ROTATIONS OF λ-LATTICES

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Abstract. In [2], J. Klimeš studied rotations of lattices. The aim of the paper is to research rotations of the so-called λ-lattices introduced in [3] by V. Snášel.

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The set of all lower (upper) bounds of a subset X of an ordered set A will be denoted by $L(X)\big(U(X)\big)$. In the case of a finite set $X = \{a, b, \ldots\}$ we write $L(a, b, \ldots)\big(U(a, b, \ldots)\big)$ instead of $L(X)\big(U(X)\big)$. As usual, under a Galois correspondence we mean a pair (f, g) of mappings between ordered sets P and Q such that f, g are antitone and the compositions gf, fg are extensive.

It is easy to prove the following

1. Lemma. *Let P, Q be ordered sets, $f\colon P \to Q$, $g\colon Q \to P$ mappings. Then the pair (f, g) is a Galois correspondence between P and Q if and only if we have, for each $x \in P$, $y \in Q$,*

$$f\big(L\big(x, g(y)\big)\big) \subseteq U\big(f(x), y\big),$$
$$g\big(L\big(f(x), y\big)\big) \subseteq U\big(x, g(y)\big).$$

2. Definition. A below directed ordered set A with a binary operation $\wedge$ is called a λ-$\wedge$-*semilattice* if it satisfies the following three axioms:

(1) $a \wedge b \in L(a, b)$ for each a, $b \in A$.

(2) If $a \leqslant b$, then $a \wedge b = a$ for each a, $b \in A$.

(3) $\wedge$ is commutative.

A λ-$\wedge$-semilattice is defined dually. An ordered set with two binary operations $\wedge$ and $\vee$ is called a λ-lattice if it is a λ-$\wedge$-semilattice and λ-$\vee$-semilattice.

3. Theorem. *Let K, L be λ-$\wedge$-semilattices, $f\colon K \to L$, $g\colon L \to K$ mappings. Then the pair of mappings (f,g) is a Galois correspondence between K and L if and only if, for each $x \in K$, $y \in L$,*

$$f\bigl(x \wedge g(y)\bigr) \in U\bigl(f(x), y\bigr),$$
$$g\bigl(f(x) \wedge y\bigr) \in U\bigl(x, g(y)\bigr).$$

P r o o f. "$\Rightarrow$": Let (f,g) be a Galois correspondence between K and L. Let $x \in K$, $y \in L$. By 1, we have $f\bigl(L(x, g(y))\bigr) \subseteq U\bigl(f(x), y\bigr)$, $g\bigl(L(f(x), y)\bigr) \subseteq U\bigl(x, g(y)\bigr)$. But $x \wedge g(y) \in L\bigl(x, g(y)\bigr)$ by 2(1), so that $f\bigl(x \wedge g(y)\bigr) \in U\bigl(f(x), y\bigr)$. Interchanging K and L, f and g, we obtain the second assertion.

"$\Leftarrow$": Let $x \in K$, $y \in L$. We have $gf(x) = g\bigl(f(x) \wedge f(x)\bigr) \in U\bigl(x, gf(x)\bigr)$, thus $gf(x) \geqslant x$ by 2(2). The mapping gf is therefore extensive. Now, let x_1, $x_2 \in K$, $x_1 \leqslant x_2$. Then, by 2(2), $x_1 = x_1 \wedge gf(x_2)$, for, by extensivity of gf, $x_1 \leqslant x_2 \leqslant gf(x_2)$. This implies $f(x_1) = f\bigl(x_1 \wedge gf(x_2)\bigr) \in U\bigl(f(x_1), f(x_2)\bigr)$ and $f(x_1) \geqslant f(x_2)$; hence the mapping f is antitone. Interchanging K and L, f and g, we obtain extensivity of fg and antitony of g. Consequently, the pair (f,g) is a Galois correspondence between K and L. $\square$

4. Definition. Let K, L be λ-lattices, $f\colon K \to L$, $g\colon L \to K$ mappings. The pair of mappings (f,g) is called

a) a *left semirotation* between K and L if

$$f\bigl(x \wedge g(y)\bigr) \in U\bigl(f(x), y\bigr) \cap L\bigl(f(x) \vee y\bigr),$$
$$g\bigl(f(x) \wedge y\bigr) \in U\bigl(x, g(y)\bigr)$$

for each $x \in K$, $y \in L$,

b) a *right semirotation* between K and L if

$$f\bigl(x \wedge g(y)\bigr) \in U\bigl(f(x), y\bigr),$$
$$g\bigl(f(x) \wedge y\bigr) \in U\bigl(x, g(y)\bigr) \cap L\bigl(x \vee g(y)\bigr)$$

for each $x \in K$, $y \in L$,

c) a *rotation* between K and L if it is a left and a right semirotation.

