satisfy the estimates

$$|g_\varrho| + |G_\varrho| \leqslant c g_0 |\nabla u_\varrho|, \ \overline{g}_\varrho + |\overline{G}_\varrho| \leqslant c(|f| + g_0 k_1 |\nabla u_\varrho|), \tag{4.3}$$

with a constant c only dependent on k_1/k_0 and k_1'/k_0. An important device in the proof is the orthonormality of the system $\{T^{(1)}, T^{(2)}, n_S\}$ everywhere in Ω. Every vector field V defined in Ω has a decomposition $V = \sum_{j=1}^{2} (T^{(j)} \cdot V) T^{(j)} + (n_S \cdot V) n_S$. We further introduce the differential operators $\partial^{(i)} := T^{(i)} \cdot \nabla$ for $i = 1, 2$ and $\partial^{(3)} := n_S \cdot \nabla$. If V_1, V_2 are two vector fields among $\{T^{(1)}, T^{(2)}, n_S\}$, the permutation formula $V_1 \cdot \nabla(V_2 \cdot \nabla u_\varrho) = V_2 \cdot \nabla(V_1 \cdot \nabla u_\varrho) + [(V_1 \cdot \nabla)V_2 - (V_2 \cdot \nabla)V_1] \cdot \nabla u_\varrho$, is valid, so that, in view of the convention (4.3),

$$\partial^{(i)} \partial^{(j)} u_\varrho = \partial^{(j)} \partial^{(i)} u_\varrho + g_\varrho \quad \text{for } i, j = 1, 2, 3. \tag{4.4}$$

Second step. Due to (4.4) and the definition (3.15),

$$\partial^{(i)} \xi_\varrho^{(j)} - \partial^{(j)} \xi_\varrho^{(i)} = g_\varrho \quad \text{for } i, j = 1, 2. \tag{4.5}$$

The definition of the function $\xi_\varrho^{(3)}$ and the property of orthonormal decomposition imply that

$$\xi_\varrho^{(3)} = \sum_{i=1}^{2} (\kappa_\varrho n_S \cdot T^{(i)}) \xi_\varrho^{(i)} + (\kappa_\varrho n_S \cdot n_S) \partial^{(3)} u_\varrho = \sum_{i=1}^{2} a_\varrho^{3,i} \xi_\varrho^{(i)} + a_\varrho^{3,3} \partial^{(3)} u_\varrho.$$

Thus

$$\partial^{(3)} u_\varrho = \frac{1}{a_\varrho^{3,3}} \left(\xi_\varrho^{(3)} - \sum_{i=1}^{2} a_\varrho^{3,i} \xi_\varrho^{(i)} \right), \tag{4.6}$$

and it follows for $i = 1, 2$ from (4.6) and (4.4) that

$$\partial^{(3)} \xi_\varrho^{(i)} = \partial^{(i)} \partial^{(3)} u_\varrho + g_\varrho = \frac{1}{a_\varrho^{3,3}} \partial^{(i)} \xi_\varrho^{(3)} - \sum_{j=1}^{2} \frac{a_\varrho^{3,j}}{a_\varrho^{3,3}} \partial^{(i)} \xi_\varrho^{(j)} + g_\varrho, \tag{4.7}$$

$$g_\varrho := \partial^{(i)} \frac{1}{a_\varrho^{3,3}} \xi_\varrho^{(3)} + \sum_{j=1}^{2} \partial^{(i)} \frac{a_\varrho^{3,j}}{a_\varrho^{3,3}} \xi_\varrho^{(j)}.$$

The properties (3.9), (3.10) yield for $i = 1, 2$

$$\left| \partial^{(i)} \frac{1}{a_\varrho^{3,3}} \right| + \sum_{j=1}^{2} \left| \partial^{(i)} \frac{a_\varrho^{3,j}}{a_\varrho^{3,3}} \right| \leqslant 3 \frac{k_1 k_1'}{k_0^2}, \tag{4.8}$$

which can be used to prove in (4.7) that g_ϱ still satisfies (4.3). Using (4.7), we also show that

$$\partial^{(i)} \xi_\varrho^{(3)} = a_\varrho^{3,3} \partial^{(3)} \xi_\varrho^{(i)} + \sum_{j=1}^{2} a_\varrho^{3,j} \partial^{(j)} \xi_\varrho^{(i)} + \overline{g}_{\varrho,i}. \tag{4.9}$$

Third step. In (4.5), (4.7) and (4.9) we have obtained permutations formula for the derivatives $\partial^{(i)}\xi_\varrho^{(j)}$ for $i \neq j$. The equation (3.12) contains additional information about the symmetrical derivatives $\partial^{(i)}\xi_\varrho^{(i)}$. Orthonormal decomposition of ∇u_ϱ, joined to the relation (4.6) yields

$$\kappa_\varrho \nabla u_\varrho = \sum_{i=1}^{2} \xi_\varrho^{(i)} \Big(\kappa_\varrho T^{(i)} - \frac{a_\varrho^{3,i}}{a_\varrho^{3,3}} \kappa_\varrho n_S\Big) + \frac{\xi_\varrho^{(3)}}{a_\varrho^{3,3}} \kappa_\varrho n_S.$$

Again decomposing the vectors $\kappa_\varrho T^{(i)}$ $(i = 1, 2)$ and $\kappa_\varrho n_S$ it follows that

$$\kappa_\varrho \nabla u_\varrho = \sum_{i,j=1}^{2} \Big(a_\varrho^{i,j} - \frac{a_\varrho^{i,3} a_\varrho^{j,3}}{a_\varrho^{3,3}}\Big) \xi_\varrho^{(i)} T^{(j)} + \sum_{j=1}^{2} \frac{a_\varrho^{3,j}}{a_\varrho^{3,3}} T^{(j)} \xi_\varrho^{(3)} + n_S \xi_\varrho^{(3)}. \tag{4.10}$$

According to Lemma 3.1, $u_\varrho \in W^{2,2}(\Omega)$, and $-\operatorname{div}(\kappa_\varrho \nabla u_\varrho) = f$ almost everywhere in Ω. Therefore, (4.10) implies that

(4.11)

$$\begin{aligned}\sum_{i,j=1}^{2} \Big(a_\varrho^{i,j} - \frac{a_\varrho^{i,3} a_\varrho^{j,3}}{a_\varrho^{3,3}}\Big) \partial^{(j)} \xi_\varrho^{(i)} + \sum_{j=1}^{2} \frac{a_\varrho^{3,j}}{a_\varrho^{3,3}} \partial^{(j)} \xi_\varrho^{(3)} + \partial^{(3)} \xi_\varrho^{(3)} = \overline{g}_\varrho \\ := -f - \sum_{i,j=1}^{2} \operatorname{div}\Big(\Big(a_\varrho^{i,j} - \frac{a_\varrho^{i,3} a_\varrho^{j,3}}{a_\varrho^{3,3}}\Big) T^{(j)}\Big) \xi_\varrho^{(i)} - \operatorname{div}\Big(\sum_{j=1}^{2} \frac{a_\varrho^{3,j}}{a_\varrho^{3,3}} T^{(j)} + n_S\Big) \xi_\varrho^{(3)}.\end{aligned}$$

Due to (3.9), $\overline{g}_\varrho$ satisfies the estimate (4.3) again (cf. the computation (4.8)). Fix an index $i \in \{1, 2\}$, and define i' by requesting that $\{i\} \cup \{i'\} = \{1, 2\}$. From (4.11), it follows for $i = 1, 2$ that

$$\begin{aligned}\Big(a_\varrho^{i,i} - \frac{[a_\varrho^{i,3}]^2}{a_\varrho^{3,3}}\Big) \partial^{(i)} \xi_\varrho^{(i)} &= \overline{g}_\varrho - \nabla \xi_\varrho^{(3)} \cdot \Big(n_S + \sum_{j=1}^{2} \frac{a_\varrho^{3,j}}{a_\varrho^{3,3}} T^{(j)}\Big) \\ &\quad - \Big(a_\varrho^{i,i'} - \frac{a_\varrho^{i,3} a_\varrho^{i',3}}{a_\varrho^{3,3}}\Big) \partial^{(i')} \xi_\varrho^{(i)} - \sum_{j=1}^{2} \Big(a_\varrho^{i',j} - \frac{a_\varrho^{i',3} a_\varrho^{j,3}}{a_\varrho^{3,3}}\Big) \partial^{(j)} \xi_\varrho^{(i')}.\end{aligned} \tag{4.12}$$

In (4.12), permutation of $\partial^{(i')}$ and $\partial^{(i)}$ together with the formula (4.4) yields

$$\begin{aligned}\Big(a_\varrho^{i,i} - \frac{[a_\varrho^{i,3}]^2}{a_\varrho^{3,3}}\Big) \partial^{(i)} \xi_\varrho^{(i)} &= \overline{g}_\varrho - \nabla \xi_\varrho^{(3)} \cdot \Big(n_S + \sum_{j=1}^{2} \frac{a_\varrho^{3,j}}{a_\varrho^{3,3}} T^{(j)}\Big) \\ &\quad - \Big(a_\varrho^{i',i'} - \frac{[a_\varrho^{i',3}]^2}{a_\varrho^{3,3}}\Big) \partial^{(i')} \xi_\varrho^{(i')} - 2\Big(a_\varrho^{i,i'} - \frac{a_\varrho^{i,3} a_\varrho^{i',3}}{a_\varrho^{3,3}}\Big) \partial^{(i)} \xi_\varrho^{(i')}.\end{aligned} \tag{4.13}$$

Using the formula (4.9) we can also re-express the term $\partial^{(i')}\xi_\varrho^{(3)}$ in the formula (4.13) to obtain, for $i = 1, 2$, the decomposition

$$\text{(4.14)} \qquad \Big(a_\varrho^{i,i} - \frac{[a_\varrho^{i,3}]^2}{a_\varrho^{3,3}}\Big)\partial^{(i)}\xi_\varrho^{(i)} = \overline{g}_\varrho - \nabla\xi_\varrho^{(3)} \cdot \Big(n_S + \frac{a_\varrho^{3,i}}{a_\varrho^{3,3}}T^{(i)}\Big) - \nabla\xi_\varrho^{(i')} \cdot \Big(a_\varrho^{i',i'}T^{(i')} + \Big[2a_\varrho^{i',i} - \frac{a_\varrho^{3,i}a_\varrho^{3,i'}}{a_\varrho^{3,3}}\Big]T^{(i)} + a_\varrho^{3,i'}n_S\Big).$$

In the case $i = 3$, we conclude from (4.11) and (4.9) that
(4.15)

$$\begin{aligned}\partial^{(3)}\xi_\varrho^{(3)} &= -\sum_{i,j=1}^{2}\Big(a_\varrho^{i,j} - \frac{a_\varrho^{i,3}a_\varrho^{j,3}}{a_\varrho^{3,3}}\Big)\partial^{(j)}\xi_\varrho^{(i)} - \sum_{i=1}^{2}\frac{a_\varrho^{3,i}}{a_\varrho^{3,3}}\partial^{(i)}\xi_\varrho^{(3)} + \overline{g}_\varrho \\ &= -\sum_{i,j=1}^{2} a_\varrho^{i,j}\partial^{(j)}\xi_\varrho^{(i)} - \sum_{i=1}^{2} a_\varrho^{3,i}\partial^{(3)}\xi_\varrho^{(i)} + \overline{g}_\varrho = -\sum_{i=1}^{2}\kappa_\varrho T^{(i)} \cdot \nabla\xi_\varrho^{(i)} + \overline{g}_\varrho.\end{aligned}$$

Fourth step. For $i = 1, 2$, the relation

$$\text{(4.16)} \qquad \Big(a_\varrho^{i,i} - \frac{[a_\varrho^{3,i}]^2}{a_\varrho^{3,3}}\Big)\partial^{(i)}\xi_\varrho^{(i)} = -\nabla\xi_\varrho^{(3)} \cdot V_\varrho - \nabla\xi_\varrho^{(i')} \cdot W_\varrho + \overline{g}_\varrho$$

follows from (4.13) for the choice
(4.17)

$$V_\varrho := n_S + \sum_{j=1}^{2}\frac{a_\varrho^{3,j}}{a_\varrho^{3,3}}T^{(j)}, \; W_\varrho := \Big(a_\varrho^{i',i'} - \frac{[a_\varrho^{3,i'}]^2}{a_\varrho^{3,3}}\Big)T^{(i')} + 2\Big(a_\varrho^{i,i'} - \frac{a_\varrho^{3,i'}a_\varrho^{3,i}}{a_\varrho^{3,3}}\Big)T^{(i)},$$

whereas (4.16) is a consequence of (4.14) for the choice

$$\text{(4.18)} \quad V_\varrho := n_S + \frac{a_\varrho^{3,i}}{a_\varrho^{3,3}}T^{(i)}, \quad W_\varrho := a_\varrho^{i',i'}T^{(i')} + \Big[2a_\varrho^{i',i} - \frac{a_\varrho^{3,i}a_\varrho^{3,i'}}{a_\varrho^{3,3}}\Big]T^{(i)} + a_\varrho^{3,i'}n_S.$$

We decompose the vector $\nabla\xi_\varrho^{(i)}$ and use the representation (4.7) to show for $i = 1, 2$ that

$$\text{(4.19)} \qquad \nabla\xi_\varrho^{(i)} = \partial^{(i)}\xi_\varrho^{(i)}\Big(T^{(i)} - \frac{a_\varrho^{3,i}}{a_\varrho^{3,3}}n_S\Big) + \partial^{(i)}\xi_\varrho^{(i')}\Big(T^{(i')} - \frac{a_\varrho^{3,i'}}{a_\varrho^{3,3}}n_S\Big) + \partial^{(i)}\xi_\varrho^{(3)}\frac{n_S}{a_\varrho^{3,3}} + g_\varrho n_S.$$

The representation (4.16) and the formula (4.19) imply for $i = 1, 2$ that

$$\text{(4.20)} \quad \nabla\xi_\varrho^{(i)} = \Big(a_\varrho^{i,i} - \frac{[a_\varrho^{3,i}]^2}{a_\varrho^{3,3}}\Big)^{-1}(-\nabla\xi_\varrho^{(3)} \cdot V_\varrho - \nabla\xi_\varrho^{(i')} \cdot W_\varrho + \overline{g}_\varrho)\Big(T^{(i)} - \frac{a_\varrho^{3,i}}{a_\varrho^{3,3}}n_S\Big) + \partial^{(i)}\xi_\varrho^{(i')}\Big(T^{(i')} - \frac{a_\varrho^{3,i'}}{a_\varrho^{3,3}}n_S\Big) + \partial^{(i)}\xi_\varrho^{(3)}\frac{n_S}{a_\varrho^{3,3}} + g_\varrho n_S.$$

