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Contents

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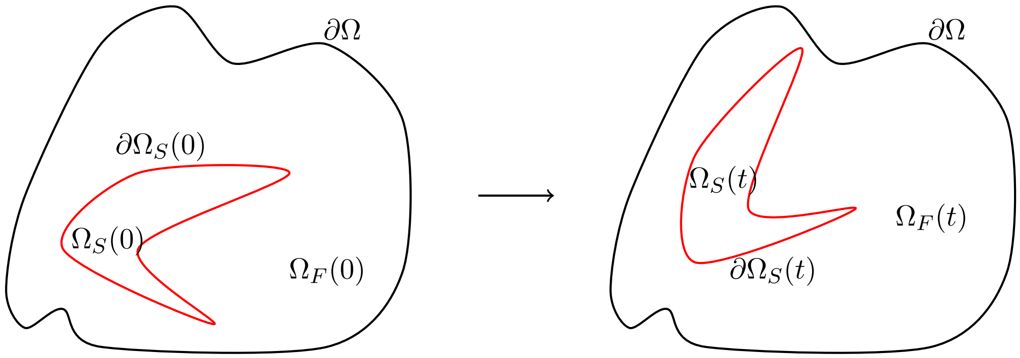

Figure 1: Domain

1 Introduction

We consider a fluid-rigid body interaction problem in $\mathbb{R}^3$ and we are focusing on developing an L^p-theory for strong solutions of the coupled system, for both Newtonian and non-Newtonian fluids with the moving rigid body.

We begin with a description of a model. We study a system of equations modelling the interaction between a fluid flow satisfying the incompressible generalized Navier-Stokes equations and a rigid body satisfying the conservation of linear and angular momentum. The rigid body moves inside the fluid and at time $t \geq 0$, occupies a bounded domain $\Omega_S(t)$, while the fluid fills a bounded domain $\Omega_F(t)$. The common boundary of $\Omega_F(t)$ and $\Omega_S(t)$ is denoted by $\partial\Omega_S(t)$. Note that

$$\Omega_S(0) \cup \Omega_F(0) \cup \partial\Omega_S(0) = \Omega_S(t) \cup \Omega_F(t) \cup \partial\Omega_S(t) =: \Omega \subsetneq \mathbb{R}^3; \quad t \geq 0$$

and

$$\partial\Omega_F(t) = \partial\Omega \cup \partial\Omega_S(t).$$

For the sake of simplicity, we assume the fluid has constant density = 1. By choosing a frame of coordinates whose origin initially coincides with the centre of mass of the rigid body, the domain $\Omega_S(t)$ at any instant t can be given by

$$\Omega_S(t) = \{\boldsymbol{h}(t) + Q(t)\boldsymbol{y} : \boldsymbol{y} \in \Omega_S(0)\}$$

where $\boldsymbol{h}(t)$ is the centre of mass of the rigid body at time t and $Q(t)$ is a rotation matrix associated to the angular velocity $\boldsymbol{\omega}(t)$ of the rigid body. The matrix $Q(t)$ is the solution of the initial value problem

$$\begin{aligned} \dot{Q}(t)Q^T(t)\boldsymbol{y} &= \boldsymbol{\omega}(t) \times \boldsymbol{y} \quad \forall \boldsymbol{y} \in \mathbb{R}^3 \\ Q(0) &= I_3. \end{aligned} \tag{1.1}$$

Here A^T denotes the transpose matrix of A and I_3 is the 3×3 identity matrix. The system of equations modelling the motion of the fluid and the rigid body is given by:

$$
\begin{cases}
\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla_x)\boldsymbol{u} = \operatorname{div}\, T(\boldsymbol{u},\pi) & \text{in } \ \Omega_F(t)\times(0,T),\\
\operatorname{div}\, \boldsymbol{u} = 0 & \text{in } \ \Omega_F(t)\times(0,T),\\
\boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad [T(\boldsymbol{u},\pi)\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \ \partial\Omega\times(0,T),\\
\boldsymbol{u}\cdot\boldsymbol{n} = \boldsymbol{u}_S\cdot\boldsymbol{n} & \text{on } \ \partial\Omega_S(t)\times(0,T),\\
[T(\boldsymbol{u},\pi)\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha\boldsymbol{u}_{\boldsymbol{\tau}} = \alpha\boldsymbol{u}_{S\boldsymbol{\tau}} & \text{on } \ \partial\Omega_S(t)\times(0,T),\\
m\boldsymbol{l}'(t) = -\int\limits_{\partial\Omega_S(t)} T(\boldsymbol{u},\pi)\boldsymbol{n}, & t\in(0,T),\\
(J\boldsymbol{\omega})'(t) = -\int\limits_{\partial\Omega_S(t)} (\boldsymbol{x}-\boldsymbol{h}(t))\times T(\boldsymbol{u},\pi)\boldsymbol{n}, & t\in(0,T),\\
\boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in } \ \Omega_F(0),\\
\boldsymbol{l}(0) = \boldsymbol{l}_0, \quad \boldsymbol{\omega}(0) = \boldsymbol{\omega}_0
\end{cases}
\tag{1.2}
$$

where $\boldsymbol{u}$ and π denote the velocity field and pressure of the fluid respectively, $T(\boldsymbol{u},\pi) := \mu(|\mathbb{D}\boldsymbol{u}|^2)\mathbb{D}\boldsymbol{u} - \pi I_3$ is the stress tensor with the viscosity function $\mu \in C^{1,1}(\mathbb{R}^+;\mathbb{R})$ satisfying the following assumptions

$$
\mu(s) > 0 \quad \text{ and } \quad \mu(s) + 2s\mu'(s) > 0 \quad \text{ for all } \ s \geq 0 \tag{1.3}
$$

and $\mathbb{D}\boldsymbol{u} := \frac{1}{2}\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\right)$ *i.e.*, $(\mathbb{D}\boldsymbol{u})_{ij} := \mathbb{D}_{ij}\boldsymbol{u} := \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right)$ denotes the deformation tensor with $|\mathbb{D}\boldsymbol{u}|^2 = \sum_{i,j=1}^3 (\mathbb{D}_{ij}\boldsymbol{u})^2$ is the Hilbert-Schmidt norm. The friction coefficient $\alpha(x) \geq 0$ is a given function and $\boldsymbol{n}(\boldsymbol{x},t)$ denotes the unit outward normal vector with respect to the domain $\Omega_F(t)$. The subscript $(\cdot)_{\boldsymbol{\tau}}$ denotes the tangential component of a vector *i.e.* $\boldsymbol{v}_{\boldsymbol{\tau}} = \boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}$. The constant $m > 0$ is the mass of the rigid body and $J(t)$ is its inertia tensor, given by

$$
J(t)a\cdot b = \int\limits_{\Omega_S(0)} \rho_S(a\times(\boldsymbol{x}-\boldsymbol{h}(t)))\cdot(b\times(\boldsymbol{x}-\boldsymbol{h}(t))) \qquad \forall\, a,b\in\mathbb{R}^3
$$

where $\rho_S > 0$ is the density of the body. Lastly, $\boldsymbol{l}(t) := \boldsymbol{h}'(t)$ denotes the translational velocity such that $\boldsymbol{u}_S(\boldsymbol{x},t) := \boldsymbol{l}(t) + (\boldsymbol{x}-\boldsymbol{h}(t))\times\boldsymbol{\omega}(t)$ is the velocity of the rigid body.

The assumptions on stress tensor T allow a wide flexibility of stress law coming from various experimentally verified physical models. In particular, it includes the power-law type fluids, namely,

$$
\mu(|\mathbb{D}\boldsymbol{u}|^2) = \mu_0\left(1+|\mathbb{D}\boldsymbol{u}|^2\right)^{\frac{d-2}{2}}, \quad \mu(|\mathbb{D}\boldsymbol{u}|^2) = \mu_0|\mathbb{D}\boldsymbol{u}|^{d-2} \quad \text{ for } \ \mu_0\in(0,\infty),\ d\in(1,\infty).
$$

The case $d = 2$ corresponds to the classical Newtonian fluids *i.e.* the case of constant viscosity. In that case, we denote the stress tensor by $\sigma(\boldsymbol{u},\pi)$ (just to distinguish) which is simply given by $\sigma(\boldsymbol{u},\pi) := 2\mathbb{D}\boldsymbol{u} - \pi I$. Of particular importance within this class are, the shear-thinning fluids, *i.e.*, the case $d \in (1,2)$ which include many important materials of interest (e.g. can be applied for modelling of blood) and also shear-thickening fluids, *i.e.*, the case $d \in (2,\infty)$. For further discussions on the related non-Newtonian fluids, we refer e.g. [6], [5]. Concerning the study of the fluid-rigid body interaction system involving non-Newtonian fluid, there are not many works done till now. The authors in [23] provides an L^p-theory for strong solutions in 3-dimension, although considering the Dirichlet boundary condition only. Also in [21], authors have considered a similar system with only power-law type fluid and the no-slip boundary condition ($\alpha = \infty$) and establish the existence of a global in time, weak solution. Let us also mention type of regularity results which was done for particular case when we consider the motion of rigid body and motion of incompressible fluid around the rigid body [38, 39].