5. R e m a r k. (1) In the case of K, L being lattices, the notion of a left semirotation, right semirotation, and rotation coincide with the corresponding notions introduced by J. Klimeš in [2].

(2) In the definition of a left semirotation, it suffices to require that K is a λ-$\wedge$-semilattice; similarly for a right semirotation.

6. Lemma. *Let K, L be λ-lattices, (f,g) a left or right semirotation between K and L. Then the pair of mappings (f,g) is a Galois correspondence between K and L.*

Proof. It follows from 3. $\square$

7. Lemma. *Let K, L be λ-lattices, $f\colon K \to L$, $g\colon L \to K$ mappings. Then the following statements are equivalent:*

(a) *(f,g) is a left semirotation between K and L.*

(b) *(f,g) is a Galois correspondence between K and L and, for each $x \in K$, $y \in L$,*

$$f\bigl(L\bigl(x \wedge g(y)\bigr)\bigr) \cap L\bigl(f(x) \vee y\bigr) \neq \emptyset.$$

Proof. (a) $\Rightarrow$ (b): Let (a) hold. Then (f,g) is a Galois correspondence between K and L by 6. For any $x \in K$, $y \in L$, $f\bigl(L\bigl(x\wedge g(y)\bigr)\bigr)\cap L\bigl(f(x)\vee y\bigr) \neq \emptyset$, for $f\bigl(x\wedge g(y)\bigr)$ belongs to this intersection by 4.

(b) $\Rightarrow$ (a): Let (b) hold. Let $x \in K$, $y \in L$. As $f\bigl(L\bigl(x \wedge g(y)\bigr)\bigr) \cap L\bigl(f(x) \vee y\bigr) \neq \emptyset$, there exists $u \in L\bigl(x \wedge g(y)\bigr)$ such that $f(u) \in L\bigl(f(x) \vee y\bigr)$. Thus $u \leqslant x \wedge g(y)$, $f(u) \leqslant f(x) \vee y$. Regarding the antitony of f we obtain $f\bigl(x \wedge g(y)\bigr) \leqslant f(u) \leqslant f(x) \vee y$, so that $f\bigl(x \wedge g(y)\bigr) \in L\bigl(f(x) \vee y\bigr)$. By 3, we have $f\bigl(x \wedge g(y)\bigr) \in U\bigl(f(x), y\bigr)$, $g\bigl(f(x)\wedge y\bigr) \in U\bigl(x, g(y)\bigr)$. Summarizing, we get $f\bigl(x\wedge g(y)\bigr) \in U\bigl(f(x), y\bigr)\cap L\bigl(f(x)\vee y\bigr)$, $g\bigl(f(x) \wedge y\bigr) \in U\bigl(x, g(y)\bigr)$, and (f,g) is a left semirotation between K and L. $\square$

8. Lemma. *Let K, L be λ-lattices, $f\colon K \to L$, $g\colon L \to K$ mappings. Then the following statements are equivalent:*

(a) *(f,g) is a right semirotation between K and L.*

(b) *(f,g) is a Galois correspondence between K and L and, for each $x \in K$, $y \in L$,*

$$g\bigl(L\bigl(f(x) \wedge y\bigr)\bigr) \cap L\bigl(x \vee g(y)\bigr) \neq \emptyset.$$

Proof. Dual to 7. $\square$

9. Theorem. *Let K, L be λ-lattices, $f\colon K \to L$, $g\colon L \to K$ mappings. Then the following statements are equivalent:*

