THE b-WEAK COMPACTNESS OF WEAK BANACH-SAKS OPERATORS

Belmesnaoui Aqzzouz, SalaAljadida, Othman Aboutafail, Taib Belghiti, Jawad H'Michane, Kénitra

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Abstract. We characterize Banach lattices on which every weak Banach-Saks operator is b-weakly compact.

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1. Introduction and notation

Many authors have studied the *Banach-Saks property*, the *weak Banach-Saks property* and operators with these properties (see [8], [10], [11], [12], [13]). The notions of *Banach-Saks property* and *weak Banach-Saks property* are introduced in [12]. Note that the later property was introduced for the first time in [8] with the name *Banach-Saks-Rosenthal property*.

A Banach space X is said to have the *Banach-Saks property* if every bounded sequence (x_n) in X has a subsequence (x_{n_k}) which is Cesàro convergent. The origin of the Banach-Saks property can be traced back to a result of S. Mazur. If a sequence (x_n) in a Banach space is weakly convergent to some point x, then there is a sequence formed by convex combinations of (x_n) that converges in norm to x. It is proved that a space with the Banach-Saks property must be reflexive but not all reflexive spaces have the Banach-Saks property.

Also, a Banach space X is said to have the weak *Banach-Saks property* if every weakly null sequence (x_n) in X has a Cesàro convergent subsequence. Note that a Banach space with the *Banach-Saks property* satisfies the weak Banach-Saks property, and that not all spaces have the weak Banach-Saks property.

We say that an operator $T\colon X \to Y$ is *weak Banach-Saks* if every weakly null sequence (x_n) in X has a subsequence such that (Tx_{n_k}) is Cesàro convergent. As examples, the identity operator of the Banach lattice l^1 is weak Banach-Saks but the identity operator of the Banach lattice l^∞ is not.

On the other hand, let us recall that an operator T from a Banach lattice E into a Banach space X is said to be b-weakly compact whenever T carries each b-order bounded subset of E into a relatively weakly compact subset of X. For information on this class of operators see [3], [4], [5].

Note that a b-weakly compact operator is not necessarily weak Banach-Saks. In fact, the identity operator of the Banach lattice $L^2(c_0)$ is b-weakly compact (because $L^2(c_0)$ is a KB-space), but it is not weak Banach-Saks (because $L^2(c_0)$ does not have the weak Banach-Saks property). Conversely, there exists a weak Banach-Saks operator which is not b-weakly compact. In fact, the identity operator of c_0 is weak Banach-Saks (because c_0 has the weak Banach-Saks property) but it is not b-weakly compact (because c_0 is not a KB-space).

The goal of this paper is to characterize Banach lattices on which each *weak Banach-Saks* operator is b-weakly compact. In another paper, we will look at the reciprocal problem. In fact, in this paper we will prove that if E and F are two Banach lattices such that the norm of E is order continuous, then each *weak Banach-Saks* operator $T\colon E \to F$ is b-weakly compact if and only if E or F is a KB-space. As consequences, we will obtain some characterizations for KB-spaces. Also, we will characterize Banach lattices under which the second power of each *weak Banach-Saks* operator is b-weakly compact.

To state our results, we need to fix some notation and recall some definitions. Let E be a vector lattice. For each $x, y \in E$ with $x \leqslant y$, the set $[x, y] = \{z \in E\colon x \leqslant z \leqslant y\}$ is called an order interval. A subset of E is said to be order bounded if it is included in some order interval. Recall that a nonzero element x of a vector lattice E is discrete if the order ideal generated by x equals the subspace generated by x. The vector lattice E is discrete, if it admits a complete disjoint system of discrete elements.

A Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leqslant |y|$, we have $\|x\| \leqslant \|y\|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm, is also a Banach lattice. A norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, the sequence (x_α) converges to 0 for the norm $\|\cdot\|$ where the notation $x_\alpha \downarrow 0$ means that the sequence (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$.

Let us recall that a Banach lattice E is said to be a KB-space whenever every increasing norm bounded sequence of E^+ is norm convergent. As an example, each reflexive Banach lattice is a KB-space.

We refer the reader to [1] for unexplained terminology on Banach lattice theory.

2. Main results

We will use the term operator $T\colon E \to F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geqslant 0$ in F whenever $x \geqslant 0$ in E. An operator $T\colon E \to F$ is regular if $T = T_1 - T_2$ where T_1 and T_2 are positive operators from E into F. It is well known that each positive linear mapping on a Banach lattice is continuous. For terminology concerning positive operators, we refer the reader to the excellent book of Aliprantis-Burkinshaw [1].