Another aspect is the boundary conditions we analyze here, the so called slip boundary conditions, introduced by Navier [37] (the linear version), later proposed independently by Maxwell [34]. These conditions describe on one hand that the physical domain is impermeable and on other hand, the fluid may slip over the solid boundary, rather than sticking to it. Mathematically, this second condition is described as the fluid velocity need not be equal to the velocity of the solid boundary, rather the tangential component of the fluid velocity is proportional to the stress exerted by the fluid on the boundary and the proportionality constant is called the friction coefficient. Observe that the friction coefficient $\alpha = \infty$ gives the no-slip Dirichlet boundary condition, formally, while $\alpha = 0$ corresponds to the full slip conditions. Although the no-slip boundary conditions are the one widely studied and accepted in the context of fluid dynamics, there are many problems at the macro scale where the no-slip condition is not applicable, examples include the moving contact line problem [18] and the corner flows [29]. These and many other paradoxes may possibly appear because of the no-slip boundary conditions. Moreover, in the context of fluid-rigid body interaction the no-slip conditions give rise to so-called no-contact paradox which says that collision between the rigid body and the boundary will not occur in finite time [26, 43]. Therefore it is necessary to study more deeply the slip boundary condition experimentally, as well as mathematically. There are very few works done on the fluid-solid interaction system where slip boundary conditions are treated. Existence of a weak solution for the Newtonian case was proved in [12, 24], while the existence and uniqueness of local-in-time strong solutions were studied in [2, 48]. In [10] the author proved uniqueness of weak solution in the $2D$ case. Weak-strong uniqueness in $3D$ was studied in [13, 35]. Finally, we mention the existence result for weak solution in the case when the elastic structure is part of the fluid boundary [36] which is a $3D - 2D$ interaction problem. All the above-mentioned results are in L^2-setting, for the Newtonian fluids and the friction coefficient α is assumed either 0 or a constant.

Also at the molecular scale, Thompson and Trojan, based on their experiments, proposed a slip boundary condition which is highly non-linear [46], even though the fluid is still considered to be Newtonian (see also [33]). This universal condition may determine the degree of slip at a fluid-solid interface as the interfacial parameters and the shear rate are varied. But this non-linear boundary condition seems out of reach with our present mathematical technique. However, we may treat the non-linearity discussed by Lewandowski et al. [31] in the context of turbulence model

$$[\sigma(\boldsymbol{u}, \pi)\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha|\boldsymbol{u}|\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0}, \tag{1.4}$$

which is not much different from its linear counterpart, at least concerning the qualitative analysis.

Note that we can also consider the Dirichlet condition $\boldsymbol{u} = \boldsymbol{0}$ on the outer boundary $\partial\Omega$ instead of $(1.2)_3$ and study the system. We will see that this makes no big difference in the analysis. Therefore, we will mention this case only particialy, we describe the differences will be made precise.

Our main goal in this work is to develop an L^p-theory for strong solutions of the fluid-rigid body interaction system with slip boundary condition at the interface, for both Newtonian and non-Newtonian fluid. The main novelty of this work is to provide a unified result considering the linear as well as some non-linear slip condition at the fluid-solid interface, where the non-constant slip coefficient depends on the space.

We start with studying the Newtonian case. Since the domain $\Omega_F(t) \times (0,T)$ depends on the motion of the rigid body, this is a moving boundary problem where the domain is also an unknown a priori. Hence it is natural to transform the system (1.2) to a fixed domain and solve the problem there. Among several possibilities for this transformation, the usual one is a global, linear transformation which says the whole space is rigidly rotated and shifted back to its original position at each time $t > 0$ (cf. [22]). This refers the equations of motion of the fluid-rigid body system in a frame attached to the rigid body, with origin in the center of mass of the latter and coinciding with an inertial frame at time $t = 0$. One conceptual difference is that in [22], the fluid occupies an exterior domain where it is reasonable to perform such a transformation. But in our case where the fluid and solid contains a bounded domain, it is not suitable to choose such transformation. Also technically, this transformation generates an extra drift term of the form $[(\boldsymbol{\omega} \times \boldsymbol{y}) \cdot \nabla]\boldsymbol{u}$ which has unbounded coefficients. This produces a fundamental difficulty as the transformed problem is no longer parabolic. To overcome this difficulty, Tucsnak, Cumsille and Takahashi (cf. [44, 45, 15]) used another non-linear, local change of variables which only acts in a bounded neighbourhood of the body *i.e.* coincides with $Q(t)\boldsymbol{y} + \boldsymbol{h}(t)$ in a neighbourhood of the rigid body and is equal to the identity far from the rigid body. This transformation preserves the solenoidal condition of the fluid velocity and do not change the regularity of the solutions, although the rigid body equations change and become non-linear.

We follow this second approach with a different point of view. We use the rotation matrix Q instead of the Jacobian J_Y in the change of variables (cf. 7.5). In that sense it lies somewhat in between the above two methods. Although this transformation does not preserve the divergence free condition as compared to the second approach involving J_Y used in [44], it makes the corresponding estimates on the non-linear terms, appeared from the change of variable, much easier. After the change of variables, our strategy is based on the maximal regularity property of the linearized system. We extend the maximal regularity result for the Stokes problem to the fluid-rigid body system. At this step, we write the full system in terms of $\mathbb{P}\boldsymbol{u}, \boldsymbol{l}, \boldsymbol{\omega}$ only, where $\mathbb{P}$ is the Helmholtz projection. This helps us to achieve further the exponential stability of the system in the Newtonian case. We finally rewrite the full non-linear transformed problem as a fixed point problem and deduce several estimates on the coordinate transform and on the extra terms appearing from the transformed system which make the fixed point mapping contractive, provided the given data are small. This gives the existence of a unique strong solution in L^p spaces of the fluid-rigid body system.

Next we discuss the non-Newtonian fluid. Using the same transformation as before, we reduce the system on fixed domain and then linearize the corresponding operator by fixing the coefficients. To prove the maximal regularity of the linearized system, we can not follow the same approach as in the Newtonian case. Because of the complicated structure of the generalized operator, writing the full system in terms of only $\mathbb{P}\boldsymbol{u}, \boldsymbol{l}, \boldsymbol{\omega}$ is not possible. Thus we follow here the approach used in [23]. In the last subsection, we show that the same analysis can be done as well for the non-linear slip condition (1.4).

2 Main results

We assume that the rigid body at the initial position does not touch the wall of the fluid domain, i.e.

$$\text{dist}\,(\Omega_S(0), \partial\Omega) \geq \beta > 0.$$

For reference purpose, we rewrite the generalized system (1.2) in case of Newtonian fluid:

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla_x)\boldsymbol{u} = \text{div}\ \sigma(\boldsymbol{u},\pi) & \text{in}\ \ \Omega_F(t)\times(0,T), \\ \text{div}\ \boldsymbol{u} = 0 & \text{in}\ \ \Omega_F(t)\times(0,T), \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad 2\,[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on}\ \ \partial\Omega\times(0,T), \\ \boldsymbol{u}\cdot\boldsymbol{n} = \boldsymbol{u}_S\cdot\boldsymbol{n} & \text{on}\ \ \partial\Omega_S(t)\times(0,T), \\ 2\,[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha\boldsymbol{u}_{\boldsymbol{\tau}} = \alpha\boldsymbol{u}_{S\boldsymbol{\tau}} & \text{on}\ \ \partial\Omega_S(t)\times(0,T), \\ m\boldsymbol{l}'(t) = -\displaystyle\int_{\partial\Omega_S(t)} \sigma(\boldsymbol{u},\pi)\boldsymbol{n}, & t\in(0,T), \\ (J\boldsymbol{\omega})'(t) = -\displaystyle\int_{\partial\Omega_S(t)} (\boldsymbol{x}-\boldsymbol{h}(t))\times\sigma(\boldsymbol{u},\pi)\boldsymbol{n}, & t\in(0,T), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in}\ \ \Omega_F(0), \\ \boldsymbol{l}(0) = \boldsymbol{l}_0, \quad \boldsymbol{\omega}(0) = \boldsymbol{\omega}_0. \end{cases} \tag{2.1}$$

We use the following function spaces. For a domain $D \in \mathbb{R}^n, n \in \mathbb{N}$, the Sobolev spaces are denoted by $W^{m,q}(D)$. For every $0 < s < m$, $m \in \mathbb{N}$ and $1 \leq q < \infty, 1 \leq p \leq \infty$, we denote the Besov spaces by $B^s_{q,p}(D)$ which can be defined (equivalently) by real interpolation of Sobolev spaces (cf. [47, Section 1.6.4, page 39])

$$B^s_{q,p}(D) := \left(L^q(D), W^{m,q}(D)\right)_{s/m,p}. \tag{2.2}$$

We also introduce the notation, the subscript σ denotes the divergence free condition in the domain and the subscript $\boldsymbol{\tau}$ over a space denotes the zero normal component on the boundary. For example, we write:

$$\boldsymbol{L}^q_{\sigma,\boldsymbol{\tau}}(D) := \{\boldsymbol{v}\in\boldsymbol{L}^q(D) : \text{div}\ \boldsymbol{v} = 0 \text{ in } D, \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial D\}.$$