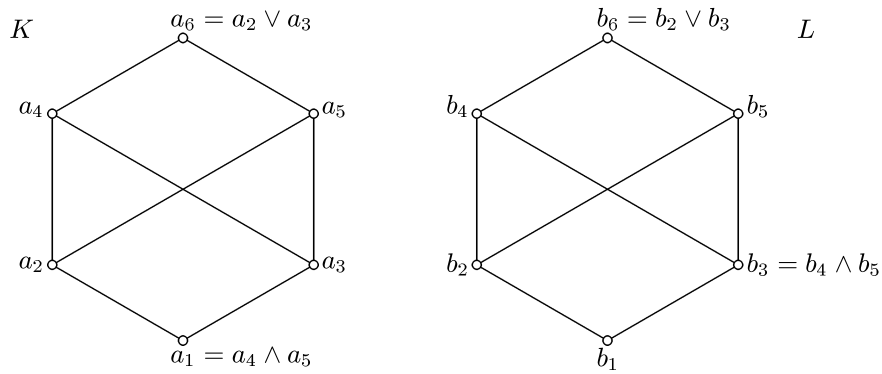
(a) *(f,g) is a rotation between K and L.*

(b) *(f,g) is a Galois correspondence between K and L and, for each $x \in K$, $y \in L$, the sets $f\bigl(L\bigl(x \wedge g(y)\bigr)\bigr) \cap L\bigl(f(x) \vee y\bigr)$ and $g\bigl(L\bigl(f(x) \wedge y\bigr)\bigr) \cap L\bigl(x \vee g(y)\bigr)$ are nonempty.*

Proof. It follows from 7 and 8. $\square$

10. R e m a r k. While in the case of lattices, both sets in (b) are singletons under the assumptions of 9, in our case any of them may contain more elements, which is shown by the following example.

11. E x a m p l e. Let K, L be λ-lattices with isomorphic Hasse diagrams:



If two elements x, y in K or L have the standard supremum or infimum, we put $x \vee y = \sup\{x, y\}$ or $x \wedge y = \inf\{x, y\}$. In the other cases the joins and meets are inscribed in the diagrams. Define a mapping $f\colon K \to L$ as follows:

$$f(a_i) = b_{7-i} \text{ for each } i \in \{1, 2, 3, 4, 5, 6\},$$

and put $g = f^{-1}$. Then (f, g) is a rotation between K and L, but

$$g\big(L\big(f(a_2) \wedge b_4\big)\big) \cap L\big(a_2 \vee g(b_4)\big) = \{a_4, a_6\}.$$

12. N o t a t i o n. Let A, B be sets, $f\colon A \to B$, $g\colon B \to A$ mappings. Denote

$$C_{gf} = \{x \in A;\ x = gf(x)\},$$
$$C_{fg} = \{y \in B;\ y = fg(y)\}.$$

13. Lemma. *Let K, L be λ-lattices, (f, g) a left semirotation between K and L. Then the set C_{fg} is an upper subset of the ordered set L such that y_1, $y_2 \in C_{fg}$ implies $fg(y_1 \wedge y_2) \in L(y_1, y_2)$.*

P r o o f. Let $y \in C_{fg}$, $s \in L$, $y \leqslant s$. By 6, (f, g) is a Galois correspondence between K and L, so that g is antitone and we have $g(y) \geqslant g(s)$, thus $g(s) = g(s) \wedge g(y)$. Using extensivity of fg we obtain $f\big(g(s) \wedge g(y)\big) = fg(s) \geqslant s$, and, moreover, $fg(y) = y$ (for $y \in C_{fg}$). As (f, g) is a left semirotation, $s \leqslant fg(s) = f\big(g(y) \wedge g(s)\big) \leqslant fg(y) \vee s = y \vee s = s$, hence $fg(s) = s$ and $s \in C_{fg}$. Further, let y_1, $y_2 \in C_{fg}$. As $y_1 \geqslant y_1 \wedge y_2$, $y_2 \geqslant y_1 \wedge y_2$, we get $g(y_1) \leqslant g(y_1 \wedge y_2)$, $g(y_2) \leqslant g(y_1 \wedge y_2)$. In view of the antitony of f, $fg(y_1 \wedge y_2) \leqslant fg(y_1) = y_1$, $fg(y_1 \wedge y_2) \leqslant fg(y_2) = y_2$, hence $fg(y_1 \wedge y_2) \in L(y_1, y_2)$. $\square$