Let $B_\varrho^{(i)}$ be the matrix that satisfies

$$B_\varrho^{(i)}\Big(T^{(i)} - \frac{a_\varrho^{3,i}}{a_\varrho^{3,3}} n_S\Big) = \Big(a_\varrho^{i,i} - \frac{[a_\varrho^{3,i}]^2}{a_\varrho^{3,3}}\Big) T^{(i)},$$
$$B_\varrho^{(i)}\Big(T^{(i')} - \frac{a_\varrho^{3,i'}}{a_\varrho^{3,3}} n_S\Big) = W_\varrho, \quad B_\varrho^{(i)} n_S = a_\varrho^{3,3} V_\varrho. \tag{4.21}$$

Multiply the relation (4.20) by $B^{(i)}$ to see that

$$\begin{aligned} B_\varrho^{(i)}\nabla\xi_\varrho^{(i)} &= (-\nabla\xi_\varrho^{(3)} \cdot V_\varrho - \nabla\xi_\varrho^{(i')} \cdot W_\varrho) T^{(i)} + \partial^{(i)}\xi_\varrho^{(i')} W_\varrho + \partial^{(i)}\xi_\varrho^{(3)} V_\varrho + \overline{G}_\varrho \\ &= (T^{(i)} \times V_\varrho) \times \nabla\xi_\varrho^{(3)} + (T^{(i)} \times W_\varrho) \times \nabla\xi_\varrho^{(i')} + \overline{G}_\varrho. \end{aligned} \tag{4.22}$$

Fifth step. In *the case* $i = 1$, the formula (4.17) yields

$$T^{(1)} \times V_\varrho = T^{(2)} - \frac{a_\varrho^{2,3}}{a_\varrho^{3,3}} n_S, \quad T^{(1)} \times W_\varrho = -\Big(a_\varrho^{2,2} - \frac{[a_\varrho^{3,2}]^2}{a_\varrho^{3,3}}\Big) n_S. \tag{4.23}$$

Moreover, the conditions (4.21) imply the identity

$$O^T B_\varrho^{(1)} O = A_\varrho + \begin{pmatrix} 0 & -b_\varrho^{(1)} & 0 \\ b_\varrho^{(1)} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad b_\varrho^{(1)} := a_\varrho^{2,1} - \frac{a_\varrho^{3,1} a_\varrho^{3,2}}{a_\varrho^{3,3}} = \frac{m_\varrho^{1,1}}{a_\varrho^{3,3}}.$$

Elementary calculations with the skew-symmetric matrix part show that

$$B_\varrho^{(1)}\nabla\xi_\varrho^{(1)} = \kappa_\varrho \nabla\xi_\varrho^{(1)} + b_\varrho^{(1)} (T^{(2)} \times T^{(1)}) \times \nabla\xi_\varrho^{(1)}. \tag{4.24}$$

Observe that $T^{(2)} \times T^{(1)} = -n_S$. Putting (4.24) and (4.23) into (4.22), the claim (4.2) follows for $\xi_\varrho^{(1)}$.

In *the case* $i = 2$, the formula (4.18) implies that

$$T^{(2)} \times V_\varrho = T^{(2)} \times n_S = -T^{(1)}, \; T^{(2)} \times W_\varrho = a_\varrho^{1,1} n_S - a_\varrho^{3,1} T^{(1)}. \tag{4.25}$$

Moreover, it can be shown easily that the matrix $B_\varrho^{(2)}$ that is uniquely defined by the conditions (4.21) is nothing else but the matrix $\tilde{\kappa}_\varrho$ introduced in Section 3. The claim (4.2) for $\xi_\varrho^{(2)}$ follows from (4.22).

In *the case* $i = 3$, orthonormal decomposition and the formula (4.9) imply that

(4.26)
$$\begin{aligned} \nabla\xi_\varrho^{(3)} &= \sum_{i=1}^{2} \Big(a_\varrho^{3,3}\partial^{(3)}\xi_\varrho^{(i)} + \sum_{j=1}^{2} a_\varrho^{3,j}\partial^{(j)}\xi_\varrho^{(i)}\Big) T^{(i)} + \partial^{(3)}\xi_\varrho^{(3)} n_S + \overline{G}_\varrho \\ &= \sum_{i=1}^{2} \kappa_\varrho n_S \cdot \nabla\xi^{(i)} T^{(i)} + \partial^{(3)}\xi_\varrho^{(3)} n_S + \overline{G}_\varrho. \end{aligned}$$

Insert (4.15) into (4.26) to obtain the equivalent representation

$$\nabla \xi_\varrho^{(3)} = \sum_{i=1}^{2} (\kappa_\varrho n_S \cdot \nabla \xi_\varrho^{(i)} T^{(i)} - \kappa_\varrho T_i \cdot \nabla \xi_\varrho^{(i)} n_S) + \overline{G}_\varrho. \tag{4.27}$$

The permutation formula (4.4) implies that

$$\begin{aligned} &\kappa_\varrho n_S \cdot \nabla \xi_\varrho^{(2)} T^{(2)} - \kappa_\varrho T_2 \cdot \nabla \xi_\varrho^{(2)} n_S \\ &\quad = (\kappa_\varrho n_S - a_\varrho^{3,1} T^{(1)}) \cdot \nabla \xi_\varrho^{(2)} T^{(2)} - (\kappa_\varrho T^{(2)} - a_\varrho^{2,1} T^{(1)}) \cdot \nabla \xi_\varrho^{(2)} n_S \\ &\qquad + a_\varrho^{3,1} T^{(2)} \cdot \nabla \xi_\varrho^{(1)} T^{(2)} - a_\varrho^{2,1} T^{(2)} \cdot \nabla \xi_\varrho^{(1)} n_S + \overline{G}_\varrho. \end{aligned} \tag{4.28}$$

From (4.27) and (4.28) it follows that

$$\begin{aligned} \nabla \xi_\varrho^{(3)} &= (\kappa_\varrho n_S - a_\varrho^{3,1} T^{(1)}) \cdot \nabla \xi_\varrho^{(2)} T^{(2)} - (\kappa_\varrho T^{(2)} \\ &\quad - a_\varrho^{2,1} T^{(1)}) \cdot \nabla \xi_\varrho^{(2)} n_S + \overline{G}_\varrho + \widetilde{M}_\varrho^{(3)} \nabla \xi_\varrho^{(1)}, \\ (\widetilde{M}_\varrho^{(3)})^{i,j} &:= T_i^{(1)} (\kappa_\varrho n_S)_j - n_{S,i} (\kappa_\varrho T^{(1)} + a_\varrho^{2,1} T^{(2)})_j + a_\varrho^{3,1} T_i^{(2)} T_j^{(2)}. \end{aligned} \tag{4.29}$$

Let $B_\varrho^{(3)}$ be a matrix that satisfies

$$B_\varrho^{(3)} T^{(2)} = \frac{\kappa_\varrho T^{(2)} - a_\varrho^{2,1} T^{(1)}}{m_\varrho^{1,1}}, \quad B_\varrho^{(3)} n_S = \frac{\kappa_\varrho n_S - a_\varrho^{3,1} T^{(1)}}{m_\varrho^{1,1}}. \tag{4.30}$$

Apply $B_\varrho^{(3)}$ to (4.29), and define $M_\varrho^{(3)} := B_\varrho^{(3)} \widetilde{M}_\varrho^{(3)}$, then

$$B_\varrho^{(3)} \nabla \xi_\varrho^{(3)} = B_\varrho^{(3)} \overline{G}_\varrho + M_\varrho^{(3)} \nabla \xi_\varrho^{(1)} + (m_\varrho^{1,1})^{-1} (B_\varrho^{(3)} n_S \times B_\varrho^{(3)} T^{(2)}) \times \nabla \xi_\varrho^{(2)}.$$

Observe that

$$\begin{aligned} B_\varrho^{(3)} n_S \times B_\varrho^{(3)} T^{(2)} &= [a_\varrho^{2,3}]^2 (T^{(2)} \times n_S) + a_\varrho^{2,2} a_\varrho^{3,3} (n_S \times T^{(2)}) \\ &= (a_\varrho^{2,2} a_\varrho^{3,3} - [a_\varrho^{2,3}]^2) T^{(1)} = m_\varrho^{1,1} T^{(1)}. \end{aligned}$$

We at last notice using (2.7) that the choice $B_\varrho^{(3)} = (m_\varrho^{1,1})^{-1} \tilde{\kappa}_\varrho$ satisfies (4.30). The claim (4.2) for $\xi_\varrho^{(3)}$ follows easily. □

In the following Lemmas we use the result of Lemma 4.1 to derive integral relations satisfied by the functions $\xi_\varrho^{(i)}$ $(i = 1, 2, 3)$.

Lemma 4.2. *Same assumptions as in Lemma* 4.1. *Then there is*

$$\overline{G}_\varrho^{(1)} \in [L^2(\Omega)]^3,\ |\overline{G}_\varrho^{(1)}| \leqslant c(|f| + g_0k_1|\nabla u_\varrho|) \text{ a.e. in } \Omega, \tag{4.31}$$

such that for all $v \in W^{2,2}(\Omega)$

$$\int_\Omega \kappa_\varrho \nabla\xi_\varrho^{(1)} \cdot \nabla v = \int_\Omega \overline{G}_\varrho^{(1)} \cdot \nabla v - \int_\Gamma (\kappa_\varrho n_\Gamma \cdot \nabla u_\varrho)(\tau^{(1)} \cdot \nabla v). \tag{4.32}$$

Proof. Choose $v \in W^{2,2}(\Omega)$ arbitrary, and multiply the relation (4.2) for $\xi_\varrho^{(1)}$ by ∇v. Due to integration by parts and to the vector identity $\operatorname{div}(a \times b) = \operatorname{curl} a \cdot b + \operatorname{curl} b \cdot a$,

$$\begin{aligned}\int_\Omega \left(T^{(2)} - \frac{a_\varrho^{2,3}}{a_\varrho^{3,3}} n_S\right) \times \nabla\xi_\varrho^{(3)} \cdot \nabla v &= -\int_\Omega \left(T^{(2)} - \frac{a_\varrho^{2,3}}{a_\varrho^{3,3}} n_S\right) \times \nabla v \cdot \nabla\xi_\varrho^{(3)} \\ &= \int_\Omega \operatorname{curl}\left(T^{(2)} - \frac{a_\varrho^{2,3}}{a_\varrho^{3,3}} n_S\right) \cdot \nabla v \xi_\varrho^{(3)} - \int_\Gamma \left(T^{(2)} - \frac{a_\varrho^{2,3}}{a_\varrho^{3,3}} n_S\right) \times \nabla v \cdot n_\Gamma \xi_\varrho^{(3)}.\end{aligned}$$

By similar arguments, and abbreviating $p_{i,\varrho} := a_\varrho^{2,i} - a_\varrho^{3,i} a_\varrho^{3,2}/a_\varrho^{3,3}$, it follows for $i = 1, 2$ that

$$-\int_\Omega p_{i,\varrho}(n_S \times \nabla\xi_\varrho^{(i)}) \cdot \nabla v = \int_\Gamma p_{i,\varrho}(n_S \times \nabla v) \cdot n_\Gamma \xi_\varrho^{(i)} - \int_\Omega \operatorname{curl}(p_{i,\varrho} n_S) \cdot \nabla v \xi_\varrho^{(i)}.$$

Choosing $G_\varrho^{(1)}$ as in Lemma 4.1, we define

$$\overline{G}_\varrho^{(1)} := G_\varrho^{(1)} - \sum_{i=1}^{2} \operatorname{curl}\left(\left(a_\varrho^{2,i} - \frac{a_\varrho^{3,i} a_\varrho^{3,2}}{a_\varrho^{3,3}}\right) n_S\right) \xi_\varrho^{(i)} + \operatorname{curl}\left(T^{(2)} - \frac{a_\varrho^{2,3}}{a_\varrho^{3,3}} n_S\right) \xi_\varrho^{(3)}.$$

For $g \in C^{0,1}(\Omega)$, observe that $\operatorname{curl}(g n_S) = g n_S + \nabla g \times n_S$. Thus, only tangential derivatives of the regularized coefficients occur in the definition of $\overline{G}_\varrho^{(1)}$, and (3.10) can be used to prove the estimate (4.31). In order to reformulate the integrals over Γ, observe that

$$\begin{aligned}(n_S \times \nabla v) \cdot n_\Gamma &= -(n_S \times n_\Gamma) \cdot \nabla v = -|n_S \times n_\Gamma| \tau^{(1)} \cdot \nabla v, \\ (T^{(2)} \times \nabla v) \cdot n_\Gamma &= -(T^{(2)} \times n_\Gamma) \cdot \nabla v = (T^{(2)} \cdot \tau^{(2)}) \tau^{(1)} \cdot \nabla v.\end{aligned}$$

Lemma C.3 in the appendix implies that

$$\begin{aligned}&\sum_{i=1}^{2} \left(a_\varrho^{2,i} - \frac{a_\varrho^{3,i} a_\varrho^{3,2}}{a_\varrho^{3,3}}\right)(n_S \times \nabla v) \cdot n_\Gamma \xi_\varrho^{(i)} - \left(T^{(2)} - \frac{a_\varrho^{2,3}}{a_\varrho^{3,3}} n_S\right) \times \nabla v \cdot n_\Gamma \xi_\varrho^{(3)} \\ &\quad = \left(-\sin\alpha\left[\sum_{i=1}^{2}\left(a_\varrho^{2,i} - \frac{a_\varrho^{3,i} a_\varrho^{3,2}}{a_\varrho^{3,3}}\right)\xi_\varrho^{(i)} + \frac{a_\varrho^{2,3}}{a_\varrho^{3,3}} \xi_\varrho^{(3)}\right] - \cos\alpha \xi_\varrho^{(3)}\right)(\tau^{(1)} \cdot \nabla v).\end{aligned}$$