Let the Stokes operator with Navier boundary conditions on $\boldsymbol{L}^q_{\sigma,\boldsymbol{\tau}}(\Omega_F(0))$ is defined as,

$$\begin{cases} \mathcal{D}(A_q) := \{\boldsymbol{u}\in\boldsymbol{W}^{2,q}_{\sigma,\boldsymbol{\tau}}(\Omega_F(0)) : 2[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0} \text{ on } \partial\Omega_F(0)\}, \\ A_q\boldsymbol{u} = \mathbb{P}\Delta\boldsymbol{u} \quad \text{ for all } \boldsymbol{u}\in\mathcal{D}(A_q) \end{cases} \tag{2.3}$$

where α is such that

$$\alpha \in \begin{cases} W^{1-\frac{1}{\frac{3}{2}+\varepsilon},\frac{3}{2}+\varepsilon}(\partial\Omega_F(0)) & \text{if } \ 1 < p \leq \frac{3}{2} \\ W^{1-\frac{1}{p},p}(\partial\Omega_F(0)) & \text{if } \ p > \frac{3}{2} \end{cases} \tag{2.4}$$

with $\varepsilon > 0$ arbitrarily small and $\mathbb{P}$ is the Helmholtz projection

$$\mathbb{P} : \mathbf{L}^q(\Omega_F(0)) \to \mathbf{L}^q_{\sigma,\boldsymbol{\tau}}(\Omega_F(0))$$

i.e. for $\varphi \in \mathbf{L}^q(\Omega_F(0))$, $\mathbb{P}\varphi = \varphi - \nabla p$ for some $p \in W^{1,q}(\Omega_F(0))$ which satisfies

$$\begin{cases} \operatorname{div}(\nabla p - \varphi) = 0 & \text{in } \Omega_F(0) \\ (\nabla p - \varphi) \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega_F(0). \end{cases}$$

Also we say that $\boldsymbol{u}_0 \in B^{2-2/p}_{q,p}(\Omega_F(0))$ satisfies the compatibility condition if

$$\boldsymbol{u}_0 - \boldsymbol{v}_0 \in \left(\boldsymbol{L}^q_{\sigma,\boldsymbol{\tau}}(\Omega_F(0)), \mathcal{D}(A_q)\right)_{1-\frac{1}{p},p} \qquad \text{for some } \boldsymbol{v}_0 \in C^2(\Omega_F(0)) \text{ satisfying}$$

$$\begin{aligned} &\operatorname{div}\, \boldsymbol{v}_0 = 0 \ \text{ in } \Omega_F(0), \\ &\boldsymbol{v}_0 \cdot \boldsymbol{n} = (\boldsymbol{l}_0 + (\boldsymbol{\omega}_0 \times y)) \cdot \boldsymbol{n} \ \text{ on } \ \partial\Omega_S(0), \quad \boldsymbol{v}_0 \cdot \boldsymbol{n} = 0 \ \text{ on } \ \partial\Omega, \\ \text{and if } p > 3, \quad &\begin{cases} 2[(\mathbb{D}\boldsymbol{v}_0)\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{v}_{0\boldsymbol{\tau}} &= \boldsymbol{0} \ \text{ on } \partial\Omega, \\ 2[(\mathbb{D}\boldsymbol{v}_0)\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{v}_{0\boldsymbol{\tau}} &= \alpha(\boldsymbol{l}_0 + \boldsymbol{\omega}_0 \times \boldsymbol{y})_{\boldsymbol{\tau}} \ \text{ on } \partial\Omega_S(0). \end{cases} \end{aligned} \tag{2.5}$$

We can now state our main results on the existence of a unique, global in time, strong solutions for the Newtonian and the generalized Newtonian system (2.1) and (1.2) under the smallness assumption on data.

Theorem 2.1. *Let $\Omega_F(0)$ be a bounded domain of class $\mathcal{C}^{2,1}$, $p, q \in (1,\infty)$ satisfy the condition $\frac{1}{p} + \frac{3}{2q} \leq \frac{3}{2}$ and $\alpha \geq 0$ be as in (2.4). Let $\eta \in (0, \eta_0)$ where η_0 is some constant (specified in Theorem 4.8). Then there exist two constants $\delta_0 > 0$ and $C > 0$, depending only on p, q, η and $\Omega_F(0)$ such that for all $\delta \in (0, \delta_0)$ and for all $(\boldsymbol{u}_0, \boldsymbol{l}_0, \boldsymbol{\omega}_0) \in B^{2(1-1/p)}_{q,p}(\Omega_F(0)) \times \mathbb{R}^3 \times \mathbb{R}^3$ satisfying the compatibility conditions (2.5) and*

$$\|\boldsymbol{u}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega_F(0))} + \|\boldsymbol{l}_0\|_{\mathbb{R}^3} + \|\boldsymbol{\omega}_0\|_{\mathbb{R}^3} \leq \delta,$$

the system (2.1) possesses a unique global strong solution $(\boldsymbol{u}, \pi, \boldsymbol{l}, \boldsymbol{\omega})$ in the class of functions satisfying

$$\begin{aligned} &\|e^{\eta(\cdot)}\boldsymbol{u}\|_{L^p(0,\infty;\boldsymbol{W}^{2,q}(\Omega_F(\cdot)))} + \|e^{\eta(\cdot)}\boldsymbol{u}\|_{W^{1,p}(0,\infty;\boldsymbol{L}^q(\Omega_F(\cdot)))} + \|e^{\eta(\cdot)}\boldsymbol{u}\|_{L^\infty(0,\infty;B^{2(1-1/p)}_{q,p}(\Omega_F(\cdot)))} \\ &+ \|e^{\eta(\cdot)}\pi\|_{L^p(0,\infty;W^{1,q}(\Omega_F(\cdot)))} + \|e^{\eta(\cdot)}\boldsymbol{l}\|_{W^{1,p}(0,\infty;\mathbb{R}^3)} + \|e^{\eta(\cdot)}\boldsymbol{\omega}\|_{W^{1,p}(0,\infty;\mathbb{R}^3)} \leq C\delta. \end{aligned} \tag{2.6}$$

Moreover, $\operatorname{dist}(\Omega_S(t), \partial(\Omega) \geq \beta/2$ *for all $t \in [0,\infty)$.*

In particular, we have,

$$\|\boldsymbol{u}(\cdot,t)\|_{B^{2(1-1/p)}_{q,p}(\Omega_F(t))} + \|\boldsymbol{l}(t)\|_{\mathbb{R}^3} + \|\boldsymbol{\omega}(t)\|_{\mathbb{R}^3} \leq C\delta e^{-\eta t}.$$

Remark 2.2. *If we consider $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega \times (0,T)$ in $(2.1)_3$ instead of the slip condition, then we obtain the same result as above, provided the compatibility condition for the initial data needs to be replaced with*

$$\boldsymbol{u}_0 - \boldsymbol{v}_0 \in \left(\boldsymbol{L}^q_{\sigma,\boldsymbol{\tau}}(\Omega_F(0)), \mathcal{D}(A^D_q)\right)_{1-\frac{1}{p},p} \qquad \textit{for some } \boldsymbol{v}_0 \in C^2(\Omega_F(0)) \textit{ satisfying}$$

$$\begin{aligned} &\operatorname{div}\, \boldsymbol{v}_0 = 0 \ \textit{ in } \Omega_F(0), \\ &\boldsymbol{v}_0 \cdot \boldsymbol{n} = (\boldsymbol{l}_0 + (\boldsymbol{\omega}_0 \times y)) \cdot \boldsymbol{n} \ \textit{ on } \ \partial\Omega_S(0), \quad \boldsymbol{v}_0 = 0 \ \textit{ on } \ \partial\Omega, \\ \textit{and} \quad &2[(\mathbb{D}\boldsymbol{v}_0)\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{v}_{0\boldsymbol{\tau}} = \alpha(\boldsymbol{l}_0 + \boldsymbol{\omega}_0 \times \boldsymbol{y})_{\boldsymbol{\tau}} \ \textit{ on } \partial\Omega_S(0) \quad \textit{ if } p > 3. \end{aligned}$$

Here $A_q^D : \boldsymbol{L}^q_{\sigma,\boldsymbol{\tau}}(\Omega_F(0)) \to \boldsymbol{L}^q_{\sigma,\boldsymbol{\tau}}(\Omega_F(0))$ *denotes the following Stokes operator with Dirichlet boundary condition at the fluid boundary:*

$$\begin{cases} \mathcal{D}(A_q^D) := \{\boldsymbol{u} \in \boldsymbol{W}^{2,q}_{\sigma,\boldsymbol{\tau}}(\Omega_F(0)); \boldsymbol{u} = \boldsymbol{0} \text{ on } \partial\Omega,\ 2[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0} \text{ on } \partial\Omega_F(0)\}, \\ A_q \boldsymbol{u} = \mathbb{P}\Delta \boldsymbol{u} \quad \text{for all } \boldsymbol{u} \in \mathcal{D}(A_q^D) \end{cases}$$