Recall that the definition of b-weakly compact operators is based on the notion of b-order bounded subsets. A subset A of a Banach lattice E is called b-order bounded if it is order bounded in the topological bidual E''. It is clear that every order bounded subset of E is b-order bounded. However, the converse is not true in general. A Banach lattice E is said to have the (b)-property if $A \subset E$ is order bounded in E whenever it is order bounded in its topological bidual E''.

Let E be a Banach lattice and let X be a Banach space. An operator $T\colon E \to X$ is said to be b-weakly compact whenever T carries each b-order bounded subset of E into a relatively weakly compact subset of X.

It follows from Aliprantis-Burkinshaw ([1], p. 222) that a Banach lattice E is lattice embeddable into another Banach lattice F whenever there exists a lattice homomorphism $T\colon E \to F$ and there exist two positive constants K and M satisfying

$$K\|x\| \leqslant \|T(x)\| \leqslant M\|x\| \quad \text{for all } x \in E.$$

T is called a lattice embedding from E into F. In this case $T(E)$ is a closed sublattice of F which can be identified with E.

Note that each KB-space has the (b)-property but a Banach lattice with the (b)-property is not necessarily a KB-space. However, by Proposition 2.1 of [3], a Banach lattice E is a KB-space if and only if it has the (b)-property and its norm is order continuous.

We note that there exists a Banach lattice with an order continuous norm without the (b)-property. In fact, the norm of c_0 is order continuous but c_0 does not have the (b)-property.

On the other hand, the norm of l^∞ is not order continuous and l^∞ has the (b)-property, but does not contain a complemented copy of c_0.

Before stating our main results, we would like to recall that "E has an order continuous norm" does not imply "*E has the weak Banach-Saks property*". In fact, it follows from [12] that E has *the Banach-Saks property* if, and only if, E has the *weak Banach-Saks property* and is reflexive. By way of contradiction, suppose that E is a KB-space implies E has the weak Banach-Saks property. Then every reflexive Banach lattice would have the *Banach-Saks property*, and this is impossible (because Baernstein's space is a reflexive Banach lattice without the *Banach-Saks property*). So, there is an operator which is not *weak Banach-Saks*, however, E has an order continuous norm. In fact, $L^2(c_0)$ has an order continuous norm, but its identity operator is not *weak Banach-Saks.*

Also, the class of weak Banach-Saks operators is a two sided ideal of the space of all operators on a Banach lattice.

Theorem 2.1. *Let E be a Banach lattice with an order continuous norm, and F a Banach lattice. Then the following assertions are equivalent:*

(1) *Each operator $T\colon E \to F$ is b-weakly compact.*
(2) *Each weak Banach-Saks operator $T\colon E \to F$ is b-weakly compact.*
(3) *Each positive weak Banach-Saks operator $T\colon E \to F$is b-weakly compact.*
(4) *One of the following assertions holds:*
  (a) *E is a KB-space.*
  (b) *F is a KB-space.*

P r o o f. (1) $\Longrightarrow$ (2) Obvious.

(2) $\Longrightarrow$ (3) Obvious.

(3) $\Longrightarrow$ (4) By way of contradiction, we suppose that neither E nor F is a KB-space and we construct a positive weak Banach-Saks operator which is not b-weakly compact. In fact, since E has an order continuous norm, Proposition 2.1 of [3] implies that E does not have the (b)-property. So it follows from Lemma 2.1 of [7] that the Banach lattice E contains a complemented copy of c_0. Denote by $P\colon E \to c_0$ the positive projection of E in c_0 and by $i\colon c_0 \to E$ the canonical injection of c_0 into E.

As F is not a KB-space, Theorem 4.61 of [1] implies that c_0 is lattice embeddable in F, so there exists a lattice embedding T from c_0 into F. Hence, there exists a constant $K > 0$ such that $\|T((\gamma_n))\| \geqslant K\|(\gamma_n)\|_\infty$ for all $(\gamma_n) \in c_0$. Note that the embedding $T\colon c_0 \to F$ is not b-weakly compact. Otherwise, as the canonical basic (e_n) of c_0 is a disjoint b-order bounded sequence, it would follow from Proposition 2.8 of [3] that $\lim\limits_n \|T((e_n))\| = 0$, but this is false because $\|T((e_n))\| \geqslant K\|(e_n)\|_\infty = K$ for each n.