Using orthonormal decomposition for the vector $-\kappa_\varrho n_\Gamma \cdot \nabla u_\varrho$, the relation (4.32) is obvious. □

Lemma 4.3. *Same assumptions as in Lemma* 4.1. *Then there are* $\overline{G}_\varrho^{(2)}, \overline{G}_\varrho^{(3)} \in [L^2(\Omega)]^3$ *such that*

$$|\overline{G}_\varrho^{(2)}| + k_0|\overline{G}_\varrho^{(3)}| \leqslant c(|f| + g_0 k_1 |\nabla u_\varrho|) \quad \text{a.e. in } \Omega, \tag{4.33}$$

and such that for all $v \in W^{2,2}(\Omega)$

$$\int_\Omega \tilde{\kappa}_\varrho \nabla \xi_\varrho^{(2)} \cdot \nabla v = \int_\Omega \{\overline{G}_\varrho^{(2)} + (a_\varrho^{1,1} n_S - a_\varrho^{1,3} T^{(1)}) \times \nabla \xi_\varrho^{(1)}\} \cdot \nabla v - \int_\Gamma \xi_\varrho^{(3)} (\tau^{(2)} \cdot \nabla v), \tag{4.34}$$

$$\int_\Omega [m_\varrho^{1,1}]^{-1} \tilde{\kappa}_\varrho \nabla \xi_\varrho^{(3)} \cdot \nabla v = \int_\Omega \{\overline{G}_\varrho^{(3)} + M_\varrho^{(3)} \nabla \xi_\varrho^{(1)}\} \cdot \nabla v + \int_\Gamma \xi_\varrho^{(2)} (\tau^{(2)} \cdot \nabla v). \tag{4.35}$$

Proof. We multiply the relation (4.2) for $\xi_\varrho^{(2)}$ by ∇v, $v \in W^{2,2}(\Omega)$ arbitrary. Integration by parts and the fact that $T^{(1)} \times n_\Gamma = \tau^{(2)}$ yield

$$\int_\Omega (T^{(1)} \times \nabla \xi_\varrho^{(3)}) \cdot \nabla v = \int_\Omega \operatorname{curl} T^{(1)} \cdot \nabla v \xi_\varrho^{(3)} + \int_\Gamma \xi_\varrho^{(3)} \tau^{(2)} \cdot \nabla v, \tag{4.36}$$

Choosing $G_\varrho^{(2)}$ as in Lemma 4.1, we define $\overline{G}_\varrho^{(2)} := G_\varrho^{(2)} - \operatorname{curl} T^{(1)} \xi_\varrho^{(3)}$. The estimate (4.33) is readily checked. The relation (4.34) is obvious.

In order to prove (4.35), multiply the relation (4.2) for $\xi_\varrho^{(3)}$ by ∇v, $v \in W^{2,2}(\Omega)$ arbitrary. As in (4.36),

$$\int_\Omega (T^{(1)} \times \nabla \xi_\varrho^{(2)}) \cdot \nabla v = \int_\Omega \operatorname{curl} T^{(1)} \cdot \nabla v \xi_\varrho^{(2)} + \int_\Gamma \xi_\varrho^{(2)} \tau^{(2)} \cdot \nabla v.$$

Define $\overline{G}_\varrho^{(3)} := G_\varrho^{(3)} + \operatorname{curl} T^{(1)} \xi_\varrho^{(2)}$. The estimate (4.33) is readily checked, completing the proof. □

We now prove two lemmas concerning the boundary data u_e and Q. The compatibility conditions (2.20), (2.21), (2.22) come here into the play.

Lemma 4.4. *In addition to the hypotheses of Lemma* 4.1, *assume that the conditions* (2.20), (2.21) *are satisfied for the problem* (P_N), *or that* (2.22) *is valid for the problem* (P_D). *Then there are* $\widetilde{Q}_{1,\varrho}, \widetilde{Q}_{2,\varrho} \in W^{1/2,2}(\Gamma)$ *and* $\widetilde{U}_{2,\varrho} \in W^{1/2,2}(\Gamma)$ *such that*

$$\frac{m_\varrho^{2,1}}{m_\varrho^{1,1}} \xi_\varrho^{(1)} + \frac{a_\varrho^{3,3}}{m_\varrho^{1,1} \sin\alpha} Q = f_d(\alpha, A_\varrho) \widetilde{Q}_{1,\varrho} + \widetilde{Q}_{2,\varrho}, \tag{4.37}$$

$$a_\varrho^{3,1} (\tau^{(1)} \cdot \nabla u_e) - \frac{a_\varrho^{3,3}}{\sin\alpha} (\tau^{(2)} \cdot \nabla u_e) = f_d(\alpha, A_\varrho) U_1 + \widetilde{U}_{2,\varrho}. \tag{4.38}$$

Moreover, there is $c = c(\Omega, k_1/k_0)$ such that

$$
\begin{aligned}
&(4.39) \quad \|\widetilde{U}_{2,\varrho}\|_{W^{1/2,2}(\Gamma)} \leqslant \|U_2\|_{V^2(\Gamma)} + ck_1 g_0 \|\nabla u_e\|_{W^{1/2,2}(\Gamma)} + C_{1,\varrho},\\
&(4.40) \quad \|\widetilde{Q}_{1,\varrho}\|_{W^{1/2,2}(\Gamma)} \leqslant \|Q_1\|_{W^{1/2,2}(\Gamma)} + cg_1 \|\xi_\varrho^{(1)}\|_{W^{1/2,2}(\Gamma)},\\
&(4.41) \quad \|\widetilde{Q}_{2,\varrho}\|_{W^{1/2,2}(\Gamma)} \leqslant ck_0^{-1}(1+g_0)(\|Q\|_{W^{1/2,2}(\Gamma)} + \|Q_1\|_{W^{1/2,2}(\Gamma)}\\
&\qquad\qquad + (1+g_1)\|\xi_\varrho^{(1)}\|_{W^{1/2,2}(\Gamma)}) + \|Q_2\|_{W^{1/2,2}(\Gamma)} + C_{2,\varrho},
\end{aligned}
$$

where $C_{1,\varrho}, C_{2,\varrho} \to 0$ as $\varrho \to 0$.

P r o o f. The condition (2.22) is by assumption valid on Γ. Recalling the definition (3.1), we multiply (2.22) by the function $I_\varrho(d_S(\cdot))$, and then add on both sides of the new relation the term $a_1^{3,1}\xi_e^{(1)} - a_1^{3,3}(\tau^{(2)} \cdot \nabla u_e)/\sin\alpha$. We obtain that

$$
\begin{aligned}
(4.42) \qquad & (a_1^{3,1} + I_\varrho[a^{3,1}]_S)\xi_e^{(1)} - a_1^{3,3} + I_\varrho[a^{3,3}]_S(\tau^{(2)} \cdot \nabla u_e)/\sin\alpha\\
&= (\cot\alpha(a_1^{3,3} + I_\varrho[a^{3,3}]_S) + (a_1^{2,3} + I_\varrho[a^{2,3}]_S))U_1 + I_\varrho U_2\\
&\quad + a_1^{3,1}\xi_e^{(1)} - a_1^{3,3}(\tau^{(2)} \cdot \nabla u_e)/\sin\alpha - (\cot\alpha a_1^{3,3} + a_1^{2,3})U_1.
\end{aligned}
$$

Due to (3.5), $a_1^{3,1} + I_\varrho[a^{3,1}]_S = a_\varrho^{3,1} = L_\varrho(a^{3,1})$ (etc.), so that

$$
\begin{aligned}
a_\varrho^{3,1}\xi_e^{(1)} - a_\varrho^{3,3}(\tau^{(2)} \cdot \nabla u_e)/\sin\alpha &= (\cot\alpha a_\varrho^{3,3} + a_\varrho^{2,3})U_1 + \widetilde{U}_{2,\varrho},\\
\widetilde{U}_{2,\varrho} := I_\varrho U_2 + a_1^{3,1}\xi_e^{(1)} - a_1^{3,3}&(\tau^{(2)} \cdot \nabla u_e)/\sin\alpha - (\cot\alpha a_1^{3,3} + a_1^{2,3})U_1,
\end{aligned}
$$

which proves (4.38) on Γ. Thanks to Lemma B.5 in the appendix, we verify that

$$
\|I_\varrho U_2\|_{W^{1/2,2}(\Gamma)} \leqslant \|U_2\|_{V^2(\Gamma)} + C_{1,\varrho},\ C_{1,\varrho} \to 0.
$$

Using also Lemma B.1, the norm estimate (4.39) follows. In order to prove (4.37), use the assumption (2.21) to define

$$
\widetilde{Q}_{1,\varrho} := Q_1 + \Big[\frac{m^{2,1}}{m^{1,1}}\Big]_S [f_d(\alpha, A)]_S^{-1}\xi_\varrho^{(1)} = Q_1 + g_1\xi_\varrho^{(1)}.
$$

Due to (B.1), we readily verify the estimate (4.40). It then follows from (2.20) that

$$
\begin{aligned}
(4.43) \qquad [f_d(\alpha, A)]_S\widetilde{Q}_{1,\varrho} &= [f_d(\alpha, A)]_S Q_1 + \Big[\frac{m^{2,1}}{m^{1,1}}\Big]_S \xi_\varrho^{(1)}\\
&= \Big[\frac{a^{3,3}}{m^{1,1}}\Big]_S \frac{Q}{\sin\alpha} - Q_2 + \Big[\frac{m^{2,1}}{m^{1,1}}\Big]_S \xi_\varrho^{(1)}.
\end{aligned}
$$

The construction (3.8) has in particular the property that

$$
\left(\cot\alpha\frac{a_1^{3,3}}{m_1^{1,1}} + \frac{a_1^{2,3}}{m_1^{1,1}}\right) + I_\varrho[f_d(\alpha, A)]_S = f_d(\alpha, A_\varrho).
$$

Similarly, $a_\varrho^{3,3}/m_\varrho^{1,1} = L_\varrho(a^{3,3}/m^{1,1})$ and $m_\varrho^{1,2}/m_\varrho^{1,1} = L_\varrho(m^{1,2}/m^{1,1})$. We multiply (4.43) by $I_\varrho(d_S(\cdot))$, then with the help of (3.8), we obtain that

$$f_d(\alpha, A_\varrho)\widetilde{Q}_{1,\varrho} = \frac{a_\varrho^{3,3}}{m_\varrho^{1,1}}\frac{Q}{\sin\alpha} + \frac{m_\varrho^{2,1}}{m_\varrho^{1,1}}\xi_\varrho^{(1)} - \widetilde{Q}_{2,\varrho},$$

$$\widetilde{Q}_{2,\varrho} := I_\varrho Q_2 + \frac{a_1^{3,3}Q}{m_1^{1,1}\sin\alpha} - \left(\cot\alpha\frac{a_1^{3,3}}{m_1^{1,1}} + \frac{a_1^{2,3}}{m_1^{1,1}}\right)\widetilde{Q}_{1,\varrho} + \frac{m_1^{2,1}}{m_1^{1,1}}\xi_\varrho^{(1)}. \tag{4.44}$$

The construction of the regularization (3.6), (3.7) plays here the essential role. The inequality (4.41) is derived in the same fashion as (4.39), using Lemma B.5, Lemma B.1 and (4.40). □

Lemma 4.5. *Same assumptions as in Lemma* 4.4. *Let* $u_\varrho \in W^{2,2}(\Omega)$ *be the weak solution to* (P_ϱ). *If* u_ϱ *satisfies the condition* (1.3), *then*

$$-\xi_\varrho^{(2)} = \left(\cot\alpha\frac{a_\varrho^{3,3}}{m_\varrho^{1,1}} + \frac{a_\varrho^{3,2}}{m_\varrho^{1,1}}\right)(\xi_\varrho^{(3)} + \widetilde{Q}_{1,\varrho}) + \widetilde{Q}_{2,\varrho} \quad \text{a.e. on } \Gamma. \tag{4.45}$$

If u_ϱ *satisfies the condition* (1.4), *then*

$$\xi_\varrho^{(3)} = (\cot\alpha a_\varrho^{3,3} + a_\varrho^{3,2})(\xi_\varrho^{(2)} + U_1) + \widetilde{U}_{2,\varrho} \quad \text{a.e. on } \Gamma. \tag{4.46}$$

Proof. We recall the notation (2.4) and (2.5). If (1.3) is satisfied in the sense of traces, then

$$Q = -\kappa_\varrho n_\Gamma \cdot \nabla u_\varrho = -(n_\Gamma \cdot n_S)\kappa_\varrho n_S \cdot \nabla u_\varrho - (n_\Gamma \cdot T^{(2)})\kappa_\varrho T^{(2)} \cdot \nabla u_\varrho, \tag{4.47}$$

thanks to orthonormal decomposition on Γ. For the same reason, the equivalence (4.6) yields

$$\kappa_\varrho T^{(2)} \cdot \nabla u_\varrho\, \xi_\varrho^{(i)} + a_\varrho^{2,3}\, n_S \cdot \nabla u_\varrho = \sum_{i=1}^{2}\left(a_\varrho^{2,i} - \frac{a_\varrho^{3,i}a_\varrho^{3,2}}{a_\varrho^{3,3}}\right)\xi_\varrho^{(i)} + \frac{a_\varrho^{3,2}}{a_\varrho^{3,3}}\xi_\varrho^{(3)}.$$

Using Lemma C.3 and the definition (3.15) of $\xi_\varrho^{(3)}$, we easily deduce from (4.47) that

$$-\xi_\varrho^{(2)} = \left(\cot\alpha\frac{a_\varrho^{3,3}}{m_\varrho^{1,1}} + \frac{a_\varrho^{3,2}}{m_\varrho^{1,1}}\right)\xi_\varrho^{(3)} + \frac{m_\varrho^{2,1}}{m_\varrho^{1,1}}\xi_\varrho^{(1)} + \frac{a_\varrho^{3,3}}{m_\varrho^{1,1}\sin\alpha}Q,$$

and (4.45) follows from Lemma 4.4, (4.37). With help of orthonormal decomposition, (4.6), and Lemma C.3

$$\begin{aligned}\tau^{(2)} \cdot \nabla u_\varrho &= (\tau^{(2)} \cdot T^{(2)})\xi_\varrho^{(2)} + (\tau^{(2)} \cdot n_S)(n_S \cdot \nabla u_\varrho)\\ &= \cos\alpha\xi^{(2)} - \frac{\sin\alpha}{a_\varrho^{3,3}}\left(\xi_\varrho^{(3)} - \sum_{i=1}^{2} a_\varrho^{3,i}\xi_\varrho^{(i)}\right).\end{aligned}$$