Theorem 2.3. *Let* $p > 5$, $\Omega_F(0)$ *be a bounded domain of class* $\mathcal{C}^{2,1}$ *and* $\alpha \geq 0$ *satisfies (2.4). Then there exists a constant* $\delta_0 > 0$ *depending only on* p *and* $\Omega_F(0)$ *such that for all* $\delta \in (0, \delta_0)$ *and for all* $(\boldsymbol{u}_0, \boldsymbol{l}_0, \boldsymbol{\omega}_0) \in W^{2-2/p,p}(\Omega_F(0)) \times \mathbb{R}^3 \times \mathbb{R}^3$ *satisfying the compatibility conditions (2.5) and*

$$\|\boldsymbol{u}_0\|_{W^{2-2/p,p}(\Omega_F(0))} + \|\boldsymbol{l}_0\|_{\mathbb{R}^3} + \|\boldsymbol{\omega}_0\|_{\mathbb{R}^3} \leq \delta,$$

the problem (1.2) admits a unique strong solution

$$\boldsymbol{u} \in L^p(0,\infty; \boldsymbol{W}^{2,p}(\Omega_F(\cdot))) \cap W^{1,p}(0,\infty; \boldsymbol{L}^p(\Omega_F(\cdot))),$$
$$\pi \in L^p(0,\infty; W^{1,p}(\Omega_F(\cdot))), \boldsymbol{l} \in W^{1,p}(0,\infty;\mathbb{R}^3), \boldsymbol{\omega} \in W^{1,p}(0,\infty;\mathbb{R}^3).$$

Remark 2.4. *Since the generalized stress tensor* T *includes the Newtonian stress tensor* σ *as the special case, Theorem 2.3 generalizes Theorem 2.1. On the other hand, less restrictive assumptions on* p, q *are needed in Theorem 2.1, compared to Theorem 2.3.*

Our last result concerns the nonlinear slip condition (1.4).

Theorem 2.5. *Let* $p > 5$, $\Omega_F(0)$ *be a bounded domain of class* $\mathcal{C}^{2,1}$ *and* $\alpha \geq 0$ *satisfies (2.4). Then there exists a constant* $\delta_0 > 0$ *depending only on* p *and* $\Omega_F(0)$ *such that for all* $\delta \in (0, \delta_0)$ *and for all* $(\boldsymbol{u}_0, \boldsymbol{l}_0, \boldsymbol{\omega}_0) \in W^{2-2/p,p}(\Omega_F(0)) \times \mathbb{R}^3 \times \mathbb{R}^3$ *satisfying the compatibility conditions (2.5) and*

$$\|\boldsymbol{u}_0\|_{W^{2-2/p,p}(\Omega_F(0))} + \|\boldsymbol{l}_0\|_{\mathbb{R}^3} + \|\boldsymbol{\omega}_0\|_{\mathbb{R}^3} \leq \delta,$$

the problem (1.2) with the boundary conditions $(1.2)_3 - (1.2)_5$ *replaced by the nonlinear slip condition (1.4) admits a unique strong solution*

$$\boldsymbol{u} \in L^p(0,\infty; \boldsymbol{W}^{2,p}(\Omega_F(\cdot))) \cap W^{1,p}(0,\infty; \boldsymbol{L}^p(\Omega_F(\cdot))),$$
$$\pi \in L^p(0,\infty; W^{1,p}(\Omega_F(\cdot))), \boldsymbol{l} \in W^{1,p}(0,\infty;\mathbb{R}^3), \boldsymbol{\omega} \in W^{1,p}(0,\infty;\mathbb{R}^3).$$

3 Preliminaries and notations

In this section, we introduce the notation used throughout this paper, in particular some results concerning maximal regularity and $\mathcal{R}$-boundedness in Banach spaces.

For Banach spaces X and Y, we denote the space of all bounded, linear operators from X to Y by $\mathcal{L}(X,Y)$. The resolvent set of a linear operator A is denoted by $\rho(A)$. The domain of an operator A is denoted by $\mathcal{D}(A)$. Whenever we consider $\mathcal{D}(A)$ as a Banach space, it is assumed to be equipped with the graph norm of A.

We consider the following problem:

$$\begin{cases} u'(t) + Au(t) = f(t),\ t \geq 0 \\ u(0) = u_0. \end{cases} \tag{3.1}$$

Definition 3.1. *Let $p \in (1,\infty)$. We say that A has the maximal L^p-regularity property on the interval I (with $I = [0,T]$ or $I = [0,\infty)$) if there exists a constant $C > 0$ such that for all $f \in L^p(I;X)$, there is a unique $u \in L^p(I;\mathcal{D}(A))$ with $u' \in L^p(I;X)$ satisfying (3.1) with $u_0 = 0$ for almost every $t \in I$ and*

$$\|u\|_{L^p(I;X)} + \|u'\|_{L^p(I;X)} + \|Au\|_{L^p(I;X)} \leq C\|f\|_{L^p(I;X)}.$$

We also need to define the notion of UMD-space (unconditional difference martingale property). Actually we give here a property of UMD-spaces which is equivalent to the original definition (for more on this subject, see [11], [9]). The Hilbert transform $\mathcal{H}f$ of a measurable function f is, whenever it exists, the limit as $\varepsilon \to 0^+$ and $T \to +\infty$ of

$$\mathcal{H}_{\varepsilon,T}f(t) = \frac{1}{\pi}\int\limits_{\varepsilon\leq|s|\leq T}\frac{f(t-s)}{s}\mathrm{d}s, \quad t \in \mathbb{R}.$$

Definition 3.2. *A complex Banach space is said to be of class UMD if the Hilbert transform $\mathcal{H}$ is bounded in $L^p(\mathbb{R};X)$ for all (or equivalently, for one) $p \in (1,\infty)$.*

These spaces are also called of class $\mathcal{HT}$. Any Hilbert space is in the class UMD. If X is a Banach space in the UMD-class, then $L^p(\Omega;X)$ for $\Omega \subset \mathbb{R}^n$ and $p \in (1,\infty)$ is also in the UMD-class.

Now we want to state an equivalent property to maximal regularity in terms of $\mathcal{R}$-boundedness of the resolvent of the operator. For further details on $\mathcal{R}$-boundedness, refer to [50].

Definition 3.3. *A set $\mathcal{T} \subset \mathcal{L}(X,Y)$ is called $\mathcal{R}$-bounded if there is a constant $C > 0$ such that for all $n \in \mathbb{N}$, $T_1,...,T_n \in \mathcal{T}$ and $x_1,...,x_n \in X$,*

$$\int\limits_0^1 \Big\|\sum_{j=1}^n r_j(s)T_jx_j\Big\|_Y \mathrm{d}s \leq C\int\limits_0^1 \big\|\sum_{j=1}^n r_j(s)x_j\big\|_X \mathrm{d}s$$

where $\{r_j\}_{j=1,...,n}$ is a sequence of independent $\{-1,1\}$-valued random variables on $[0,1]$.

The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$, we denote it by $\mathcal{R}(\mathcal{T})$.

We also collect some useful properties of $\mathcal{R}$-boundedness which will be used later. For proof, see [40, Remark 4.1.3, Proposition 4.1.6].

Proposition 3.4.

1. *If $\mathcal{T} \subset \mathcal{L}(X,Y)$ is $\mathcal{R}$-bounded, then it is uniformly bounded with*

$$\sup\{|T| : T \in \mathcal{T}\} \leq \mathcal{R}(\mathcal{T}).$$

2. *If X and Y are Hilbert spaces, a set $\mathcal{T} \subset \mathcal{L}(X,Y)$ is $\mathcal{R}$-bounded if and only if it is bounded.*

3. *Let X,Y be Banach spaces and $\mathcal{T},\mathcal{S} \subset \mathcal{L}(X,Y)$ be $\mathcal{R}$-bounded. Then $\mathcal{T}+\mathcal{S}$ is $\mathcal{R}$-bounded as well and*

$$\mathcal{R}\left(\mathcal{T}+\mathcal{S}\right) \leq \mathcal{R}(\mathcal{T}) + \mathcal{R}(\mathcal{S}).$$

4. Let X, Y, Z be Banach spaces and $\mathcal{T} \subset \mathcal{L}(X,Y)$ and $\mathcal{S} \subset \mathcal{L}(Y,Z)$ be $\mathcal{R}$-bounded. Then $\mathcal{ST}$ is also $\mathcal{R}$-bounded and

$$\mathcal{R}(\mathcal{ST}) \leq \mathcal{R}(\mathcal{S})\mathcal{R}(\mathcal{T}).$$

To this end, let us introduce another notion for the sake of being in line with the references. Let us denote the sector in the complex plane

$$\Sigma_\theta := \{\lambda \in \mathbb{C}\backslash\{0\} : |\arg \lambda| < \theta\}, \quad \theta \in (0, \pi).$$

Definition 3.5. [30, *page* 417]. *Let X be a complex Banach space and $A : \mathcal{D}(A) \subseteq X \to X$ be a densely defined, closed, linear operator. A is said to be sectorial if $(0, \infty) \subset \rho(A)$, has dense range and there exists some $\theta > 0$ such that*

$$|\lambda(\lambda I - A)^{-1}| \leq C, \quad \lambda \in \Sigma_\theta.$$

for some constant $C < \infty$. Moreover, A is called $\mathcal{R}$-sectorial if $\{\lambda(\lambda I - A)^{-1} : \lambda \in \Sigma_\theta\}$ is $\mathcal{R}$-bounded.