Now, we consider the composed operator $T \circ P\colon E \to c_0 \to F$. Since $T \circ P = T \circ \mathrm{Id}_{c_0} \circ P$ and the identity operator $\mathrm{Id}_{c_0}\colon c_0 \to c_0$ is weak Banach-Saks, hence $T \circ P$ is also weak Banach-Saks. But it is not a b-weakly compact operator. Otherwise, the composed operator $T \circ P \circ i$, which is exactly the embedding $T\colon c_0 \to F$, would be b-weakly compact, but this is a contradiction.

(4a) $\Longrightarrow$ (1) Follows from Proposition 2.1 of [4].

(4b) $\Longrightarrow$ (1) Follows from Corollary 2.3 of [5].

Remark. The assumption "E with an order continuous norm" is essential in Theorem 2.1. In fact, each positive operator T from l^∞ into c_0 is b-weakly compact, but neither l^∞ nor c_0 is a KB-space.

As consequences, we obtain the following characterizations of KB-spaces.

Corollary 2.2. *Let E be a Banach lattice with an order continous norm. Then the following assertions are equivalent:*

(1) *Each operator $T\colon E \to E$ is b-weakly compact.*
(2) *Each weak Banach-Saks operator $T\colon E \to E$ is b-weakly compact.*
(3) *Each positive weak Banach-Saks operator $T\colon E \to E$ is b-weakly compact.*
(4) *E is a KB-space.*

Corollary 2.3. *Let E be a Banach lattice with an order continous norm. Then the following assertions are equivalent:*

(1) *Each operator $T\colon E \to c_0$ is b-weakly compact.*
(2) *Each weak Banach-Saks operator $T\colon E \to c_0$ is b-weakly compact.*
(3) *Each positive weak Banach-Saks operator $T\colon E \to c_0$ is b-weakly compact.*
(4) *E is a KB-space.*

Corollary 2.4. *Let F be a Banach lattice. Then the following assertions are equivalent:*

(1) *Each operator $T\colon c_0 \to F$ is b-weakly compact.*
(2) *Each weak Banach-Saks operator $T\colon c_0 \to F$ is b-weakly compact.*
(3) *Each positive weak Banach-Saks operator $T\colon c_0 \to F$ is b-weakly compact.*
(4) *F is a KB-space.*

Now, we note that there exists an operator which is *weak Banach-Saks* but its second power is not b-weakly compact. In fact, the identity operator of the Banach lattice c_0 is *weak Banach-Saks*, but its second power which is also the identity operator of c_0 is not b-weakly compact.

In the next result we give necessary and sufficient conditions under which the second power of each *weak Banach-Saks* operator is b-weakly compact.

Theorem 2.5. *Let E be a Banach lattice with an order continuous norm. Then the following assertions are equivalent:*

(1) *For all positive operators S and T from E into E with $0 \leqslant S \leqslant T$ and T weak Banach-Saks, S is b-weakly compact.*

(2) *Each positive weak Banach-Saks operator $T\colon E \to E$ is b-weakly compact.*

(3) *For each positive weak Banach-Saks operator $T\colon E \to E$, the second power T^2 is b-weakly compact.*

(4) *E is a KB-space.*

Proof. (1) $\Longrightarrow$ (2) Let $T\colon E \to E$ be a positive weak Banach-Saks operator. Since $0 \leqslant T \leqslant T$, by our hypothesis T is b-weakly compact.

(2) $\Longrightarrow$ (3) By our hypothesis T is b-weakly compact and hence T^2 is b-weakly compact.

(3) $\Longrightarrow$ (4) By way of contradiction, suppose that E is not a KB-space. As the norm of E is order continuous, it follows from Proposition 2.4 of [4] and Lemma 2.1 of [7] that E contains a complemented copy of c_0, and there exists a positive projection $P\colon E \to c_0$. Denote by $i\colon c_0 \to E$ the canonical injection.

Consider the operator $T = i \circ P\colon E \to c_0 \to E$. Clearly the operator T is weak Banach-Saks (because $T = i \circ \mathrm{Id}_{c_0} \circ P$) but it is not b-weakly compact. Otherwise, the operator $P \circ T \circ i = \mathrm{Id}_{c_0}$ would be b-weakly compact, and this is a contradiction. Hence, the operator $T^2 = T$ is not b-weakly compact.

(4) $\Longrightarrow$ (1) Follows from Corollary 2.2 and [3], Corollary 2.9.

Recall from [6] that an operator T from a Banach lattice E into a Banach space X is said to be b-AM-compact if it carries each b-order bounded subset of E into a relatively compact subset of X.