14. Lemma. *Let K, L be λ-lattices, (f,g) a right semirotation between K and L. Then the set C_{gf} is an upper subset of the ordered set K such that $x_1, x_2 \in C_{gf}$ implies $gf(x_1 \wedge x_2) \in L(x_1, x_2)$.*

P r o o f. Dual to 13. $\square$

15. Theorem. *Let K, L be λ-lattices, (f,g) a rotation between K and L. Then:*
(1) *C_{gf} is an upper subset of the ordered set K.*
(2) *C_{fg} is an upper subset of the ordered set L.*
(3) *$x_1, x_2 \in C_{gf}$ implies $gf(x_1 \wedge x_2) \in L(x_1, x_2)$, $f(x_1 \wedge x_2) \in U\big(f(x_1), f(x_2)\big) \cap L\big(f(x_1) \vee f(x_2)\big)$.*
(4) *$y_1, y_2 \in C_{fg}$ implies $fg(y_1 \wedge y_2) \in L(y_1, y_2)$, $g(y_1 \wedge y_2) \in U\big(g(y_1), g(x_2)\big) \cap L\big(g(y_1) \vee g(y_2)\big)$.*
(5) *$f \restriction C_{gf}$ is an order antiisomorphism of C_{gf} onto C_{fg}.*
(6) *$g \restriction C_{fg}$ is an order antiisomorphism of C_{fg} onto C_{gf}.*

P r o o f. (1) follows from 14.

(2) follows from 13.

(3) The first part follows from 14. Further, let $x_1, x_2 \in C_{gf}$. Then $f(x_1 \wedge x_2) \in U\big(f(x_1), f(x_2)\big)$, for f is antitone. We have $f(x_1 \wedge x_2) = f\big(x_1 \wedge gf(x_2)\big) \in L\big(f(x_1) \vee f(x_2)\big)$ by 4. Hence $f(x_1 \wedge x_2) \in U\big(f(x_1), f(x_2)\big) \cap L\big(f(x_1) \vee f(x_2)\big)$.

(4) Dual to (3).

(5) and (6) hold for any Galois correspondence and are well-known. $\square$

16. Theorem. *Let K, L be λ-lattices, $f\colon K \to L$, $g\colon L \to K$ mappings such that gf and fg are extensive on K and L, respectively. If, for any $x, u \in K$, $y, v \in L$, $x \wedge g(y) \leqslant u \vee g(v)$ is equivalent to $f(x) \vee y \geqslant f(u) \wedge v$, then (f,g) is a rotation between K and L.*

P r o o f. First, we shall show that f is antitone. Let $x_1, x_2 \in K$, $x_1 \leqslant x_2$. Then $x_1 \wedge gf(x_1) \leqslant x_1 \leqslant x_2 \leqslant x_2 \vee gf(x_2)$, thus $f(x_1) = f(x_1) \vee f(x_1) \geqslant f(x_2) \wedge f(x_2) = f(x_2)$, and f is antitone. Interchanging K and L, f and g, we obtain antitony of g. Hence the pair (f,g) is a Galois correspondence between K and L. Hence, by 3, $f\big(x \wedge g(y)\big) \in U\big(f(x), y\big)$, $g\big(f(x) \wedge y\big) \in U\big(x, g(y)\big)$ for any $x \in K$, $y \in L$. Further, we have $x \wedge g(y) \leqslant gf\big(x \wedge g(y)\big) = gf\big(x \wedge g(y)\big) \vee gf\big(x \wedge g(y)\big)$, consequently $f(x) \vee y \geqslant fgf\big(x \wedge g(y)\big) \wedge f\big(x \wedge g(y)\big)$. But $fgf\big(x \wedge g(y)\big) \geqslant f\big(x \wedge g(y)\big)$, so that $fgf\big(x \wedge g(y)\big) \wedge f\big(x \wedge g(y)\big) = f\big(x \wedge g(y)\big)$, and we get $f\big(x \wedge g(y)\big) \leqslant f(x) \vee y$. Again, interchanging K and L, f and g, we have $g\big(f(x) \wedge y\big) \leqslant x \vee g(y)$. This yields $f\big(x \wedge g(y)\big) \in U\big(f(x), y\big) \cap L\big(f(x) \vee y\big)$, $g\big(f(x) \wedge y\big) \in U\big(x, g(y)\big) \cap L\big(x \vee g(y)\big)$ for any $x \in K$, $y \in L$, and (f,g) is a rotation between K and L. $\square$

17. Lemma. *Let K, L be λ-lattices, $f\colon K \to L$, $g\colon L \to K$ mappings. If, for any x, $a \in K$, y, $b \in L$,*
(1) *$f(x) \geqslant f(a) \wedge b$ implies $x \leqslant a \vee g(b)$, and*
(2) *$g(y) \geqslant a \wedge g(b)$ implies $y \leqslant f(a) \vee b$,*
then (f, g) is a rotation between K and L.