If (1.4) is satisfied in the sense of traces, then

$$\xi_\varrho^{(3)} = (\cot\alpha a_\varrho^{3,3} + a_\varrho^{3,2})\xi_\varrho^{(2)} + a_\varrho^{3,1}\xi_e^{(1)} - \frac{a_\varrho^{3,3}}{\sin\alpha}\tau^{(2)}\cdot\nabla u_e,$$

and (4.46) follows from Lemma 4.4, (4.38). □

5. $W^{2,2}$ regularity

In this section we prove the convergence of the approximation method (P_ϱ) in the space $W^{2,2}$. In order to abbreviate our estimates, we introduce for the problem (P_N) the quantities

$$\begin{aligned} N_q &:= k_0^{-1}(\|f\|_{L^q(\Omega)} + \|Q\|_{W^{1/q',q}(\Gamma)}),\\ \widetilde{N}_q &:= k_0^{-1}(\|f\|_{L^q(\Omega)} + \|Q_1\|_{W^{1/q',q}(\Gamma)} + \|Q_2\|_{V^q(\Gamma)}),\end{aligned}$$

and for the problem (P_D) the quantities

$$\begin{aligned} N_q &:= k_0^{-1}\|f\|_{L^q(\Omega)} + \|\nabla u_e\|_{W^{1,q}(\Omega)},\\ \widetilde{N}_q &:= N_q + \|U_1\|_{W^{1/q',q}(\Gamma)} + \|U_2\|_{V^q(\Gamma)}.\end{aligned}$$

Here, the functions Q_i and U_i are taken from (2.20), (2.21) and (2.22). The main result of the section is the following:

Theorem 5.1. *Assume that $S\in\mathcal{C}^2$ and $f\in L^2(\Omega)$. Let u be the weak solution to (P). Assume that the condition (2.19) is valid, and that one of the following assumptions is satisfied:*
(1) *u satisfies (1.3) on Γ, and the conditions (2.20), (2.21) hold with $q=2$.*
(2) *u satisfies (1.4) on Γ, and the condition (2.22) holds with $q=2$.*

Then u belongs to $W^{2,2}(\Omega_i)$ for $i=1,2$ and, moreover, satisfies the continuous estimate

$$\|D^2u\|_{L^2(\Omega_i)} \leqslant c(1+g_0)\widetilde{N}_2 \tag{5.1}$$

with a constant c that depends on Ω, k_1/k_0, k_1'/k_0, and additionally on g_1 for the problem (P_N).

The proof of the theorem is carried out in the following four propositions.

Proposition 5.2. *Assume that* $S \in \mathcal{C}^2$ *and* $f \in L^2(\Omega)$. *Let* u_ϱ *be the weak solution to* (P_ϱ). *Assume that* u_ϱ *satisfies either* (1.3) *with* $Q \in W^{1/2,2}(\Gamma)$ *or* (1.4) *with* $u_e \in W^{3/2,2}(\Gamma)$. *Then there is a constant* c, *depending only on* Ω *and on* k_1/k_0, *such that*

$$\|\nabla \xi_\varrho^{(1)}\|_{L^2(\Omega)} \leqslant c(1+g_0)N_2. \tag{5.2}$$

P r o o f. Let u_ϱ denote the solution to the problem (P_ϱ). We first consider the boundary condition (1.3). For $v \in W^{2,2}(\Omega)$, introduce the linear functional

$$F_Q^{(1)}(v) := \int_\Gamma Q(\tau^{(1)} \cdot \nabla v). \tag{5.3}$$

The continuity estimate

$$|F_Q^{(1)}(v)| \leqslant c g_0 \|Q\|_{W^{1/2,2}(\Gamma)} \|\nabla v\|_{L^2(\Omega)} \tag{5.4}$$

follows from Lemma C.1, and implies that the functional $F_Q^{(1)}$ extends by density to $W^{1,2}(\Omega)$. Due to (4.32),

$$\int_\Omega \kappa_\varrho \nabla \xi_\varrho^{(1)} \cdot \nabla v = \int_\Omega \overline{G}_\varrho^{(1)} \cdot \nabla v + F_Q^{(1)}(v) \quad \forall v \in W^{1,2}(\Omega). \tag{5.5}$$

In (5.5), we are allowed to choose $v := \xi_\varrho^{(1)}$. To derive (5.2) from the estimates (5.4) and (4.31) and Lemma 3.1 is a straightforward exercise on Young's inequality.

For the boundary condition (1.4), we introduce the extension $u_e \in W^{2,2}(\Omega)$ of the boundary data, and $\xi_e^{(1)} := \tau^{(1)} \cdot \nabla u_e \in W^{1,2}(\Omega)$. Due to (4.32),

$$\int_\Omega \kappa_\varrho \nabla(\xi_\varrho^{(1)} - \xi_e^{(1)}) \cdot \nabla v = \int_\Omega (\overline{G}_\varrho^{(1)} - \kappa_\varrho \nabla \xi_e^{(1)}) \cdot \nabla v \quad \forall v \in W_0^{1,2}(\Omega), \tag{5.6}$$

and (5.2) follows. □

Berfore stating the following lemma, we recall the definition (2.12) of the function f_d.

Lemma 5.3. *Let the hypotheses of Proposition* 5.2 *be valid. Assume in addition that the condition* (2.19) *is valid. For* $u \in W^{1,2}(\Omega)$, $v \in W^{2,2}(\Omega)$, *define*

$$(B_\varrho(u), v) := -\int_\Gamma f_d(\alpha, A_\varrho) u(\tau^{(2)} \cdot \nabla v). \tag{5.7}$$

Then, the mapping B_ϱ *extends to an element of* $\mathcal{L}(W^{1,2}(\Omega), [W^{1,2}(\Omega)]^*)$. *Moreover, there is* $\varrho_0 = \varrho_0(S, \kappa_2, \kappa_1, \alpha)$ *such that for all* $\varrho \leqslant \varrho_0$ *the inequalities*

$$(B_\varrho(u), (u-m)^+) \leqslant \tilde{c}(1+g_0) \int_\Gamma u(u-m)^+, \tag{5.8}$$

$$(B_\varrho(u), (u+m)^-) \leqslant \tilde{c}(1+g_0) \int_\Gamma u(u+m)^- \tag{5.9}$$

are valid for all $u \in W^{1,2}(\Omega)$ *and all* $m \in \mathbb{N}$, *with* $\tilde{c} := ck_0^{-1}$ *for* (P_N), *and* $\tilde{c} := ck_1$ *for* (P_D).

P r o o f. Due to Lemma C.1 and Lemma B.1,

$$\begin{aligned} |(B_\varrho(u), v)| &\leqslant c g_0 \|f_d(\alpha, A_\varrho) u\|_{W^{1/2,2}(\Gamma)} \|\nabla v\|_{L^2(\Omega)} \\ &\leqslant c_\varrho \|u\|_{W^{1/2,2}(\Gamma)} \|\nabla v\|_{L^2(\Omega)} \end{aligned} \tag{5.10}$$

for all $u \in W^{1,2}(\Omega)$, $v \in W^{2,2}(\Omega)$. Therefore, the mapping B_ϱ extends by density to an element of $\mathcal{L}(W^{1,2}(\Omega), [W^{1,2}(\Omega)]^*)$.

For $u \in W^{2,2}(\Omega)$, $m \in \mathbb{N}$,

$$(B_\varrho(u), (u-m)^+) = \frac{-1}{2} \int_\Gamma f_d(\alpha, A_\varrho) \tau^{(2)} \cdot \nabla((u+m)(u-m)^+). \tag{5.11}$$

For the (P_N)-case of (2.12), integration by parts yields

$$\begin{aligned} (B_\varrho(u), (u-m)^+) &= \int_\Gamma \left(\cot\alpha \tau^{(2)} \cdot \nabla \frac{a_\varrho^{3,3}}{m_\varrho^{1,1}} + \tau^{(2)} \cdot \nabla \frac{a_\varrho^{2,3}}{m_\varrho^{1,1}} \right) \frac{(u+m)}{2} (u-m)^+ \\ &+ \int_\Gamma \left(\operatorname{div}_\Gamma(\cot\alpha \tau^{(2)}) \frac{a_\varrho^{3,3}}{m_\varrho^{1,1}} + \operatorname{div}_\Gamma(\tau^{(2)}) \frac{a_\varrho^{2,3}}{m_\varrho^{1,1}} \right) \frac{(u+m)}{2} (u-m)^+. \end{aligned} \tag{5.12}$$

Using (3.4), the fact that $\tau^{(2)} \cdot n_S = -\sin\alpha$ on Γ, and (3.8), we compute

$$\begin{aligned} \cot\alpha \tau^{(2)} \cdot \nabla \frac{a_\varrho^{3,3}}{m_\varrho^{1,1}} + \tau^{(2)} \cdot \nabla \frac{a_\varrho^{2,3}}{m_\varrho^{1,1}} &= -\sin\alpha b_\varrho \varrho^{-1} [f_d(\alpha, A)]_S \\ &+ \cot\alpha L_\varrho \left(\tau^{(2)} \cdot \nabla \frac{a^{3,3}}{m^{1,1}} \right) + L_\varrho \left(\tau^{(2)} \cdot \nabla \frac{a^{2,3}}{m^{1,1}} \right). \end{aligned} \tag{5.13}$$

Due to the uniform continuity of the data A_1, A_2, α there is a neighborhood D of the curve $\Gamma \cap S$ such that (2.19) is valid in the domain $\overline{D \cap \Omega}$. Therefore, if $\varrho \leqslant \varrho_0(A_1, A_2, \alpha)$, then

$$-\sin\alpha b_\varrho \varrho^{-1}[f_d(\alpha, A)]_S(u+m)(u-m)^+ \leqslant 0. \tag{5.14}$$

The estimate

$$\cot\alpha L_\varrho\left(\tau^{(2)}\cdot\nabla\frac{a^{3,3}}{m^{1,1}}\right) + L_\varrho\left(\tau^{(2)}\cdot\nabla\frac{a^{2,3}}{m^{1,1}}\right) \leqslant \frac{k_1^2 k_1'}{k_0^4}$$

together with (5.13) and (5.14) yields

$$\left(\cot\alpha\tau^{(2)}\cdot\nabla\frac{a_\varrho^{3,3}}{m_\varrho^{1,1}} + \tau^{(2)}\cdot\nabla\frac{a_\varrho^{2,3}}{m_\varrho^{1,1}}\right)(u+m)(u-m)^+ \leqslant \frac{2k_1^2k_1'}{k_0^4}u(u-m)^+. \tag{5.15}$$

The estimate (5.8) follows from (5.12) and (5.15). For the problem (P_D), we can reformulate

$$\begin{aligned}(B_\varrho(u),(u-m)^+) &= \frac{1}{2}\int_\Gamma(\cot\alpha\tau^{(2)}\cdot\nabla a_\varrho^{3,3} + \tau^{(2)}\cdot\nabla a_\varrho^{2,3})(u+m)(u-m)^+ \\ &+ \frac{1}{2}\int_\Gamma(\mathrm{div}_\Gamma(\cot\alpha\tau^{(2)})a_\varrho^{3,3} + \mathrm{div}_\Gamma(\tau^{(2)})a_\varrho^{2,3})(u+m)(u-m)^+.\end{aligned} \tag{5.16}$$

Under the assumption (2.19), we verify for $\varrho \leqslant \varrho_0$ (cf. (5.14)) that

$$(\cot\alpha\tau^{(2)}\cdot\nabla a_\varrho^{3,3} + \tau^{(2)}\cdot\nabla a_\varrho^{3,2}) \leqslant ck_1'. \tag{5.17}$$

Here again, the estimate (5.8) follows from (5.16) thanks to standard inequalities. Due to the formula

$$(B_\varrho(u),(u+m)^-) = -\frac{1}{2}\int_\Gamma f_d(\alpha, A_\varrho)\tau^{(2)}\cdot\nabla((u-m)(u+m)^-),$$

we similarly verify (5.9). Finally, in view of the continuity property (5.10), the inequalities (5.8) and (5.9) hold true for all $u \in W^{1,2}(\Omega)$. □

Proposition 5.4. *Assume that $S \in \mathcal{C}^2$ and $f \in L^2(\Omega)$. Let u_ϱ be the weak solution to (P_ϱ). Assume that u_ϱ satisfies (1.3), and that the conditions (2.19), (2.20), (2.21) are valid with $q = 2$. Then there is a constant $c = c(\Omega, k_1/k_0, k_1'/k_0)$ and a sequence of numbers $\{C_\varrho\}$ that tends to zero, such that*

$$\|\nabla \xi_\varrho^{(3)}\|_{L^2(\Omega)} \leqslant c(1+g_0)\widetilde{N}_2 + C_\varrho. \tag{5.18}$$

Proof. Thanks to the relation (4.45), the operator B_ϱ of Lemma 5.3, and to the functional

$$F^{(2)}_{\widetilde{Q}_{2,\varrho}} := -\int_\Gamma \widetilde{Q}_{2,\varrho}(\tau^{(2)} \cdot \nabla v) \tag{5.19}$$

(cf. (5.3), and (5.4) for a norm estimate on $F^{(2)}$), (4.35) is equivalent to
(5.20)

$$\begin{aligned}\int_\Omega (m_\varrho^{1,1})^{-1}\tilde{\kappa}_\varrho \nabla \xi_\varrho^{(3)} \cdot \nabla v = \int_\Omega \{\overline{G}_\varrho^{(3)} + M_\varrho^{(3)} \nabla \xi_\varrho^{(1)}\} \cdot \nabla v + (B_\varrho(\xi_\varrho^{(3)} + \widetilde{Q}_{1,\varrho}), v)\\ + F^{(2)}_{\widetilde{Q}_{2,\varrho}}(v), \quad \forall\, v \in W^{1,2}(\Omega),\end{aligned}$$

or, for the variable $w_\varrho := \xi_\varrho^{(3)} + \widetilde{Q}_{1,\varrho}$, to

$$\begin{aligned}\int_\Omega (m_\varrho^{1,1})^{-1}\tilde{\kappa}_\varrho \nabla w_\varrho \cdot \nabla v = \int_\Omega \{\overline{G}_\varrho^{(3)} + M_\varrho^{(3)} \nabla \xi_\varrho^{(1)} + [m_\varrho^{1,1}]^{-1}\tilde{\kappa}_\varrho \nabla \widetilde{Q}_{1,\varrho}\} \cdot \nabla v\\ + (B_\varrho(w_\varrho), v) + F^{(2)}_{\widetilde{Q}_{2,\varrho}}(v), \quad \forall\, v \in W^{1,2}(\Omega).\end{aligned} \tag{5.21}$$