The $\mathcal{R}$-angle of A is defined by

$$\theta_r(A) := \inf\{\theta \in (0, \pi) : \mathcal{R}(\{\lambda(\lambda I - A)^{-1} : \lambda \in \Sigma_{\pi-\theta}\}) < \infty\}.$$

The next characterization which is due to Weis [50, Theorem 4.2], is the key tool to prove the existence of a strong solution of (2.1).

Theorem 3.6. *Let X be an UMD-space and A be a generator of a bounded analytic semigroup. Then A has maximal L^p-regularity if and only if there exists a $\theta > 0$ such that*

$$\mathcal{R}\left(\{\lambda(\lambda I - A)^{-1} : \lambda \in \Sigma_{\frac{\pi}{2}+\theta}\}\right) < \infty.$$

In other words, A has maximal L^p-regularity if and only if A is $\mathcal{R}$-sectorial of angle $\theta_r(A) > \pi/2$.

Recall that A generates a bounded analytic semigroup in X if and only if $\{\lambda(\lambda I - A)^{-1} : \lambda \in \Sigma_{\frac{\pi}{2}+\theta}\}$ is bounded for some $\theta > 0$ (cf. [19, Theorem 4.6, Section II, page 101]).

Also if X is an UMD-space and the operator A has bounded imaginary powers, then A has maximal L^p-regularity, by the well-known Dore-Venni result [17, Theorem 3.2].

The above characterization in Theorem 3.6 provides a convenient tool to check maximal regularity for concrete operators, as we will show in the next section. We will also need some perturbation results which we state below (although we did not find the proof and its exact reference; A slight variation has been proved in [30, Corollary 2]).

Theorem 3.7. [49, *Corollary*, *page* 207]. *Let A generates a bounded analytic semigroup on an UMD-space X and B be a linear operator satisfying $\mathcal{D}(B) \supset \mathcal{D}(A)$ and*

$$\|Bx\| \leq a\|Ax\| + b\|x\|, \quad x \in \mathcal{D}(A).$$

If A has maximal L^p-regularity and a is small enough, e.g. $a < (1 + C)^{-2}$ where C is the $\mathcal{R}$-bound of $\{A(\lambda I - A)^{-1} : \lambda \in \Sigma_{\pi-\theta}\}$, then $A + B$ has maximal L^p-regularity on $[0, T]$ for all $T < \infty$.

We conclude this section by stating the following well-known result (see for example [25, Theorem 2.3], [3, Theorem 4.10.7, Chapter III]) which deals with the maximal L^p-regularity of the Cauchy problem (3.1).

Proposition 3.8. *Suppose X be a Banach space of class UMD, $p \in (1, \infty)$ and let A be a $\mathcal{R}$-sectorial operator with $\theta_r(A) > \pi/2$. Then (3.1) has a unique solution $u \in W^{1,p}(0, \infty; X) \cap L^p(0, \infty; \mathcal{D}(A))$ if and only if $f \in L^p(0, \infty; X)$ and $u_0 \in (X, \mathcal{D}(A))_{1-\frac{1}{p},p}$.*

4 Linear problem

After changing the full non-linear system (2.1) on a fixed domain (see Appendix), we would like to first study the corresponding linearized problem. Fixing all the non-linear terms in (7.7), the remaining linear system reduces to the following form (for notational convenience, in this section, we omit the tilda on the variables):

$$\begin{cases} \partial_t \boldsymbol{u} - \operatorname{div} \sigma(\boldsymbol{u}, \pi) = \boldsymbol{f} & \text{in } \Omega_F(0) \times (0,T), \\ \operatorname{div} \boldsymbol{u} = \operatorname{div} \boldsymbol{h} & \text{in } \Omega_F(0) \times (0,T), \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad 2\left[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \partial\Omega \times (0,T), \\ \boldsymbol{u} \cdot \boldsymbol{n} = (\boldsymbol{l} + \boldsymbol{\omega} \times \boldsymbol{y}) \cdot \boldsymbol{n} & \text{on } \partial\Omega_S(0) \times (0,T), \\ 2\left[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \alpha(\boldsymbol{l} + \boldsymbol{\omega} \times \boldsymbol{y})_{\boldsymbol{\tau}} & \text{on } \partial\Omega_S(0) \times (0,T), \\ m\boldsymbol{l}' = -\displaystyle\int_{\partial\Omega_S(0)} \sigma(\boldsymbol{u}, \pi)\boldsymbol{n} + \boldsymbol{g}_1, & t \in (0,T), \\ J(0)\boldsymbol{\omega}' = -\displaystyle\int_{\partial\Omega_S(0)} \boldsymbol{y} \times \sigma(\boldsymbol{u}, \pi)\boldsymbol{n} + \boldsymbol{g}_2, & t \in (0,T), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in } \Omega_F(0), \\ \boldsymbol{l}(0) = \boldsymbol{l}_0, \quad \boldsymbol{\omega}(0) = \boldsymbol{\omega}_0. \end{cases} \tag{4.1}$$

We want to re-formulate the system (4.1) in the form:

$$z'(t) = Az(t) + f(t), \quad z(0) = z_0$$

or equivalently, to approach via semigroup theory, we want to consider the corresponding resolvent problem. First we treat the system with divergence-free condition and then return to the full inhomogeneous divergence condition.

As the classical approach, we need to eliminate the pressure from both the fluid and the structure equations. The standard way to eliminate pressure from the fluid equations is to invoke the Helmholtz projection (cf. [28]). But we also decompose the velocity field into $\mathbb{P}\boldsymbol{u}$ and $(I_3 - \mathbb{P})\boldsymbol{u}$ which is crucial since the pressure which is eliminated from the fluid equations using the projector $\mathbb{P}$, also appears in the structure equations.

4.1 Resolvent problem

Given $\lambda \in \mathbb{C}$, $\boldsymbol{f} \in \mathbf{L}^q(\Omega_F(0))$ and $(\mathbf{g}_1, \mathbf{g}_2) \in \mathbb{C}^3 \times \mathbb{C}^3$, consider the system

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega_F(0), \\ \text{div } \boldsymbol{u} = 0 & \text{in } \Omega_F(0), \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad 2\left[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \mathbf{0} & \text{on } \partial\Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = (\boldsymbol{l} + \boldsymbol{\omega} \times \boldsymbol{y}) \cdot \boldsymbol{n} & \text{on } \partial\Omega_S(0), \\ 2\left[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \alpha(\boldsymbol{l} + \boldsymbol{\omega} \times \boldsymbol{y})_{\boldsymbol{\tau}} & \text{on } \partial\Omega_S(0), \\ \lambda m \boldsymbol{l} = -\displaystyle\int_{\partial\Omega_S(0)} \sigma(\boldsymbol{u}, \pi)\boldsymbol{n} + \boldsymbol{g}_1, & \\ \lambda J(0)\boldsymbol{\omega} = -\displaystyle\int_{\partial\Omega_S(0)} \boldsymbol{y} \times \sigma(\boldsymbol{u}, \pi)\boldsymbol{n} + \boldsymbol{g}_2. & \end{cases} \tag{4.2}$$

The following existence result governing the steady fluid equations is required to reformulate the fluid part in the above system.