Note that each b-AM-compact operator is b-wealy compact but the converse is false in general. In fact, the identity operator of the Banach lattice $L^1[0,1]$ is b-weakly compact (because $L^1[0,1]$ is a KB-space), but it is not b-AM-compact (because $L^1[0,1]$ is not a discrete KB-space). Also, there exists a weak Banach-Saks operator which is not b-AM-compact. In fact, the identity operator of the Banach lattice c_0 is weak Banach-Saks but it is not b-AM-compact (because c_0 is not a discrete KB-space).

However, we have the following necessary conditions.

Theorem 2.6. *Let E be a Banach lattice with an order continuous norm and let F be a Banach lattice. If each positive weak Banach-Saks operator $T\colon E \to F$ is b-AM-compact, then one of the following assertions holds:*

(1) *E is a KB-space.*

(2) *F is a KB-space.*

P r o o f. Suppose that neither E nor F is a KB-space. Consider the same operator $T \circ P$ as that used in the proof of Theorem 2.1. This operator is positive and *weak Banach-Saks* but it is not b-AM-compact (because it is not b-weakly compact).

R e m a r k. The assumption "E with an order continuous norm" is essential in Theorem 2.6. In fact, each positive operator $T\colon l^\infty \to c_0$ is b-AM-compact, but neither l^∞ nor c_0 is a KB-space.

R e m a r k. The converse of Theorem 2.6 is false, i.e. there exist KB-spaces E and F such that a positive weak Banach-Saks operator $T\colon E \to F$ is not necessarily b-AM-compact. In fact, it follows from Theorem 5 of [10] that there exists a positive operator $T\colon L^1[0,1] \to l^\infty$ which is not b-AM-compact. However, the operator $T\colon L^1[0,1] \to l^\infty$ is weak Banach-Saks and $L^1[0,1]$ is a KB-space. As another example, put $E = L^1[0,1] \oplus l^2$; the identiy operator of the Banach lattice E is weak Banach-Saks, but it is not b-AM-compact. However, E is KB-space.

References

[1] *C. D. Aliprantis, O. Burkinshaw*: Positive Operators. Reprint of the 1985 original. Springer, Berlin, 2006. zbl

[2] *D. J. Aldous*: Unconditional bases and martingales in $L_p(F)$. Math. Proc. Camb. Philos. Soc. *85* (1979), 117–123. zbl

[3] *S. Alpay, B. Altin, C. Tonyali*: On property (b) of vector lattices. Positivity *7* (2003), 135–139. zbl

[4] *B. Altin*: Some property of b-weakly compact operators. G.U. Journal of Science *18* (2005), 391–395.

[5] *B. Aqzzouz, A. Elbour, J. Hmichane*: The duality problem for the class of b-weakly compact operators. Positivity *13* (2009), 683–692. zbl

[6] *B. Aqzzouz, J. Hmichane*: The class of b-AM-compact operators. Quaest. Math. *36* (2013), 1–11.

[7] *B. Aqzzouz, J. Hmichane*: The b-weak compactness of order weakly compact operators. Complex Anal. Oper. Theory *7* (2013), 3–8.

[8] *B. Beauzamy*: Propriété de Banach-Saks et modèles étalés. Semin. Geom. des Espaces de Banach, Ec. Polytech., Cent. Math., 1977–1978, Expose No. 3, 16 pp. (1978). (In French.) zbl

[9] *B. Beauzamy*: Propriété de Banach-Saks. Stud. Math. *66* (1980), 227–235. zbl

[10] *N. Cheng, Z.-L. Chen*: b-AM-compact operators on Banach lattices. Chin. J. Eng. Math. *27* (2010). zbl

[11] *J. Flores, P. Tradacete*: Factorization and domination of positive Banach-Saks operators. Stud. Math. *189* (2008), 91–101. zbl

[12] *H. P. Rosenthal*: Weakly independent sequences and the weak Banach-Saks property. Proceedings of the Durham Symposium on the Relations Between Infinite Dimensional and Finite-Dimentional Convexity (July 1975).

[13] *J. Zhenglu, F. Xiaoyong*: The Banach-Saks property of the Banach product spaces. Arxiv: math/0702538V1 [math. FA] 19 feb 2007.

Authors' addresses: *Belmesnaoui Aqzzouz*, Université Mohammed V-Souissi, Faculté des Sciences Economiques, Juridiques et Sociales, Département d'Economie, B. P. 5295, SalaAl-jadida, Morocco, e-mail: `baqzzouz@hotmail.com`; *Othman Aboutafail*, *Taib Belghiti*, *Jawad H'Michane*, Université Ibn Tofail, Faculté des Sciences, Département de Mathématiques, B. P. 133, Kénitra, Morocco.