In the relation (5.21), it is possible to choose $v := w_\varrho$. In view of (5.8) and (5.9) with $m = 0$, and of the interpolation inequality (C.2), we have

$$\begin{aligned}(B_\varrho(w_\varrho), w_\varrho) &\leqslant c\kappa_0^{-1}(1+g_0)\|w_\varrho\|^2_{L^2(\Gamma)}\\ &\leqslant cc_0^2 k_0^{-1}(1+g_0)\|w_\varrho\|_{L^2(\Omega)}\|\nabla w_\varrho\|_{L^2(\Omega)}.\end{aligned} \tag{5.22}$$

Employing from now on Young's inequality as in the proof of Proposition 5.2, Lemma 4.4 and Proposition 5.2 to bound the quantities $\widetilde{Q}_{1,\varrho}$, $\widetilde{Q}_{2,\varrho}$ and $\xi_\varrho^{(1)}$, the estimate (5.18) immediately follows. □

Proposition 5.5. *Assume that $S \in \mathcal{C}^2$ and $f \in L^2(\Omega)$. Let u_ϱ be the weak solution to (P_ϱ). Assume that* (2.19) *is valid, that u_ϱ satisfies* (1.4) *on Γ, and that the condition* (2.22) *holds with $q = 2$. Then there is a constant $c = c(\Omega, k_1/k_0, k_1'/k_0)$, and a sequence C_ϱ that converges to zero, such that*

$$\|\nabla \xi_\varrho^{(2)}\|_{L^2(\Omega)} \leqslant c(1 + g_0)\widetilde{N}_2 + C_\varrho. \tag{5.23}$$

Proof. The proof is very similar to the proof of Proposition 5.4.

Using the relation (4.46), the operator B_ϱ of Lemma 5.3 and the functional $F^{(2)}_{\widetilde{U}_{2,\varrho}}$ (cf. (5.19)), the relation (4.35) is equivalent to

$$\begin{aligned}\int_\Omega \tilde{\kappa}_\varrho \nabla \xi_\varrho^{(2)} \cdot \nabla v = \int_\Omega \{\overline{G}_\varrho^{(2)} + (a_\varrho^{1,1} n_S - a_\varrho^{3,1} T^{(1)}) \times \nabla \xi_\varrho^{(1)}\} \cdot \nabla v \\ + (B_\varrho(\xi_\varrho^{(2)} + U_1), v) + F^{(2)}_{\widetilde{U}_{2,\varrho}}(v), \quad \forall\, v \in W^{1,2}(\Omega).\end{aligned} \tag{5.24}$$

For the variable $w_\varrho := \xi_\varrho^{(2)} + U_1$, it follows that

$$\begin{aligned}\int_\Omega \tilde{\kappa}_\varrho \nabla w_\varrho \cdot \nabla v = \int_\Omega \{\overline{G}_\varrho^{(2)} + (a_\varrho^{1,1} n_S - a_\varrho^{3,1} T^{(1)}) \times \nabla \xi_\varrho^{(1)} + \tilde{\kappa}_\varrho \nabla U_1\} \cdot \nabla v \\ + (B_\varrho(w_\varrho), v) + F^{(2)}_{\widetilde{U}_{2,\varrho}}(v), \quad \forall\, v \in W^{1,2}(\Omega),\end{aligned} \tag{5.25}$$

where it is possible to choose $v := w_\varrho$. The estimate (5.23) follows with arguments similar to the proof of Proposition 5.4. □

Proposition 5.6. *Let the assumptions of Theorem* 5.1 *be satisfied. If u denotes the weak solution to (P), then $u \in W^{2,2}(\Omega_i)$ for $i = 1, 2$. Moreover,*

$$\xi_\varrho^{(i)} \rightharpoonup T^{(i)} \cdot \nabla u \quad \text{in } W^{1,2}(\Omega) \text{ (for } i = 1, 2), \quad \xi_\varrho^{(3)} \rightharpoonup \kappa n_S \cdot \nabla u \quad \text{in } W^{1,2}(\Omega).$$

Proof. Proposition 5.2, and either Proposition 5.4 in the case of $(P_{N,\varrho})$, or Proposition 5.5 in the case of $(P_{D,\varrho})$, provide uniform bounds for the sequences $\{\xi_\varrho^{(1)}\}$, and either $\{\xi_\varrho^{(2)}\}$ or $\{\xi_\varrho^{(3)}\}$, in the space $W^{1,2}(\Omega)$. Due to the gradient representations of Lemma 4.1, it then follows for both problems that there is $C > 0$ independent of ϱ such that $\|\nabla \xi_\varrho^{(i)}\|_{L^2(\Omega)} \leqslant C$ for $i = 1, 2, 3$. Thanks to the reflexivity of $W^{1,2}(\Omega)$, we find $\xi^{(i)} \in W^{1,2}(\Omega)$ such that

$$\xi_\varrho^{(i)} \rightharpoonup \xi^{(i)} \quad \text{in } W^{1,2}(\Omega) \text{ for } i = 1, 2, 3.$$

On the other hand, $\xi_\varrho^{(i)} \to T^{(i)} \cdot \nabla u$ for $i = 1, 2$ and $\xi_\varrho^{(3)} \to \kappa n_S \cdot \nabla u$ almost everywhere in Ω (cp. Lemma 3.1). Thus

$$T^{(i)} \cdot \nabla u = \xi^{(i)} \in W^{1,2}(\Omega) \quad (i = 1, 2), \quad \kappa n_S \cdot \nabla u \in W^{1,2}(\Omega).$$

□

Theorem 6.1. *Same assumptions as in Theorem* 5.1. *Assume that there is* $q_0 > 3$ *such that* $f \in L^{q_0}(\Omega)$. *For the problem* (1.3), *let* Q *satisfy* (2.20) *with* $q = q_0$; *for the problem* (1.4), *let* u_e *satisfy* (2.22) *with* $q = q_0$. *Then* $\nabla u \in L^\infty(\Omega)$, *and the estimate* $\|\nabla u\|_{L^\infty(\Omega)} \leqslant c\widetilde{N}_{q_0}$ *is valid, with a constant* c *that depends continuously on* Ω, *on* g_0, *on* k_1/k_0, *on* k_1'/k_0, *and also on* g_1 *for the problem* (P_N).

For the proof, we will show that the functions $\xi^{(i)}$, $i = 1, 2, 3$, belong to $L^\infty(\Omega)$. However, we cannot prove that the approximation method (P_ϱ) converges in the space $W^{1,\infty}(\Omega)$. Fortunately, once the result of Theorem 5.1 is ensured, we can derive in the limit new regularity properties that turn out to be sufficient for the result.

Proposition 6.2. *Assume that* $S \in \mathcal{C}^2$. *Let* u *denote the weak solution to* (P). *Assume that there is* $q_0 > 3$ *such that* $f \in L^{q_0}(\Omega)$, *and such that* u *either satisfies* (1.3) *with* $Q \in W^{1/q_0',q_0}(\Gamma)$ *or* (1.4) *with* $u_e \in W^{2,q_0}(\Omega)$. *Then* $\xi^{(1)}$ *belongs to* $W^{1,q_0}(\Omega)$ *and satisfies the estimate*

$$\|\xi^{(1)}\|_{W^{1,q_0}(\Omega)} \leqslant c\widetilde{N}_{q_0}. \tag{6.1}$$

Here, the constant c *depends continuously on* Ω, *on* g_0, *on* k_1/k_0, *and on* k_1'/k_0.

P r o o f. We let $\varrho \to 0$ in (4.32) to see in the case of the boundary condition (1.3) that $\xi^{(1)} \in W^{1,2}(\Omega)$ satisfies

$$\int_\Omega \kappa \nabla \xi^{(1)} \cdot \nabla v = \int_\Omega \{\overline{G}^{(1)} + \operatorname{curl}(Q(\tau^{(1)} \times n_\Gamma))\} \cdot \nabla v \quad \forall\, v \in W^{2,2}(\Omega), \tag{6.2}$$

where Lemma C.1 is used to rewrite the functional $F_Q^{(1)}$. The estimate (4.31) ensures that

$$\|\overline{G}^{(1)}\|_{L^{q_0}(\Omega)} \leqslant \|f\|_{L^{q_0}(\Omega)} + cg_0k_1\|\nabla u\|_{L^{q_0}(\Omega)}.$$

Since we can obtain a bound on $\|\nabla u\|_{L^{q_0}(\Omega)}$ with the arguments of Lemma A.1, the right-hand side of (6.2) belongs to $[W^{1,q_0'}(\Omega)]^*$, with a corresponding norm estimate. The result now follows in principle from Theorem 1.2 in [4]. We give the idea of the proof in the appendix, Lemma A.1. In the case of the boundary condition (1.4), introduce $\xi_e^{(1)} := \tau^{(1)} \cdot \nabla u_e$ to see that the function $\xi^{(1)} - \xi_e^{(1)}$ satisfies

$$\int_\Omega \kappa \nabla(\xi^{(1)} - \xi_e^{(1)}) \cdot \nabla v = \int_\Omega \{\overline{G}^{(1)} - \kappa \nabla \xi_e^{(1)}\} \cdot \nabla v, \quad \forall\, v \in W_0^{1,2}(\Omega). \tag{6.3}$$

Here again, the right-hand side of (6.3) extends by continuity to an element of the space $[W_0^{1,q_0'}(\Omega)]^*$, and the regularity follows from the same fundamental result in [4].
□

For the regularity of $\xi^{(2)}$ and $\xi^{(3)}$, we need to state some further properties valid on the surface Γ. We introduce a weighted space (cf. (2.14))

$$\begin{aligned} V_\alpha^q(\Gamma) &:= \{u \in W^{1/q',q}(\Gamma) \colon\ f_d(\alpha, A)u \in W^{1/q',q}(\Gamma)\}, \\ \|u\|_{V_\alpha^q(\Gamma)} &:= \|u\|_{W^{1/q',q}(\Gamma)} + \|f_d(\alpha, A)u\|_{W^{1/q',q}(\Gamma)}. \end{aligned} \tag{6.4}$$

Lemma 6.3. *Let $u \in W^{1,2}(\Omega)$ be the weak solution to (P). Assume that the hypotheses of Theorem 6.1 are valid. If u is associated with (P_N), then there are $\widetilde{Q}_1, \widetilde{Q}_2 \in W^{1/q_0',q_0}(\Gamma)$ such that*

$$\xi^{(2)} = -f_d(\alpha, A)(\xi^{(3)} + \widetilde{Q}_1) - \widetilde{Q}_2 \quad \text{a.e. on } \Gamma, \tag{6.5}$$

$$f_d(\alpha, A)(\xi^{(3)} + \widetilde{Q}_1) \in W^{1/2,2}(\Gamma). \tag{6.6}$$

If u is associated with (P_D), then there is $\widetilde{U}_2 \in W^{1/q_0',q_0}(\Gamma)$ such that

$$\xi^{(3)} = f_d(\alpha, A)(\xi^{(2)} + U_1) + \widetilde{U}_2 \quad \text{a.e. on } \Gamma, \tag{6.7}$$

$$f_d(\alpha, A)(\xi^{(2)} + U_1) \in W^{1/2,2}(\Gamma). \tag{6.8}$$

Proof. The relations (6.5) and (6.7) are easy consequences of Lemma 4.5 and of the convergence in Proposition 5.6. We recall the notation (2.13). Defining $\widetilde{Q}_1, \widetilde{Q}_2, \widetilde{U}_2$ as accumulation points of the sequences $\widetilde{Q}_{1,\varrho}, \widetilde{Q}_{2,\varrho}, \widetilde{U}_{2,\varrho}$, the representations derived in Lemma 4.4 yield

$$\begin{aligned} \widetilde{Q}_1 &= Q_1 + g_1\xi^{(1)}, \\ \widetilde{Q}_2 &= \gamma^+(Q_2) + \frac{a_1^{3,3}}{m_1^{1,1}} Q/\sin\alpha - f_d(\alpha, A_1)\widetilde{Q}_1 + \frac{m_1^{2,1}}{m_1^{1,1}}\xi^{(1)}, \\ \widetilde{U}_2 &= \gamma^+(U_2) + a_1^{3,1}\xi_e^{(1)} - a_1^{3,3}(\tau^{(2)} \cdot \nabla u_e)/\sin\alpha - f_d(\alpha, A_1)U_1. \end{aligned}$$

Thus, the assumptions on Q_1, Q_2, U_1, U_2, the result of Proposition 6.2, and the property (B.1) are sufficient to verify the $W^{1/q_0',q_0}$ regularity of $\widetilde{Q}_1, \widetilde{Q}_2, \widetilde{U}_2$.

Since $\xi^{(2)} + \widetilde{Q}_2 \in W^{1/2,2}(\Gamma)$, (6.5) directly proves (6.6). The proof of (6.8) is completely similar. □

Lemma 6.4. *For $u \in V_\alpha^2(\Gamma)$ and $v \in W^{2,2}(\Omega)$, define the bilinear form*

$$(B(u), v) = -\int_\Gamma f_d(\alpha, A)u(\tau^{(2)} \cdot \nabla v). \tag{6.9}$$

Then B extends by density to an element of $\mathcal{L}(V^2_\alpha(\Gamma), [W^{1,2}(\Omega)]^*)$, and for $2 \leqslant q_0 \leqslant 6$, the inequalities

$$(B(u), (u-m)^+) \leqslant \tilde{c}(1+g_0)\|u\|_{L^{2q_0/3}(\Gamma)}\|\nabla(u-m)^+\|_{L^{q_0'}(\Omega)}, \tag{6.10}$$

$$(B(u), (u+m)^-) \leqslant \tilde{c}(1+g_0)\|u\|_{L^{2q_0/3}(\Gamma)}\|\nabla(u+m)^-\|_{L^{q_0'}(\Omega)} \tag{6.11}$$

are valid for all $u \in W^{1,2}(\Omega)$ such that $u \in V^2_\alpha(\Gamma)$, and for all $m \in \mathbb{N}$. Here $\tilde{c} := ck_0^{-1}$ for (P_N), and $\tilde{c} := ck_1$ for (P_D).