Proof. First, we shall show extensivity of the mapping gf. For any $a \in K$, we have $f(a) \geqslant fgf(a) \wedge f(a)$. By (1), we obtain $a \leqslant gf(a) \vee gf(a) = gf(a)$. Now, let us show antitony of the mapping f. Let x_1, $x_2 \in K$, $x_1 \leqslant x_2$. As gf is extensive, $x_2 \leqslant gf(x_2)$, so that $gf(x_2) \geqslant x_1 \wedge gf(x_1)$. This implies, by (2), $f(x_2) \leqslant f(x_1) \vee f(x_1) = f(x_1)$. Interchanging K and L, f and g, we get extensivity of fg and antitony of g. By 3, we have $f\bigl(x \wedge g(y)\bigr) \in U\bigl(f(x), y\bigr)$ for any $x \in K$, $y \in L$. As gf is extensive, $gf\bigl(x \wedge g(y)\bigr) \geqslant x \wedge g(y)$, and by (2), $f\bigl(x \wedge g(y)\bigr) \leqslant f(x) \vee y$, i.e. $f\bigl(x \wedge g(y)\bigr) \in L\bigl(f(x) \vee y\bigr)$ for any $x \in K$, $y \in L$. Summarizing, we obtain $f\bigl(x \wedge g(y)\bigr) \in U\bigl(f(x), y\bigr) \cap L\bigl(f(x) \vee y\bigr)$ for any $x \in K$, $y \in L$. Similarly $g\bigl(f(x) \wedge y\bigr) \in U\bigl(x, g(y)\bigr) \cap L\bigl(x \vee g(y)\bigr)$ for any $x \in K$, $y \in L$ and (f, g) is a rotation between K and L. $\square$

18. Lemma. *Let K, L be λ-lattices, (f, g) a left semirotation between K and L. Then, for any $a \in K$, y, $b \in L$, $g(y) \geqslant a \wedge g(b)$ implies $y \leqslant f(a) \vee b$.*

Proof. By 6, (f, g) is a Galois correspondence between K and L. Let $a \in K$, y, $b \in L$, $g(y) \geqslant a \wedge g(b)$. Then $y \leqslant fg(y) \leqslant f\bigl(a \wedge g(b)\bigr) \leqslant f(a) \vee b$ in view of extensivity of fg, antitony of f, and Definition 4. $\square$

19. Lemma. *Let K, L be λ-lattices, (f, g) a right semirotation between K and L. Then, for any x, $a \in K$, $b \in L$, $f(x) \geqslant f(a) \wedge b$ implies $x \leqslant a \vee g(b)$.*

Proof. Dual to 18. $\square$

20. Theorem. *Let K, L be λ-lattices, $f\colon K \to L$, $g\colon L \to K$ mappings. Then the following statements are equivalent:*
(1) *(f, g) is a rotation between K and L.*
(2) *For each x, $a \in K$, y, $b \in L$, $f(x) \geqslant f(a) \wedge b$ implies $x \leqslant a \vee g(b)$, and $g(y) \geqslant a \wedge g(b)$ implies $y \leqslant f(a) \vee b$.*
(3) *fg and gf are extensive and, for any x, $a \in K$, y, $b \in L$, $a \geqslant x \wedge g(y)$ implies $f(a) \leqslant f(x) \vee y$, and $b \geqslant f(x) \wedge y$ implies $g(b) \leqslant x \vee g(y)$.*
(4) *fg and gf are extensive and, for any $x \in K$, $y \in L$, $f\Bigl(U\bigl(x \wedge g(y)\bigr)\Bigr) \subseteq L\bigl(f(x) \vee y\bigr)$, $g\Bigl(U\bigl(f(x) \wedge y\bigr)\Bigr) \subseteq L\bigl(x \vee g(y)\bigr)$.*

Proof. (1)$\Leftrightarrow$ (2): It follows form 17, 18, and 19.