Proposition 4.1. *Let $q \in (1, \infty)$ and $\alpha \geq 0$ be as in (2.4). Given $(\boldsymbol{l}, \boldsymbol{\omega}) \in \mathbb{C}^3 \times \mathbb{C}^3$, there exists a unique solution $(\boldsymbol{v}, \psi) \in \boldsymbol{W}^{2,q}(\Omega_F(0)) \times W^{1,q}(\Omega_F(0))$ of the following Stokes problem*

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla \psi = \mathbf{0} & \text{in } \Omega_F(0), \\ \text{div } \boldsymbol{v} = 0 & \text{in } \Omega_F(0), \\ \boldsymbol{v} \cdot \boldsymbol{n} = 0, \quad 2\left[(\mathbb{D}\boldsymbol{v})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha \boldsymbol{v}_{\boldsymbol{\tau}} = \mathbf{0} & \text{on } \partial\Omega, \\ \boldsymbol{v} \cdot \boldsymbol{n} = (\boldsymbol{l} + \boldsymbol{\omega} \times \boldsymbol{y}) \cdot \boldsymbol{n} & \text{on } \partial\Omega_S(0), \\ 2\left[(\mathbb{D}\boldsymbol{v})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha \boldsymbol{v}_{\boldsymbol{\tau}} = \alpha(\boldsymbol{l} + \boldsymbol{\omega} \times \boldsymbol{y})_{\boldsymbol{\tau}} & \text{on } \partial\Omega_S(0). \end{cases} \tag{4.3}$$

Proof. See [1, Theorem 2.1]. ■

Let us denote $S(\boldsymbol{l}, \boldsymbol{\omega}) := \boldsymbol{v}$, $S_{pr}(\boldsymbol{l}, \boldsymbol{\omega}) := \psi$. Also denote the Neumann operator

$$N : W^{1-1/q,q}(\partial\Omega_F(0)) \to W^{2,q}(\Omega_F(0))$$
$$h \mapsto \boldsymbol{\varphi}$$

where $\boldsymbol{\varphi}$ solves $\Delta \boldsymbol{\varphi} = 0$ in $\Omega_F(0)$, $\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} = h$ on $\partial\Omega_F(0)$. Set $N_S(h) := N(\mathbb{1}_{\partial\Omega_S(0)} h)$ for any $h \in W^{1-1/q,q}(\partial\Omega_S(0))$.

By extrapolation, we extend the Stokes operator A_q defined in (2.3) to an unbounded operator $\tilde{A}_q$ with domain $\mathcal{D}(\tilde{A}_q) := \boldsymbol{L}^q_{\sigma,\boldsymbol{\tau}}(\Omega_F(0))$ on $\mathcal{D}((A_q)^*)' = \mathcal{D}(A_q)'$, so that $(\tilde{A}_q, \mathcal{D}(A_q)')$ be the infinitesimal generator of a strongly continuous semigroup on $\mathcal{D}(A_q)'$, satisfying

$$\tilde{A}_q \boldsymbol{\varphi} = A_q \boldsymbol{\varphi} \quad \forall\ \boldsymbol{\varphi} \in \mathcal{D}(A_q).$$

Here $A*$ denotes the adjoint operator of A and X' denotes the dual space of X.

In the following proposition, we write an equivalent formulation of the fluid part of the resolvent problem (4.2). We decompose the fluid velocity into $\mathbb{P}\boldsymbol{u}$ and $(I_3 - \mathbb{P})\boldsymbol{u}$ which was introduced for Stokes problem in [42]. This decoupling enables us to write the pressure in terms of $\mathbb{P}\boldsymbol{u}, \boldsymbol{l}, \boldsymbol{\omega}$ which will be useful to eliminate the pressure from the structure equation.

Proposition 4.2. *Let $q \in (1,\infty)$, $\alpha \geq 0$ be as in (2.4) and $(\boldsymbol{f}, \boldsymbol{l}, \omega) \in \boldsymbol{L}^q_{\sigma,\boldsymbol{\tau}}(\Omega_F(0)) \times \mathbb{C}^3 \times \mathbb{C}^3$. Then $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,q}(\Omega_F(0)) \times W^{1,q}(\Omega_F(0))$ satisfies the system*

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega_F(0), \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad 2\left[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \partial\Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = (\boldsymbol{l} + \boldsymbol{\omega} \times \boldsymbol{x}) \cdot \boldsymbol{n}, \quad 2\left[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \alpha(\boldsymbol{l} + \boldsymbol{\omega} \times \boldsymbol{x})_{\boldsymbol{\tau}} & \text{on } \partial\Omega_S(0) \end{cases} \tag{4.4}$$

iff

$$\begin{cases} \lambda \mathbb{P}\boldsymbol{u} - \tilde{A}_q \mathbb{P}\boldsymbol{u} + \tilde{A}_q \mathbb{P} S(\boldsymbol{l}, \boldsymbol{\omega}) = \mathbb{P}\boldsymbol{f} \\ (I_3 - \mathbb{P})\boldsymbol{u} = (I_3 - \mathbb{P}) S(\boldsymbol{l}, \boldsymbol{\omega}) \\ \pi = N(\Delta \mathbb{P}\boldsymbol{u} \cdot \boldsymbol{n}) - \lambda N_S((\boldsymbol{l} + \boldsymbol{\omega} \times \boldsymbol{x}) \cdot \boldsymbol{n}). \end{cases} \tag{4.5}$$

Proof. Let $(\boldsymbol{u}, \pi)$ satisfies (4.4). Denote $(\tilde{\boldsymbol{u}}, \tilde{\pi}) := (\boldsymbol{u} - S(\boldsymbol{l}, \boldsymbol{\omega}), \pi - S_{pr}(\boldsymbol{l}, \boldsymbol{\omega}))$. Then $(\tilde{\boldsymbol{u}}, \tilde{\pi})$ satisfies

$$\begin{cases} \lambda \tilde{\boldsymbol{u}} - \Delta \tilde{\boldsymbol{u}} + \nabla \tilde{\pi} = \boldsymbol{f} - \lambda S(\boldsymbol{l}, \boldsymbol{\omega}), \quad \operatorname{div} \tilde{\boldsymbol{u}} = 0 & \text{in } \Omega_F(0), \\ \tilde{\boldsymbol{u}} \cdot \boldsymbol{n} = 0, \quad 2\left[(\mathbb{D}\tilde{\boldsymbol{u}})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha \tilde{\boldsymbol{u}}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \partial\Omega_F(0). \end{cases}$$

This shows $\tilde{\boldsymbol{u}} \in \mathcal{D}(A_q)$ and $\mathbb{P}\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}$. Therefore, applying the projection $\mathbb{P}$ on the 1st equation of the above system, we get

$$\lambda \mathbb{P}(\tilde{\boldsymbol{u}} + S(\boldsymbol{l}, \boldsymbol{\omega})) - A_q \tilde{\boldsymbol{u}} + \mathbb{P}\nabla\tilde{\pi} = \mathbb{P}\boldsymbol{f}. \tag{4.6}$$

But, note that

$$\begin{aligned} -A_q \tilde{\boldsymbol{u}} + \mathbb{P}\nabla\tilde{\pi} = \mathbb{P}(-\Delta\tilde{\boldsymbol{u}} + \nabla\tilde{\pi}) &= \mathbb{P}\mathbb{P}(-\Delta\tilde{\boldsymbol{u}}) \\ &= \mathbb{P}(-\Delta\tilde{\boldsymbol{u}}) = -A_q\tilde{\boldsymbol{u}} = -A_q\mathbb{P}\tilde{\boldsymbol{u}} = -A_q\mathbb{P}(\boldsymbol{u} - S(\boldsymbol{l}, \boldsymbol{\omega})). \end{aligned}$$

So we obtain from (4.6), $\lambda \mathbb{P}\boldsymbol{u} - A_q \mathbb{P}\boldsymbol{u} + A_q \mathbb{P} S(\boldsymbol{l}, \boldsymbol{\omega}) = \mathbb{P}\boldsymbol{f}$ in $\Omega_F(0)$.

Also, as $\mathbb{P}\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}$ *i.e.* $(I_3 - \mathbb{P})\tilde{\boldsymbol{u}} = \boldsymbol{0}$, we deduce $(I_3 - \mathbb{P})\boldsymbol{u} = (I_3 - \mathbb{P})S(\boldsymbol{l}, \boldsymbol{\omega})$.

Furthermore, from (4.4), taking divergence in the 1st equation yields, $\Delta\pi = 0$ in $\Omega_F(0)$; And since $\Delta(I_3 - \mathbb{P})\boldsymbol{u} = \boldsymbol{0}$ in $\Omega_F(0)$ (follows from the properties of Helmholtz projection),

$$\frac{\partial \pi}{\partial \boldsymbol{n}}\Big|_{\partial\Omega_F(0)} = \Delta\mathbb{P}\boldsymbol{u} \cdot \boldsymbol{n} - \lambda \boldsymbol{u} \cdot \boldsymbol{n} = \begin{cases} \Delta \mathbb{P}\boldsymbol{u} \cdot \boldsymbol{n} & \text{on } \partial\Omega \\ \Delta \mathbb{P}\boldsymbol{u} \cdot \boldsymbol{n} - \lambda(\boldsymbol{l} + \boldsymbol{\omega} \times \boldsymbol{x}) \cdot \boldsymbol{n} & \text{on } \partial\Omega_S(0). \end{cases}$$

Therefore, the expression of π in (4.5) follows from the definition of the operators N and N_S.