P r o o f. For $u \in V^2_\alpha(\Gamma)$ and $v \in W^{2,2}(\Omega)$, Lemma C.1 implies the inequality

$$|(B(u), v)| \leqslant cg_0\|f_d(\alpha, A)u\|_{W^{1/2,2}(\Gamma)}\|\nabla v\|_{L^2(\Omega)}, \tag{6.12}$$

so that B extends by density to an element of $\mathcal{L}(V^2_\alpha(\Gamma), [W^{1,2}(\Omega)]^*)$.

For $u \in W^{2,2}(\Gamma)$ such that $u \in V^2_\alpha(\Gamma)$ and for $m \in \mathbb{N}$ (cf. (5.7)) we have

$$(B(u), (u-m)^+) = \lim_{\varrho\to 0}(B_\varrho(u), (u-m)^+). \tag{6.13}$$

The inequalities (6.10) and (6.11) therefore immediately follow from (5.8) and (5.9) and Hölder's inequality. Due to the density Lemma B.4 these inequalities remain valid for all $u \in W^{1,2}(\Omega)$ such that $u \in V^2_\alpha(\Gamma)$. □

Proposition 6.5. *Same assumptions as in Theorem 6.1 for the problem (P_N). Then $\xi^{(3)}$ belongs to $L^\infty(\Omega)$ with an estimate*

$$\sup_\Omega |\xi^{(3)}| \leqslant c\widetilde{N}_{q_0}. \tag{6.14}$$

P r o o f. Denote $w := \xi^{(3)} + \widetilde{Q}_1$. Passing to the limit $\varrho \to 0$ in the relation (5.21) for test functions $v \in W^{2,2}(\Omega)$, it follows that

$$\begin{aligned}\int_\Omega [m^{1,1}]^{-1}\tilde{\kappa}\nabla w\cdot\nabla v &= \int_\Omega \{\overline{G}^{(3)} + M^{(3)}\nabla\xi^{(1)} + [m^{1,1}]^{-1}\tilde{\kappa}\nabla\widetilde{Q}_1\}\cdot\nabla v\\ &\quad - \int_\Gamma\Big(\cot\alpha\frac{a^{3,3}}{m^{1,1}} + \frac{a^{3,2}}{m^{1,1}}\Big)w(\tau^{(2)}\cdot\nabla v) - \int_\Gamma \widetilde{Q}_2(\tau^{(2)}\cdot\nabla v).\end{aligned} \tag{6.15}$$

In view of Lemma 6.4, (6.15) is equivalent to

$$\begin{aligned}\int_\Omega (m^{1,1})^{-1}\tilde{\kappa}\nabla w\cdot\nabla v = &\int_\Omega \{\overline{G}^{(3)} + M^{(3)}\nabla\xi^{(1)} + [m^{1,1}]^{-1}\tilde{\kappa}\nabla\widetilde{Q}_1\}\cdot\nabla v\\ &+ (B(w), v) + F^{(2)}_{\widetilde{Q}_2}(v),\end{aligned}$$

where the choices $v := (w-m)^+$ and $v := (w+m)^-$ are possible for all $m \in \mathbb{N}$. The claim follows using Lemma C.4, in connection with the estimates (6.10), (6.11), (5.4), as well as (4.33) and Proposition 6.2. □

Proposition 6.6. *Let the hypotheses of Theorem* 6.1 *for the problem* (P_D) *be valid. Then* $\xi^{(2)}$ *belongs to* $L^\infty(\Omega)$ *and satisfies the estimate*

$$\sup_\Omega |\xi^{(2)}| \leqslant c\widetilde{N}_{q_0}. \tag{6.16}$$

P r o o f. Define $w := \xi^{(2)} + U_1$. Passage to the limit in the relation (5.25) for test functions $v \in W^{2,2}_{\Gamma_2}(\Omega)$, and Lemma 6.4 yield

$$\begin{aligned}\int_\Omega \tilde\kappa \nabla w \cdot \nabla v = \int_\Omega \{\overline{G}^{(2)} + (a^{1,3}T^{(1)} - a^{1,1}n_S) \times \nabla\xi^{(1)} + \tilde\kappa\nabla U_1\} \cdot \nabla v \\ + (B(w), v) + F^{(2)}_{\widetilde{U}_2}(v).\end{aligned} \tag{6.17}$$

We complete the proof as in Proposition 6.5. □

We are now able to complete the proof of Theorem 6.1.

P r o o f of Theorem 6.1. We first consider the case of the boundary condition (1.3). Due to Propositions 6.2 and 6.5, $\xi^{(1)}, \xi^{(3)}$ are globally bounded in the domain Ω. The relation (6.5) and the triangle inequality yield

$$\sup_\Gamma |\xi^{(2)}| \leqslant \frac{k_1}{k_0^2} \sup_\Omega (|\xi^{(3)}| + |\widetilde{Q}_1|) + \|\widetilde{Q}_2\|_{L^\infty(\Gamma)}. \tag{6.18}$$

On the other hand, we can pass to the limit in the relation (4.34) to see that $\xi^{(2)} \in W^{1,2}(\Omega)$ satisfies, for all $v \in W^{1,2}_0(\Omega)$,

$$\int_\Omega \tilde\kappa \nabla \xi^{(2)} \cdot \nabla v = \int_\Omega \{\overline{G}^{(2)} + (a^{3,1}T^{(1)} - a^{1,1}n_S) \times \nabla\xi^{(1)}\} \cdot \nabla v.$$

Lemma C.4 implies that

$$\|\xi^{(2)}\|_{L^\infty(\Omega)} \leqslant \sup_\Gamma |\xi^{(2)}| + c(\|\overline{G}^{(2)}\|_{L^{q_0}(\Omega)} + \|\nabla\xi^{(1)}\|_{L^{q_0}(\Omega)}),$$

and the claim follows from the estimate (6.18) and Proposition 6.2.

In the case of the boundary condition (1.4), Propositions 6.2 and 6.6 yield the global boundedness of the components $\xi^{(1)}$, $\xi^{(2)}$. Using the relation (6.7) and the triangle inequality, we obtain

$$\sup_\Gamma |\xi^{(3)}| \leqslant k_1 \sup_\Omega (|\xi^{(2)}| + |U_1|) + \|\widetilde{U}_2\|_{L^\infty(\Gamma)}), \tag{6.19}$$

and the claim follows from (4.35) and Proposition 6.2. □

This section is essentially devoted to the proof of Theorem 2.3. In the case that the compatibility condition (2.19) is violated, it is still possible to prove the existence of second weak derivatives for the weak solution to (P). This is based on the following observation.

Lemma 7.1. *Let $g \in C^1(\mathbb{R})$ be nonnegative and nondecreasing, and assume moreover that $M_g := \int_{-\infty}^{+\infty} |t|g'(t)\,\mathrm{d}t < \infty$. Then the mapping B_ϱ from Lemma 5.3 satisfies for all $u \in W^{1,2}(\Omega)$ the inequality*

$$(B_\varrho(u), g(u)) \leqslant cM_g. \tag{7.1}$$

P r o o f. For $t \in \mathbb{R}$, define $G(t) := \int_0^t sg'(s)\,\mathrm{d}s$. The function G is by assumption bounded by the number M_g, and for $u \in W^{2,2}(\Omega)$ arbitrary, the identity

$$(B_\varrho(u), g(u)) = \int_\Gamma f_d(\alpha, A_\varrho)\tau^{(2)} \cdot \nabla G(u) \tag{7.2}$$

is valid. For the (P_N)-case of (2.12), integration by parts yields (cf. (5.12))

$$\begin{aligned}(B_\varrho(u), g(u)) = &\int_\Gamma \Big(\cot\alpha\tau^{(2)} \cdot \nabla\frac{a_\varrho^{3,3}}{m_\varrho^{1,1}} + \tau^{(2)} \cdot \nabla\frac{a_\varrho^{2,3}}{m_\varrho^{1,1}}\Big)G(u)\\ &+ \int_\Gamma \Big(\operatorname{div}_\Gamma(\cot\alpha\tau^{(2)})\frac{a_\varrho^{3,3}}{m_\varrho^{1,1}} + \operatorname{div}_\Gamma(\tau^{(2)})\frac{a_\varrho^{2,3}}{m_\varrho^{1,1}}\Big)G(u).\end{aligned} \tag{7.3}$$

Observe that under the assumptions of the present lemma

$$\begin{aligned}\varrho^{-1}\int_{\{x\in\Gamma:\ \operatorname{dist}(x,\Gamma\cap S)\leqslant\varrho\}} G(u) &\leqslant M_g\varrho^{-1}\operatorname{meas}(\{x \in \Gamma:\ \operatorname{dist}(x, \Gamma\cap S) \leqslant \varrho\})\\ &\to M_g\operatorname{meas}(\Gamma\cap S).\end{aligned} \tag{7.4}$$

Arguing as in (5.13), (5.15), the inequality (7.1) follows. The arguments for (P_D) are completely similar. In Lemma 5.3, we have already proved that the mapping B_ϱ extends by density to an element of $\mathcal{L}(W^{1,2}(\Omega), [W^{1,2}(\Omega)]^*)$. In view of the continuity property (5.10), the inequality (7.1) is valid for all $u \in W^{1,2}(\Omega)$. □

P r o o f of Theorem 2.3. For $\delta \in\,]0, 1[$, consider the function

$$g_\delta(t) := \operatorname{sign}(t)\Big(1 - \frac{1}{(1+|t|)^{1+\delta}}\Big). \tag{7.5}$$

Then $g_\delta'(t) = (1+\delta)(1+|t|)^{-2-\delta}$, and it follows that $M_{g_\delta} < \infty$. We consider the relation (5.21) in the case of (P_N). In the case of (P_D), we start from (5.25) and the arguments are completely similar. In (5.21), choose $v := g_\delta(w_\varrho)$ as the test function. Using in particular Lemma 7.1, we can prove that there is C independent of ϱ such that

$$\int_\Omega (m_\varrho^{1,1})^{-1} g_\delta'(w_\varrho)\tilde{\kappa}_\varrho \nabla w_\varrho \cdot \nabla w_\varrho \leqslant C. \tag{7.6}$$

It is to note here that the uniform bounds on $\overline{G}_\varrho^{(3)}$ (Lemma 4.3), on $\widetilde{Q}_{2,\varrho}$ (Lemma 4.4) and on $\nabla \xi_\varrho^{(1)}$ (Prop. 5.2) are still valid since they were obtained independently of the condition (2.19). Denote $h_\delta(t) := \int_0^t \sqrt{g_\delta'(s)}\,\mathrm{d}s$. The function h_δ is globally bounded, and the inequality (7.6) shows that there is $\widetilde{C}$ independent of ϱ such that $\|\nabla h_\delta(w_\varrho)\|_{L^2(\Omega)} \leqslant \widetilde{C}$. Therefore, $h_\delta(w_\varrho) \to \chi \in W^{1,2}(\Omega)$ weakly. Moreover, using Lemma 3.1 and Lemma 4.4, we can show that $\chi = h_\delta(w)$, where $w = \xi^{(3)} + \widetilde{Q}_1$. Using the lower semicontinuity of the norm, the latest result and (7.6) yield

$$\int_\Omega g_\delta'(w)|\nabla w|^2 \leqslant \widetilde{C}. \tag{7.7}$$

Let $p < 2$. Then Hölder's inequality and (7.7) imply that

$$\begin{aligned} \int_\Omega |\nabla w|^p &\leqslant \Big(\int_\Omega g_\delta'(w)|\nabla w|^2\Big)^{p/2}\Big(\int_\Omega |g_\delta'(w)|^{-p/(2-p)}\Big)^{(2-p)/2} \\ &\leqslant \widetilde{C}^{2/p}\Big(\int_\Omega |1+|w||^{p(2+\delta)/(2-p)}\Big)^{(2-p)/2}. \end{aligned} \tag{7.8}$$

The main theorem of [4] implies, via arguments similar to Lemma A.1, that there is $q_0 > 3$ such that the weak solution to (P) satisfies $u \in W^{1,q_0}(\Omega)$. This yields $\xi^{(3)} \in L^{q_0}(\Omega)$. Thanks to Lemma 4.4, $\widetilde{Q}_1 \in L^6(\Omega)$. Therefore, $w \in L^s(\Omega)$, $s = \min\{q_0, 6\}$. If $p < 2s/(s+2)$, then there is $\delta > 0$ such that the right-hand side of (7.8) is finite, which implies that $\nabla w \in L^p(\Omega)$. We obtain that $\xi^{(3)} \in L^s(\Omega) \cap W^{1,p}(\Omega)$. Due to Lemma 4.1, also $\xi^{(2)} \in L^s(\Omega) \cap W^{1,p}(\Omega)$. Therefore, $\nabla u \in W^{1,p}(\Omega_i)$ for $i = 1, 2$. □

8. Conclusion

In this paper we derived sufficient conditions under which the weak solution to the transmission problem (P) satisfies higher ($W^{2,2}(\Omega_i)$, or the optimal $W^{1,\infty}(\Omega)$) regularity near the intersection of a $\mathcal{C}^2$ surface with the smooth outer boundary of the domain. These sufficient conditions are essentially compatibility conditions that involve the angle of contact of the two surfaces, the anisotropic coefficient matrix κ, the type of the boundary condition and the boundary data. The *compatibility function of the problem* f_d defined in (2.12) plays an essential role (cf. Section 2.3 for the representation of f_d in the case of a scalar coefficient κ).

The proof of the regularity results relies on one simple fact: according to its sign, the quantity $[f_d]$ (= jump of f_d at the intersection of the two surfaces S and Γ) regularizes a certain component of the gradient. The function f_d associated with the Neumann problem regularizes the oblique component $\kappa n_S \cdot \nabla u = \xi^{(3)}$ (cf. Proposition 5.4), the function f_d associated with the Dirichlet problem regularizes the conormal component $T^{(2)} \cdot \nabla u = \xi^{(2)}$ (cf. Proposition 5.5) of the gradient.