(1)$\Rightarrow$(3): By 6, (f, g) is a Galois correspondence between K and L, thus the mappings fg and gf are extensive. Let $a \geqslant x \wedge g(y)$. Then, by antitony of f and

4, $f(a) \leqslant f\bigl(x \wedge g(y)\bigr) \leqslant f(x) \vee y$. Interchanging K and L, f and g, we obtain the other implication.

(3)$\Rightarrow$(1): First, let us show antitony of f. Let x_1, $x_2 \in K$, $x_1 \leqslant x_2$. Then $x_2 \geqslant x_1 = x_1 \wedge gf(x_1)$, thus $f(x_2) \leqslant f(x_1) \vee f(x_1) = f(x_1)$. Again, interchanging K and L, f and g, we obtain antitony of g. Let $x \in K$, $y \in L$. As $x \wedge g(y) \leqslant x \wedge g(y)$, we have $f\bigl(x \wedge g(y)\bigr) \leqslant f(x) \vee y$. Further, $x \wedge g(y) \leqslant x$, $x \wedge g(y) \leqslant g(y)$, hence $f\bigl(x \wedge g(y)\bigr) \geqslant f(x)$, $f\bigl(x \wedge g(y)\bigr) \geqslant fg(y) \geqslant y$, so that $f\bigl(x \wedge g(y)\bigr) \in U\bigl(f(x), y\bigr)$. Altogether, $f\bigl(x \wedge g(y)\bigr) \in U\bigl(f(x), y\bigr) \cap L\bigl(f(x) \vee y\bigr)$. Analogously, $g\bigl(f(x) \wedge y\bigr) \in U\bigl(x, g(y)\bigr) \cap L\bigl(x \vee g(y)\bigr)$ and (f, g) is a rotation between K and L.

(3)$\Leftrightarrow$(4): Trivial. $\square$

21. Definition. A bounded ordered set A with two mappings $\bigwedge$ and $\bigvee$ of the power set $\mathcal{R}(A)$ of A into A is called a *complete λ-lattice* if it satisfies the following three conditions:

(i) If $X_1 \subseteq X_2 \subseteq A$, then $\bigwedge X_1 \geqslant \bigwedge X_2$, $\bigvee X_1 \leqslant \bigvee X_2$.
(ii) If $X \subseteq A$ has a least element x, then $\bigwedge X = x$.
(iii) $\bigvee X \in U(X)$ for each $X \subseteq A$.

Instead of $\bigwedge\{a, b\}$ we write $a \wedge b$ for any a, $b \in A$; similarly with $\bigvee$.

22. Remark. A complete λ-lattice need not be a λ-lattice with regard to the binary operations $\wedge$ and $\vee$. It becomes a λ-lattice, if we add the condition
(iv) If a, $b \in A$, $a \leqslant b$, then $a \vee b = b$.

23. Theorem. *Let K, L be complete λ-lattices, $f\colon K \to L$ a mapping satisfying the conditions*
$f(\bigvee X) \geqslant \bigwedge f(X)$ *for each $X \subseteq K$, and*
$f(x \wedge y) \geqslant f(x) \vee f(y)$ *for each x, $y \in K$.*
Then there exists a unique mapping $g\colon L \to K$ such that (f, g) is a Galois correspondence between K and L.

Proof. Define a mapping $g\colon L \to K$ as follows:

$$g(y) = \bigvee\{x \in K\,;\, f(x) \geqslant y\} \text{ for any } y \in L.$$

We have $fg(y) = f\bigl(\bigvee\{x \in K;\, f(x) \geqslant y\}\bigr) \geqslant \bigwedge\{f(x);\, x \in K, f(x) \geqslant y\} \geqslant \bigwedge U(y) = y$ for any $y \in L$, because $\{f(x);\, x \in K, f(x) \geqslant y\} \subseteq U(y)$. Thus fg is extensive. Now, let y_1, $y_2 \in L$, $y_1 \leqslant y_2$. Then

$$\{x \in K\,;\, f(x) \geqslant y_1\} \supseteq \{x \in K\,;\, f(x) \geqslant y_2\}$$

and $g(y_1) = \bigvee\{x \in K\,;\, f(x) \geqslant y_1\} \geqslant \bigvee\{x \in K\,;\, f(x) \geqslant y_2\} = g(y_2)$ and g is antitone. Let $x \in K$. Then $gf(x) = \bigvee\{x_1 \in K\,;\, f(x_1) \geqslant f(x)\} \geqslant x$, because