Conversely, let $\boldsymbol{u} \in \boldsymbol{W}^{2,q}(\Omega_F(0))$ satisfies the system (4.5). Since we have $(I_3 - \mathbb{P})\boldsymbol{u} = (I_3 - \mathbb{P})S(\boldsymbol{l}, \boldsymbol{\omega})$, defining $\tilde{\boldsymbol{u}} := \boldsymbol{u} - S(\boldsymbol{l}, \boldsymbol{\omega})$ we get $\tilde{\boldsymbol{u}} \in \boldsymbol{L}^q_{\sigma,\boldsymbol{\tau}}(\Omega_F(0))$ and $\mathbb{P}\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}$. Thus the 1st equation of (4.5) can be written as

$$\tilde{A}_q \tilde{\boldsymbol{u}} = \mathbb{P}(\lambda \boldsymbol{u} - \boldsymbol{f}) =: \boldsymbol{h}.$$

But since, $\boldsymbol{h} \in \boldsymbol{L}^q_{\sigma,\boldsymbol{\tau}}(\Omega)$ and $\tilde{A}_q$ is the generator of a strongly continuous semigroup (in fact, the maximal monotone property of the operator A_q and $\tilde{A}_q$ is sufficient), then $\tilde{\boldsymbol{u}} \in \mathcal{D}(A_q)$ and hence, the boundary conditions in (4.4) is satisfied by $\boldsymbol{u}$.

[The proof of the above statement is very simple and holds for general unbounded operators. For completeness, we mention it here: Let A be a maximal dissipative operator in a

Banach space X with dense domain $D(A)$ and A_{-1} be its extension by extrapolation in X_{-1} with domain X. If $x \in X$ is such that $A_{-1}x \in X$, then $x \in D(A)$ and $A_{-1}x = Ax$.

Proof: Define, $f = x + A_{-1}x \in X$. Since A is maximum dissipative, there exists $y \in D(A)$ such that $y + Ay = f$. Hence, $y + A_{-1}y = f$. But as A_{-1} is dissipative, it follows $x = y \in D(A)$.]

Now we write once again the 1st equation of (4.5) in terms of $\tilde{\boldsymbol{u}}$ as,

$$\lambda\tilde{\boldsymbol{u}} - A_q\tilde{\boldsymbol{u}} = \mathbb{P}(\boldsymbol{f} - \lambda S(\boldsymbol{l}, \boldsymbol{\omega})).$$

Therefore, from the characterization of $(I - \mathbb{P})$, there exists $\tilde{\pi} \in W^{1,q}(\Omega_F(0))$ such that

$$\lambda\tilde{\boldsymbol{u}} - \Delta\tilde{\boldsymbol{u}} + \nabla\tilde{\pi} = \boldsymbol{f} - \lambda S(\boldsymbol{l}, \boldsymbol{\omega}).$$

Then $(\boldsymbol{u}, \pi)$ with $\pi = \tilde{\pi} + S_{pr}(\boldsymbol{l}, \boldsymbol{\omega})$ satisfies (4.4). ■

Now using the expression of the pressure obtained above, we can re-write the two equations in (4.2) satisfied by $\boldsymbol{l}$ and $\boldsymbol{\omega}$.

$$\begin{aligned}
\lambda m\boldsymbol{l} &= -2\int_{\partial\Omega_S(0)} (\mathbb{D}\boldsymbol{u})\boldsymbol{n} + \int_{\partial\Omega_S(0)} \pi\boldsymbol{n} + \boldsymbol{g}_1 \\
&= -2\left[\int_{\partial\Omega_S(0)} (\mathbb{D}(\mathbb{P}\boldsymbol{u}))\boldsymbol{n} + \int_{\partial\Omega_S(0)} \mathbb{D}((I_3 - \mathbb{P})S(\boldsymbol{l}, \boldsymbol{\omega}))\boldsymbol{n}\right] \\
&\quad + \int_{\partial\Omega_S(0)} N(\Delta\mathbb{P}\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n} - \lambda\int_{\partial\Omega_S(0)} N_S((\boldsymbol{l} + \boldsymbol{\omega}\times\boldsymbol{y})\cdot\boldsymbol{n})\boldsymbol{n} + \boldsymbol{g}_1
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
\lambda J\boldsymbol{\omega} &= -2\int_{\partial\Omega_S(0)} \boldsymbol{y}\times(\mathbb{D}\boldsymbol{u})\boldsymbol{n} + \int_{\partial\Omega_S(0)} \boldsymbol{y}\times\pi\boldsymbol{n} + \boldsymbol{g}_2 \\
&= -2\left[\int_{\partial\Omega_S(0)} \boldsymbol{y}\times(\mathbb{D}(\mathbb{P}\boldsymbol{u}))\boldsymbol{n} + \int_{\partial\Omega_S(0)} \boldsymbol{y}\times\mathbb{D}((I_3 - \mathbb{P})S(\boldsymbol{l}, \boldsymbol{\omega}))\boldsymbol{n}\right] \\
&\quad + \int_{\partial\Omega_S(0)} \boldsymbol{y}\times N(\Delta\mathbb{P}\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n} - \lambda\int_{\partial\Omega_S(0)} \boldsymbol{y}\times N_s((\boldsymbol{l} + \boldsymbol{\omega}\times\boldsymbol{y})\cdot\boldsymbol{n})\boldsymbol{n} + \boldsymbol{g}_2.
\end{aligned} \tag{4.8}$$

So, (4.7) and (4.8) can be written combindedly in the following form:

$$\lambda K\begin{pmatrix}\boldsymbol{l}\\ \boldsymbol{\omega}\end{pmatrix} = C_1\mathbb{P}\boldsymbol{u} + C_2\begin{pmatrix}\boldsymbol{l}\\ \boldsymbol{\omega}\end{pmatrix} + \begin{pmatrix}\boldsymbol{g}_1\\ \boldsymbol{g}_2\end{pmatrix}$$

where

$$K = \mathbb{I} + M$$

with

$$\mathbb{I} = \begin{pmatrix} mI_3 & 0 \\ 0 & J \end{pmatrix}_{6\times 6}$$

be the constant momentum matrix,

$$M \begin{pmatrix} \boldsymbol{l} \\ \boldsymbol{\omega} \end{pmatrix} = \begin{pmatrix} \int_{\partial\Omega_S(0)} N_S((\boldsymbol{l} + \boldsymbol{\omega} \times \boldsymbol{y}) \cdot \boldsymbol{n})\boldsymbol{n} \\ \int_{\partial\Omega_S(0)} \boldsymbol{y} \times N_S((\boldsymbol{l} + \boldsymbol{\omega} \times \boldsymbol{y}) \cdot \boldsymbol{n})\boldsymbol{n} \end{pmatrix}_{6\times 1}$$

be the added mass matrix,

$$C_1 \mathbb{P}\boldsymbol{u} = \begin{pmatrix} -2\int_{\partial\Omega_S(0)} (\mathbb{D}(\mathbb{P}\boldsymbol{u}))\boldsymbol{n} + \int_{\partial\Omega_S(0)} N(\Delta\mathbb{P}\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n} \\ -2\int_{\partial\Omega_S(0)} \boldsymbol{y} \times (\mathbb{D}(\mathbb{P}\boldsymbol{u}))\boldsymbol{n} + \int_{\partial\Omega_S(0)} \boldsymbol{y} \times N(\Delta\mathbb{P}\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n} \end{pmatrix}_{6\times 1}$$

and

$$C_2 \begin{pmatrix} \boldsymbol{l} \\ \boldsymbol{\omega} \end{pmatrix} = \begin{pmatrix} \int_{\partial\Omega_S(0)} \mathbb{D}((I_3 - \mathbb{P})S(\boldsymbol{l}, \boldsymbol{\omega}))\boldsymbol{n} \\ \int_{\partial\Omega_S(0)} \boldsymbol{y} \times \mathbb{D}((I_3 - \mathbb{P})S(\boldsymbol{l}, \boldsymbol{\omega}))\boldsymbol{n} \end{pmatrix}_{6\times 1}.$$

Lemma 4.3. *The matrix K defined above is an invertible matrix.*

Proof. The main point is that M is a positive semi-definite, symmetric matrix. Then K being the sum of an invertible matrix and a semi-definite matrix, is itself invertible. The proof is essentially same as [23, Lemma 4.3]. We briefly explain it.