According to this theory, the changes of sign of the quantity $[f_d]$ are critical. This is also the topic of the representation conditions (2.20), (2.21), and (2.22) (see the section 2.3) that impose a decay of the boundary data as $[f_d]$ approaches zero on the curve $\Gamma \cap S$.

It is obvious and easy to motivate that representation conditions of the type (2.20), (2.21), and (2.22) at the contact line are *necessary* for every higher regularity of u that implies the existence of traces for ∇u on manifolds. In the context of our method, we have not yet been able to discuss the question whether also (2.19) is necessary. This will be the object of further investigations. Nevertheless, we could prove with our method in Section 7 that a certain $W^{2,p}$ regularity is preserved independently of the latest compatibility condition.

Appendix A. An auxiliary regularity result

Lemma A.1. *Let $\kappa \in [C(\overline{\Omega_i})]^9$ for $i = 1, 2$. Let $F \in [L^{q_0}(\Omega)]^3$ with $3 < q_0 \leqslant 3+\delta$ (δ = a positive constant defined in the paper [4]). Assume that $u \in W^{1,2}(\Omega)$ satisfies*

$$\int_\Omega \kappa \nabla u \cdot \nabla v = \int_\Omega F \cdot \nabla v, \quad \forall\, v \in W^{1,2}(\Omega). \tag{A.1}$$

Then u belongs to $W^{1,q_0}(\Omega)$, and it satisfies the estimate

$$\|u\|_{W^{1,q_0}(\Omega)} \leqslant c(\|F\|_{L^{q_0}(\Omega)} + c_S\{\|F\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}\}). \tag{A.2}$$

The constant c depends on Ω, k_0 and k_1. The constant c_S depends on the surface S only upon its $\mathcal{C}^1$-norm, and on the matrices κ_i $(i = 1, 2)$ upon their C-norm.

Proof. For simplicity, we only prove the regularity in a neighborhood D ($D \subset \mathbb{R}^3$ open) of the curve $\Gamma \cap S$, which is clearly the challenging point. For $x_0 \in \Gamma \cap S$ there are, due to the definition of $\mathcal{C}^2$ surfaces, a neighborhood U of x_0 and a C^2-diffeomorphism φ that maps U onto the unit cube Q_1, and such that $\varphi(x_0) = 0$, $\varphi(\Gamma \cap U) = \,]-1,1[\times\{0\}\times\,]-1,1[$ and $\varphi(S \cap U) = \,]-1,1[\times]-1,1[\times\{0\}$. Define $\psi := \varphi^{-1}$.

To attain the model configuration of the paper [4], consider for $0 < r < 1$ a prism $P_r := \triangle_r \times\,]-r, r[\subset Q_1$, where $\triangle_r$ is an equilateral triangle with sidelength $= r$, and with its base located in the line $]-1,1[\times\{0\}\times\{0\}$. Denote $\Gamma_r := \partial P_r \cap]-1,1[\times\{0\}\times\,]-1,1[$ and $\Sigma_r := \partial P_r \setminus \Gamma_r$. Due to the choice of P_r, $\psi(P_r) \subset U$ for all $r \leqslant 1$.

Transforming the formula (A.1), we obtain that

$$\int_{P_r} \mu \nabla \tilde{u} \cdot \nabla \tilde{v} = \int_{P_r} \widetilde{F} \cdot \nabla \tilde{v}, \quad \forall \tilde{v} \in W^{1,2}_{\Sigma_r}(P_r) \tag{A.3}$$

where $\tilde{u} = u \circ \psi$, and μ is the piecewise Lipschitz continuous, symmetric, and uniformely positive definite matrix $|\det \psi'|[\psi']^{-1}\kappa \circ \psi[\psi']^{-T}$, and $\widetilde{F}$ is the vector field $|\det \psi'|[\psi']^{-T} F \circ \psi$.

Introduce in P_r the piecewise constant matrix μ^0 such that $\mu^0_i := \mu_i(0)$ for $i = 1, 2$. If $w \in W^{1,2}_{\Sigma_r}(P_r)$ satisfies

$$\int_{P_r} \mu^0 \nabla w \cdot \nabla \tilde{v} = \int_{P_r} \widetilde{F} \cdot \nabla \tilde{v}, \quad \forall \tilde{v} \in W^{1,2}_{\Sigma_r}(P_r), \tag{A.4}$$

Theorem 1.2 in [4] implies that there is a constant $c_0 = c_0(\mu^0)$ such that

$$\|w\|_{W^{1,q_0}(P_r)} \leqslant c_0 \|\widetilde{F}\|_{L^{q_0}(P_r)}. \tag{A.5}$$

(The independence of c_0 of r is easy to check: use the transformation $\Psi_r(x) := rx$ from the unit prism P_1 onto P_r, and apply on P_1 Theorem 1.2 of [4]).

It has been shown for instance in [2] that the Banach perturbation argument implies the existence of a positive $r_0 = r_0(\mu)$ such that for all $r \leqslant r_0$ and for $\tilde{u}$ satisfying (A.3)

$$\|\nabla \tilde{u}\|_{W^{1,q_0}(P_r)} \leqslant \frac{c_0}{1 - c_0 f(r)} \Big(\|\overline{F}\|_{L^{q_0}(P_r)} + \frac{1}{r}\{\|\widetilde{F}\|_{L^2(P_r)} + \|\nabla \tilde{u}\|_{L^2(P_r)}\} \Big),$$

where $f(r) := \|\mu - \mu_0\|_{L^\infty(P_r)}$.

The minimal necessary size of r depends only on the surfaces S, Γ and on the uniform continuity of the matrices κ_i, so that a finite covering of a neighborhood of the curve $\Gamma \cap S$ is possible. $\square$

Appendix B. Auxiliary propositions concerning trace spaces

We first note a useful elementary property of the spaces $W^{1/q',q}(\Gamma)$.

Lemma B.1. *Let $1 \leqslant q \leqslant \infty$ be arbitrary. If $u \in W^{1/q',q}(\Gamma)$ and $g \in C^{0,1}(\Gamma)$, then gu belongs to $W^{1/q',q}(\Gamma)$, and there is a constant $c = c(q,\Gamma)$ such that*

$$\|gu\|_{W^{1/q',q}(\Gamma)} \leqslant c_q \|g\|_{C^{0,1}(\Gamma)} \|u\|_{W^{1/q',q}(\Gamma)}.$$

P r o o f. For $q = \infty$ the claim is obvious. Otherwise, the triangle inequality implies that

$$\begin{aligned}\|gu\|^q_{W^{1/q',q}(\Gamma)} &= \int_\Gamma \int_\Gamma \frac{|u(x)g(x) - u(y)g(y)|^q}{|x-y|^{2+q/q'}} \,\mathrm{d}y \,\mathrm{d}x \\ &\leqslant \int_\Gamma |u(x)|^q \Big(\int_\Gamma \frac{|g(x)-g(y)|^q}{|x-y|^{2+q/q'}} \,\mathrm{d}y \Big) \,\mathrm{d}x + \int_\Gamma |g(x)|^q \Big(\int_\Gamma \frac{|u(x)-u(y)|^q}{|x-y|^{2+q/q'}} \,\mathrm{d}y \Big) \,\mathrm{d}x.\end{aligned}$$

Define $\tilde{c}_q := \sup\limits_{x\in\Gamma} (\int_\Gamma |x-y|^{-1} \,\mathrm{d}y)^{1/q}$. Due to Lipschitz continuity of g, it follows that

$$\|gu\|_{W^{1/q',q}(\Gamma)} \leqslant \tilde{c}_q \|\nabla g\|_{L^\infty(\Gamma)} \|u\|_{L^q(\Gamma)} + \|g\|_{L^\infty(\Gamma)} \|u\|_{W^{1/q',q}(\Gamma)},$$

and (B.1) follows easily. □

The following lemma states basic properties of the spaces $V^q(\Gamma)$, and of the operators γ^+ and γ^- (cf. (2.13)).

Lemma B.2. *Let $\mu \in L^\infty(\Omega)$ be piecewise Lipschitz continuous, that is, $\mu := \mu_i \in C^{0,1}(\Omega_i)$ for $i = 1,2$. Then:*

(1) *The mapping $u \mapsto \mu u$ is continuous from $V^q(\Gamma)$ into $W^{1/q',q}(\Gamma)$ for all $1 \leqslant q \leqslant \infty$.*

(2) *Define $d(x) := \operatorname{dist}(x, \Gamma \cap S)$ for $x \in \Gamma$. For $1 \leqslant q < \infty$, a function $u \in W^{1/q',q}(\Gamma)$ belongs to $V^q(\Gamma)$ if, and only if, $\|u/d^{1/q'}\|_{L^q(\Gamma_1)} < \infty$.*

P r o o f. (1): On Γ, one has $\mu u = \mu_1 \gamma^-(u) + \mu_2 \gamma^+(u)$. Due to Lemma (B.1) and the triangle inequality, it follows that

$$\text{(B.1)} \qquad \begin{aligned}\|\mu u\|_{W^{1/q',q}(\Gamma)} &\leqslant \|\mu_1 \gamma^-(u)\|_{W^{1/q',q}(\Gamma)} + \|\mu_2 (u - \gamma^-(u))\|_{W^{1/q',q}(\Gamma)} \\ &\leqslant c(\|\mu_1\|_{W^{1,\infty}(\Gamma)} + \|\mu_2\|_{W^{1,\infty}(\Gamma)}) \|u\|_{V^q(\Gamma)}.\end{aligned}$$

(2): The definition of γ^- implies that

$$\|\gamma^-(u)\|^q_{W^{1/q',q}(\Gamma)} = \int_{\Gamma_1}\int_{\Gamma_1} \frac{|u(x)-u(y)|^q}{|x-y|^{2+q/q'}}\,\mathrm{d}x\,\mathrm{d}y + 2\int_{\Gamma_1} |u(x)|^q \bar{d}_{\Gamma_1}(x)^{q/q'}\,\mathrm{d}x,$$

$$\bar{d}_{\Gamma_1}(x) := \Bigl(\int_{\Gamma_2} |x-y|^{-(2+q/q')}\,\mathrm{d}y\Bigr)^{q'/q}, \ x\in\Gamma_1.$$

There are constants c_1, c_2 such that $c_1 d^{-1}(x) \leqslant \bar{d}_{\Gamma_1}(x) \leqslant c_2 d^{-1}(x)$ on Γ_1, proving the claim. □

Remark B.3. The elements of the space $V^q(\Gamma)$ satisfy a critical decay property $u/d^{1/q'} \in L^q(\Gamma_1)$ (cf. [10], Cor. 5.1). In the case $q=2$, it is possible to relate the space $V^2(\Gamma)$ to the space $W^{1/2,2}_{00}$.

In the following lemma, we note a density property of the space $V^q_\alpha(\Gamma)$ (cf. (6.4)).

Lemma B.4. *Assume that* $u\in V^2_\alpha(\Gamma)$. *Then there is a sequence* $\{v_k\}_{k\in\mathbb{N}} \subset C^\infty(\overline{\Omega})\cap V^2_\alpha(\Gamma)$ *such that* $v_k\to u$ *in* $V^2_\alpha(\Gamma)$.

Proof. We first show some preliminaries. With the abbreviation $\mu := f_d(\alpha, A)$, the definition of V^2_α implies that

$$\|u\|_{V^2_\alpha(\Gamma)} = \|u\|_{W^{1/2,2}(\Gamma)} + \|\mu u\|_{W^{1/2,2}(\Gamma)},$$

and since $\mu u = \mu_1\gamma^-(u) + \mu_2\gamma^+(u)$, it follows that

$$\text{(B.2)}\qquad \|u\|_{V^2_\alpha(\Gamma)} \leqslant \|u\|_{W^{1/2,2}(\Gamma)} + \|\mu_1\gamma^-(u)\|_{W^{1/2,2}(\Gamma)} + \|\mu_2\gamma^+(u)\|_{W^{1/2,2}(\Gamma)}.$$

Lemma B.2, (2) and Lemma B.1 yield

$$\begin{aligned}\|\mu_1\gamma^-(u)\|_{W^{1/2,2}(\Gamma)} &\leqslant \|\mu_1 u\|_{W^{1/2,2}(\Gamma_1)} + \|\mu_1 u/d^{1/2}\|_{L^2(\Gamma_1)}\\ &\leqslant c\|\mu_1\|_{C^{0,1}(\Gamma)}\|u\|_{W^{1/2,2}(\Gamma)} + \|\mu_1 u/d^{1/2}\|_{L^2(\Gamma_1)}.\end{aligned}$$

By similar arguments, it follows from (B.2) that

$$\text{(B.3)}\qquad \|u\|_{V^2_\alpha(\Gamma)} \leqslant c_1(\|u\|_{W^{1/2,2}(\Gamma)} + \|\mu_1 u/d^{1/2}\|_{L^2(\Gamma_1)} + \|\mu_2 u/d^{1/2}\|_{L^2(\Gamma_2)}).$$

To start the proof of the approximation property, consider first the truncation $T_k(u) := \operatorname{sign}(u)\min\{|u|,k\}$, at level $k\in\mathbb{N}$. Due to the dominated convergence, note that

$$\|\mu_1(T_k(u)-u)/d^{1/2}\|^2_{L^2(\Gamma_1)} = \int_{\{x\in\Gamma\colon |u(x)|>k\}} \frac{\mu_1^2 u^2}{d} \to 0.$$

Since it is well-known that $T_k(u) \to u$ in $W^{1,2}(\Omega)$, or in $W^{1/2,2}(\Gamma)$, the inequality (B.3) shows that $T_k(u) \to u$ in $V^2_\alpha(\Gamma)$ as $k \to \infty$. Therefore, there is no loss of generality in assuming $u \in V^2_\alpha(\Gamma) \cap L^\infty(\Gamma)$.

Since $\mu \in L^\infty(\Gamma)$, (B.3) implies immediately that

$$\|u\|_{V^2_\alpha(\Gamma)} \leqslant c_2 \|u\|_{V^2(\Gamma)}, \ V^2(\Gamma) \subseteq V^2_\alpha(\Gamma). \tag{B.4}$$

In the second step, we prove that $V^2(\Gamma)$ is dense in $V^2_\alpha(\Gamma)$.