$x \in \{x_1 \in K;\, f(x_1) \geqslant f(x)\}$, and gf is extensive. Further, let x_1, $x_2 \in K$, $x_1 \leqslant x_2$. Then $x_1 = x_1 \wedge x_2$, consequently $f(x_1) = f(x_1 \wedge x_2) \geqslant f(x_1) \vee f(x_2) \geqslant f(x_2)$ and f is antitone. Therefore (f, g) is a Galois correspondence between K and L. Let (f, g') be a Galois correspondence between K and L as well. Then, by 3, $g'(y) = g'\bigl(y \wedge fg(y)\bigr) \geqslant g(y)$ for any $y \in L$. Similarly $g(y) \geqslant g'(y)$ for any $y \in L$. Hence g is unique such that (f, g) is a Galois correspondence between K and L. $\square$

24. Theorem. *Let K, L be complete λ-lattices, $f\colon K \to L$ a surjective mapping satisfying the conditions*

$f(\bigvee X) = \bigwedge f(X)$ for any $X \subseteq K$, and

$f(x \wedge y) = f(x) \vee f(y)$ for any x, $y \in K$.

Then there exists a unique mapping $g\colon L \to K$ such that (f, g) is a left semirotation between K and L; moreover, $fg = \mathrm{id}_L$.

P r o o f. Define a mapping $g\colon L \to K$ as follows:

$$g(y) = \bigvee \{x \in K;\, f(x) = y\} \text{ for any } y \in L.$$

We have $fg(y) = f\bigl(\bigvee \{x \in K;\, f(x) = y\}\bigr) = \bigwedge \{f(x);\, x \in K, f(x) = y\} = y$. Thus $fg = \mathrm{id}_L$ and fg is extensive. Now, let y_1, $y_2 \in L$, $y_1 \leqslant y_2$. As $y_1 = y_1 \wedge y_2 = fg(y_1) \wedge fg(y_2) = f\bigl(g(y_1) \vee g(y_2)\bigr)$, we obtain $g(y_1) \vee g(y_2) \in \{x \in K;\, f(x) = y_1\}$. Hence $g(y_1) = \bigvee \{x \in K;\, f(x) = y_1\} \geqslant g(y_1) \vee g(y_2) \geqslant g(y_2)$ and g is antitone. Let $x \in K$. Then $gf(x) = \bigvee \{x_1 \in K;\, f(x_1) = f(x)\} \geqslant x$, because $x \in \{x_1 \in K;\, f(x_1) = f(x)\}$, and gf is extensive. Further, let x_1, $x_2 \in K$, $x_1 \leqslant x_2$. Then $x_1 = x_1 \wedge x_2$, so that $f(x_1) = f(x_1 \wedge x_2) = f(x_1) \vee f(x_2) \geqslant f(x_2)$ and f is antitone. Consequently, (f, g) is a Galois correspondence between K and L. The uniqueness of g follows from 23. By 3, we have $f\bigl(x \wedge g(y)\bigr) \in U\bigl(f(x), y\bigr)$, $g\bigl(f(x) \wedge y\bigr) \in U\bigl(x, g(y)\bigr)$ for any $x \in K$, $y \in L$. It remains to show that $f\bigl(x \wedge g(y)\bigr) \leqslant f(x) \vee y$ for any $x \in K$, $y \in L$. But we have $f\bigl(x \wedge g(y)\bigr) = f(x) \vee fg(y) = f(x) \vee y$, and (f, g) is a left semirotation between K and L. $\square$

References

[1] *M. Erné, J. Koslowski, A. Melton, G.E. Strecker*: A primer on Galois connections. Papers on general topology and applications (Madison, WI, 1991), 704. Ann. New York Acad. Sci., pp. 103–125.

[2] *J. Klimeš*: Rotations between lattices. Math. Slovaca (Submitted).

[3] *V. Snášel*: λ-lattices. Thesis. Masaryk University, Brno, 1991. (In Czech.)

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