For $k \in \mathbb{N}$, we choose a Lipschitz continuous function $\psi_k \in C^{0,1}(\overline{\Omega})$ such that

$$\psi_k(x) \begin{cases} = 1 & \text{if } \operatorname{dist}(x, \Gamma \cap S) > 1/k, \\ \in [0,1] & \text{if } 1/2k \leqslant \operatorname{dist}(x, \Gamma \cap S) \leqslant 1/k, \\ = 0 & \text{if } \operatorname{dist}(x, \Gamma \cap S) < 1/2k, \end{cases}$$
$$|\nabla \psi_k| \leqslant k, \ \operatorname{supp}(\nabla \psi_k) \subseteq \{x \in \Omega \colon \operatorname{dist}(x, \Gamma \cap S) \leqslant 1/k\}. \tag{B.5}$$

Then the sequence $\{\psi_k u\}$ is uniformly bounded in $W^{1,2}(\Omega)$ and in $W^{1/2,2}(\Gamma)$, since

$$\|u \nabla \psi_k\|_{L^2(\Omega)} \leqslant \|u\|_{L^\infty(\Omega)} k \operatorname{meas}(\operatorname{supp}(\nabla \psi_k))^{1/2} \leqslant C.$$

Since also $\psi_k |u| \leqslant |u|$ on Γ, the inequality (B.3) shows that the sequence $\{\psi_k u\}$ is uniformly bounded in $V^2_\alpha(\Gamma)$ as well. Due to the Hilbert space structure of $V^2_\alpha(\Gamma)$, $\psi_k u \rightharpoonup u$ in $V^2_\alpha(\Gamma)$ for a subsequence.

Weak and strong closures coincide for convex sets (an argument sometimes called Mazur's Lemma), and we can extract a sequence of convex combinations of $\psi_k u$ that strongly converges to u in V^2_α.

In the third step, we show that $C^\infty(\overline{\Omega}) \cap V^2(\Gamma)$ is dense in $V^2(\Gamma)$. If $\tilde{u} \in V^2(\Gamma)$, then the extension by zero to S (same notation) satisfies $\tilde{u} \in W^{1/2,2}(\partial\Omega_i)$ for $i = 1, 2$. Therefore, via extension into Ω with the trace theorem, there is a sequence $\{\zeta_k\} \subset C^\infty_c(\Omega \setminus S)$ such that $\zeta_k \to \tilde{u}$ in $W^{1,2}(\Omega_i)$. Thus, by the argument of Lemma B.2, (2) we have

$$\|\gamma^-(\zeta_k - \tilde{u})\|_{W^{1/2,2}(\Gamma)} = \|\zeta_k - \tilde{u}\|_{W^{1/2,2}(\partial\Omega_1)} \to 0,$$

establishing the density in V^2.

For $\varepsilon > 0$, thanks to the first and second steps of this proof there is a $\tilde{u}_\varepsilon \in V^2(\Gamma)$, such that $\|u - \tilde{u}_\varepsilon\|_{V^2_\alpha(\Gamma)} \leqslant \varepsilon$. Due to the third step, there is $\zeta_\varepsilon \in C^\infty(\overline{\Omega}) \cap V^2(\Gamma)$ such that $\|\zeta_\varepsilon - \tilde{u}_\varepsilon\|_{V^2(\Gamma)} \leqslant \varepsilon$. It follows from (B.4) that

$$\|u - \zeta_\varepsilon\|_{V^2_\alpha(\Gamma)} \leqslant \|u - \tilde{u}_\varepsilon\|_{V^2_\alpha(\Gamma)} + \|\zeta_\varepsilon - \tilde{u}_\varepsilon\|_{V^2_\alpha(\Gamma)} \leqslant (1 + c_2)\varepsilon,$$

proving the approximation property. □

Lemma B.5. *Let $\mu_\varrho := L_\varrho(\mu)$ denote the approximation* (3.2) *of the coefficient μ. Assume that $u \in V^2(\Gamma)$. Then $\mu_\varrho u \to \mu u$ in $W^{1/2,2}(\Gamma)$.*

Proof. For $x \in \Gamma$, define $g_\varrho(x) := \mu_\varrho(x) - \mu(x)$. Due to Lemma B.2, (2), it is clear that $g_\varrho u \in W^{1/2,2}(\Gamma)$. Denote $\Gamma_\varrho := \{x \in \Gamma\colon \operatorname{dist}(x,S) \leqslant \varrho\}$. Then $g_\varrho(x) = 0$ for all $x \in \Gamma \setminus \Gamma_\varrho$. Therefore, we have

$$\|g_\varrho u\|^2_{W^{1/2,2}(\Gamma)} = 2\int_{\Gamma\setminus\Gamma_\varrho}\int_{\Gamma_\varrho} \frac{u^2(x)g_\varrho^2(x)}{|x-y|^2}\,\mathrm{d}y\,\mathrm{d}x + \int_{\Gamma_\varrho}\int_{\Gamma_\varrho} \frac{|(ug_\varrho)(x)-(ug_\varrho)(y)|^2}{|x-y|^3}\,\mathrm{d}y\,\mathrm{d}x.$$

By assumption, $u^2\bar{d}_{\Gamma_1} \in L^1(\Gamma)$ (cf. Lemma B.2), and therefore

$$\int_{\Gamma\setminus\Gamma_\varrho}\int_{\Gamma_\varrho} \frac{u^2(x)g_\varrho^2(x)}{|x-y|^3}\,\mathrm{d}y\,\mathrm{d}x \leqslant \int_{\Gamma\setminus\Gamma_\varrho} u^2 g_\varrho^2 \bar{d}_{\Gamma_1} \to 0,$$

by the dominated convergence theorem. On the other hand,

$$\int_{\Gamma_\varrho}\int_{\Gamma_\varrho} \frac{|(ug_\varrho)(x)-(ug_\varrho)(y)|^2}{|x-y|^3}\,\mathrm{d}y\,\mathrm{d}x \leqslant \int_{\Gamma_\varrho} |u(x)|^2\Bigl(\int_{\Gamma_\varrho}\frac{|g_\varrho(x)-g_\varrho(y)|^2}{|x-y|^3}\,\mathrm{d}y\Bigr)\,\mathrm{d}x + \int_{\Gamma_\varrho} |g_\varrho(x)|^2\Bigl(\int_{\Gamma_\varrho}\frac{|u(x)-u(y)|^2}{|x-y|^3}\,\mathrm{d}y\Bigr)\,\mathrm{d}x.$$

We estimate $|g_\varrho(x)-g_\varrho(y)|^2 \leqslant 4\|\mu\|^2_{L^\infty(\Omega)}$ and obtain that

$$\int_{\Gamma_\varrho} |u(x)|^2\Bigl(\int_{\Gamma_\varrho}\frac{|g_\varrho(x)-g_\varrho(y)|^2}{|x-y|^3}\,\mathrm{d}y\Bigr)\,\mathrm{d}x \leqslant 4\|\mu\|^2_{L^\infty(\Omega)}\int_{\Gamma_\varrho} u^2\bar{d}_{\Gamma_1} \to 0,$$

due to the absolute continuity of the integral. On the other hand, denoting $f(x) := \int_\Gamma |u(x)-u(y)|^2/|x-y|^3\,\mathrm{d}y \in L^1(\Gamma)$, we have the majoration

$$\int_{\Gamma_\varrho} |g_\varrho(x)|^2\Bigl(\int_{\Gamma_\varrho}\frac{|u(x)-u(y)|^2}{|x-y|^3}\,\mathrm{d}y\Bigr)\,\mathrm{d}x \leqslant \int_{\Gamma_\varrho} |g_\varrho(x)|^2 f(x)\,\mathrm{d}x \to 0.$$

□

Lemma C.1. *Let $1 \leqslant q \leqslant \infty$ be arbitrary, let $g \in W^{1,q}(\Omega)$, and let $\tau \in \{\tau^{(1)}, \tau^{(2)}\}$ where $\tau^{(i)}$ is defined by* (2.4). *For $v \in W^{2,q'}(\Omega)$,*

$$\int_\Gamma g(\tau \cdot \nabla v) = \int_\Omega \operatorname{curl}(g(\tau \times n_\Gamma)) \cdot \nabla v, \tag{C.1}$$

$$\left| \int_\Gamma g(\tau \cdot \nabla v) \right| \leqslant (g_0 \|g\|_{L^q(\Omega)} + \|\nabla g\|_{L^q(\Omega)}) \|\nabla v\|_{L^{q'}(\Omega)}. \tag{C.2}$$

P r o o f. The representation (C.1) follows from integration by parts. The estimate (C.2) is obvious due to Hölder's inequality. □

Lemma C.2. *For all $u \in W^{1,2}(\Omega)$, we have the estimate*

$$\|u\|^2_{L^2(\Gamma)} \leqslant c_0 \|\nabla u\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}. \tag{C.3}$$

P r o o f. The inequality (C.3) is proved in [6], Chapter 2, Paragraph 2. □

The proof of the following Lemma follows from elementary vector identities.

Lemma C.3. *Let $T^{(1)}, T^{(2)}$ be given by* (2.5), *and let $\tau^{(1)}, \tau^{(2)}$ be given by* (2.4). *Then we have on Γ*

$$T^{(1)} \cdot \tau^{(1)} = 1,\ T^{(1)} \cdot \tau^{(2)} = 0,\ T^{(1)} \cdot n_\Gamma = 0, \tag{C.4}$$

$$T^{(2)} \cdot \tau^{(1)} = 0,\ T^{(2)} \cdot \tau^{(2)} = \cos\alpha,\ T^{(2)} \cdot n_\Gamma = \sin\alpha, \tag{C.5}$$

$$n_S \cdot \tau^{(1)} = 0,\ n_S \cdot \tau^{(2)} = -\sin\alpha,\ n_S \cdot n_\Gamma = \cos\alpha. \tag{C.6}$$

Lemma C.4. *Let $q_0 > 3$ be an arbitrary real number, and let $m_0 \in \mathbb{N}$. For all $m \in \mathbb{N}$ such that $m \geqslant m_0$, let $u \in W^{1,2}(\Omega)$ satisfy*

$$\int_\Omega |\nabla(u-m)^+|^2 \leqslant K \|\nabla(u-m)^+\|_{L^{q_0'}(\Omega)}. \tag{C.7}$$

Then there is a consant c depending on Ω such that $\sup_\Omega u \leqslant m_0 + cK$. Under the same conditions, let u satisfy

$$\int_\Omega |\nabla(u+m)^-|^2 \leqslant K \|\nabla(u+m)^-\|_{L^{q_0'}(\Omega)}. \tag{C.8}$$

Then $\inf_\Omega u \geqslant -m_0 - cK$ with a constant c depending on Ω.

P r o o f. Lemma C.4 follows from a (nowadays classical) lemma by G. Stampacchia [16]. Supplements to the original proof are to be found, for instance, in [17], Chapter 2, Section 2.3. Similar results were obtained in [6], Chapter 3, Paragraph 13.

□

References

[1] *C. Bacuta, A. L. Mazzucato, V. Nistor, L. Zikatanov*: Interface and mixed boundary value problems on n-dimensional polyhedral domains. Doc. Math., J. DMV *15* (2010), 687–745. zbl zbl

[2] *J. Elschner, J. Rehberg, G. Schmidt*: Optimal regularity for elliptic transmission problems including C^1 interfaces. Interfaces Free Bound. *9* (2007), 233–252. zbl zbl

[3] *D. Gilbarg, N. S. Trudinger*: Elliptic Partial Differential Equations of Second Order. Reprint of the 1998 ed. Classics in Mathematics. Springer, Berlin, 2001. zbl zbl

[4] *R. Haller-Dintelmann, H.-C. Kaiser, J. Rehberg*: Elliptic model problems including mixed boundary conditions and material heterogeneities. J. Math. Pures Appl. (9) *89* (2008), 25–48. zbl zbl

[5] *A. Kufner, O. John, S. Fučík*: Function Spaces. Academia, Praha, 1977. zbl zbl

[6] *O. A. Ladyzhenskaya, N. N. Ural'tseva*: Linear and Quasilinear Elliptic Equations. Mathematics in Science and Engeneering 46. Academic Press. New York, 1968. zbl zbl

[7] *O. A. Ladyzhenskaya, V. Ya. Rivkind, N. N. Ural'tseva*: The classical solvability of diffraction problems. Tr. Mat. Inst. Steklova *92* (1966), 116–146. (In Russian.) zbl

[8] *O. A. Ladyzhenskaya, V. A. Solonnikov*: Solutions of some non-stationary problems of magnetohydrodynamics for a viscous incompressible fluid. Tr. Mat. Inst. Steklova *59* (1960), 115–173. (In Russian.)

[9] *Y. Y. Li, M. Vogelius*: Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients. Arch. Ration. Mech. Anal. *153* (2000), 91–151. zbl zbl

[10] *J. L. Lions, E. Magenes*: Problèmes aux limites non homogènes. IV. Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser. *15* (1961), 311–326. (In French.) zbl zbl

[11] *J. L. Lions, E. Magenes*: Inhomogenous Boundary Problems and Applications. Vol. 1, 2. Dunod, Paris, 1968. (In French.) zbl zbl

[12] *D. Mercier*: Minimal regularity of the solutions of some transmission problems. Math. Methods Appl. Sci. *26* (2003), 321–348. zbl zbl

[13] *S. Nicaise, A.-M. Sändig*: Transmission problems for the Laplace and elasticity operators: Regularity and boundary integral formulation. Math. Models Methods Appl. Sci. *9* (1999), 855–898. zbl zbl

[14] *G. Savaré*: Regularity results for elliptic equations in Lipschitz domains. J. Funct. Anal. *152* (1998), 176–201. zbl zbl

[15] *G. Stampacchia*: Su un probleme relativo alle equazioni di tipo ellitico del secondo ordine. Ricerche Mat. *5* (1956), 1–24. (In Italian.) zbl zbl

[16] *G. Stampacchia*: Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. Ann. Inst. Fourier *15* (1965), 189–257 (In French.); Colloques Int. Centre Nat. Rech. Sci. *146* (1965), 189–258. zbl zbl

[17] *G. M. Troianiello*: Elliptic Differential Equations and Obstacle Problems. The University Series in Mathematics. Plenum Press, New York, 1987. zbl zbl

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