First we derive an explicit representation formula for M. For that, let $\{e_i\}$ be the basis vectors of $\mathbb{C}^3$ and let v^i, V^i be solutions of the weak Neumann problems:

$$\begin{cases} \Delta v^i = 0 \quad \text{in } \Omega_F(0), \\ \dfrac{\partial v^i}{\partial \boldsymbol{n}} = 0 \quad \text{on } \partial\Omega, \quad \dfrac{\partial v^i}{\partial \boldsymbol{n}} = e_i \cdot \boldsymbol{n} \quad \text{on } \partial\Omega_S(0); \end{cases}$$

and

$$\begin{cases} \Delta V^i = 0 \quad \text{in } \Omega_F(0), \\ \dfrac{\partial V^i}{\partial \boldsymbol{n}} = 0 \quad \text{on } \partial\Omega, \quad \dfrac{\partial V^i}{\partial \boldsymbol{n}} = (e_i \times \boldsymbol{y}) \cdot \boldsymbol{n} \quad \text{on } \partial\Omega_S(0). \end{cases}$$

Therefore, from the definition of the operator N_S, we can write,

$$N_S((\boldsymbol{l} + \boldsymbol{\omega} \times \boldsymbol{y}) \cdot \boldsymbol{n}) = \sum_{i=1}^{3} l_i v^i + \sum_{j=1}^{3} \omega_i V^i.$$

By defining,

$$m_{ij} = \begin{cases} \displaystyle\int_{\partial\Omega_S(0)} v^i n_j \quad \text{for } 1 \le i \le 3, \quad 1 \le j \le 3, \\ \displaystyle\int_{\partial\Omega_S(0)} V^{i-3} n_j \quad \text{for } 4 \le i \le 6, \quad 1 \le j \le 3, \\ \displaystyle\int_{\partial\Omega_S(0)} v^i (e_{j-3} \times \boldsymbol{y}) \cdot n \quad \text{for } 1 \le i \le 3, \quad 4 \le j \le 6, \\ \displaystyle\int_{\partial\Omega_S(0)} V^{i-3} (e_{j-3} \times \boldsymbol{y}) \cdot n \quad \text{for } 4 \le i \le 6, \quad 4 \le j \le 6, \end{cases}$$

we get $M = (m_{ij})_{1\le i,j\le 6}$. Now to show M is symmetric, observe that, by Gauss' theorem, for $1 \le i, j \le 3$,

$$m_{ij} = \int\limits_{\partial\Omega_S(0)} v^i n_j = \int\limits_{\partial\Omega_S(0)} v^i \frac{\partial v^j}{\partial \boldsymbol{n}} = \int\limits_{\Omega_F(0)} \operatorname{div}(v^i \nabla v^j) = \int\limits_{\Omega_F(0)} \nabla v^i \cdot \nabla v^j = \sum_{k=1}^{3} \int\limits_{\Omega_F(0)} \partial_k v^i \partial_k v^j$$

which is symmetric; Similarly, for $4 \le i, j \le 6$,

$$m_{ij} = \int\limits_{\partial\Omega_S(0)} V^{i-3}(e_{j-3} \times \boldsymbol{y}) \cdot n = \int\limits_{\partial\Omega_S(0)} V^{i-3} \frac{\partial V^{j-3}}{\partial \boldsymbol{n}} = \int\limits_{\Omega_F(0)} \nabla V^{i-3} \cdot \nabla V^{j-3} = m_{ji},$$

and for $4 \le i \le 6$, $1 \le j \le 3$,

$$m_{ij} = \int\limits_{\partial\Omega_S(0)} V^{i-3} n_j = \int\limits_{\partial\Omega_S(0)} V^{i-3} \frac{\partial v^j}{\partial \boldsymbol{n}} = \int\limits_{\Omega_F(0)} \nabla V^{i-3} \cdot \nabla v^j = m_{ji},$$

and the same for $1 \le i \le 3$, $4 \le j \le 6$. Finally, for any $z \in \mathbb{C}^6$, we obtain

$$\begin{aligned} z^T M z &= \sum_{i,j=1}^{3} m_{ij} z_i z_j + \sum_{i=4}^{6} \sum_{j=1}^{3} m_{ij} z_i z_j + \sum_{i=1}^{3} \sum_{j=4}^{6} m_{ij} z_i z_j + \sum_{i,j=4}^{6} m_{ij} z_i z_j \\ &= \sum_{k=1}^{3} \int\limits_{\Omega_F(0)} \left(\sum_{i=1}^{3} \partial_k v^i x_i + \sum_{i=1}^{3} \partial_k V^i x_i \right)^2 \ge 0. \end{aligned}$$

This completes the proof. ■

Let us now define the fluid-structure operator $\mathcal{A}_{FS} : \mathcal{D}(\mathcal{A}_{FS}) \subset X \to X$ with

$$X := \boldsymbol{L}^q_{\sigma,\boldsymbol{\tau}}(\Omega_F(0)) \times \mathbb{C}^3 \times \mathbb{C}^3,$$

and

$$\begin{cases} \mathcal{D}(\mathcal{A}_{FS}) := \{(\mathbb{P}\boldsymbol{u}, \boldsymbol{l}, \boldsymbol{\omega}) \in X : \mathbb{P}\boldsymbol{u} - \mathbb{P}S(\boldsymbol{l}, \boldsymbol{\omega}) \in \mathcal{D}(A_q)\}, \\ \qquad \mathcal{A}_{FS} = \begin{pmatrix} A_q & -A_q \mathbb{P} S \\ K^{-1} C_1 & K^{-1} C_2 \end{pmatrix}_{9\times 9}. \end{cases}$$

Combining the above results, we obtain below an equivalent formulation of the resolvent problem (4.2).

Proposition 4.4. *Let $q \in (1, \infty)$, $\alpha \ge 0$ be as in (2.4) and $(\boldsymbol{f}, \boldsymbol{g}_1, \boldsymbol{g}_2) \in X$. Then $(\boldsymbol{u}, \pi, \boldsymbol{l}, \boldsymbol{\omega}) \in W^{2,q}(\Omega_F(0)) \times W^{1,q}(\Omega_F(0)) \times \mathbb{C}^3 \times \mathbb{C}^3$ satisfies the resolvent problem (4.2) iff*

$$\begin{aligned} &(\lambda I - \mathcal{A}_{FS}) \begin{pmatrix} \mathbb{P}\boldsymbol{u} \\ \boldsymbol{l} \\ \boldsymbol{\omega} \end{pmatrix} = \begin{pmatrix} \mathbb{P}\boldsymbol{f} \\ \tilde{\boldsymbol{g}}_1 \\ \tilde{\boldsymbol{g}}_2 \end{pmatrix} \\ &(I_3 - \mathbb{P})\boldsymbol{u} = (I_3 - \mathbb{P}) S(\boldsymbol{l}, \boldsymbol{\omega}) \\ &\pi = N(\Delta \mathbb{P}\boldsymbol{u} \cdot \boldsymbol{n}) - \lambda N_S((\boldsymbol{l} + \boldsymbol{\omega} \times \boldsymbol{x}) \cdot \boldsymbol{n}) \end{aligned} \tag{4.9}$$

where $(\tilde{\boldsymbol{g}}_1, \tilde{\boldsymbol{g}}_2)^T = K^{-1}(\boldsymbol{g}_1, \boldsymbol{g}_2)^T$

The following lemma, gives an equivalent norm on the domain of the above operator.

Lemma 4.5. *The map*

$$(\mathbb{P}\boldsymbol{u}, \boldsymbol{l}, \boldsymbol{\omega}) \mapsto \|\mathbb{P}\boldsymbol{u}\|_{\boldsymbol{W}^{2,q}(\Omega_F(0))} + \|\boldsymbol{l}\|_{\mathbb{C}^3} + \|\boldsymbol{\omega}\|_{\mathbb{C}^3}$$

is a norm on $\mathcal{D}(\mathcal{A}_{FS})$ equivalent to the graph norm.

Proof. The proof is similar to the one im [41, Proposition 3.3] for Dirichlet condition. Nonetheless we briefly repeat it for the sake of completeness.

For $\lambda > 0$, $(\lambda I_6 - K^{-1}C_2)$ is an isomorphism from $\mathbb{C}^6$ to $\mathbb{C}^6$. Thus $(\boldsymbol{l}, \boldsymbol{\omega}) \mapsto \|(\boldsymbol{l}, \boldsymbol{\omega})\|_{\mathbb{C}^6} + \|K^{-1}C_2(\boldsymbol{l}, \boldsymbol{\omega})\|_{\mathbb{C}^6}$ is an equivalent norm to $(\boldsymbol{l}, \boldsymbol{\omega}) \mapsto \|\boldsymbol{l}\|_{\mathbb{C}^3} + \|\boldsymbol{\omega}\|_{\mathbb{C}^3}$. Also since A_q is an isomorphism from $\mathcal{D}(A_q)$ to $\boldsymbol{L}^q_{\sigma,\boldsymbol{\tau}}(\Omega_F(0))$, there exist positive constants C_1 and C_2 such that

$$\begin{aligned} C_1\|\mathbb{P}\boldsymbol{u} - \mathbb{P}S(\boldsymbol{l}, \boldsymbol{\omega})\|_{\boldsymbol{W}^{2,q}(\Omega_F(0))} &\leq \|A_q\mathbb{P}\boldsymbol{u} - A_q\mathbb{P}S(\boldsymbol{l}, \boldsymbol{\omega})\|_{\boldsymbol{L}^q(\Omega_F(0))} \\ &\leq C_2\|\mathbb{P}\boldsymbol{u} - \mathbb{P}S(\boldsymbol{l}, \boldsymbol{\omega})\|_{\boldsymbol{W}^{2,q}(\Omega_F(0))}. \end{aligned}$$

Now, using the fact that $S(\boldsymbol{l}, \boldsymbol{\omega}) \in \mathcal{L}\left(\mathbb{C}^6; \boldsymbol{W}^{2,q}(\Omega_F(0))\right)$ and $K^{-1}C_2 \in \mathcal{L}\left(\mathbb{C}^6; \mathbb{C}^6\right)$, we get that

$$\begin{aligned} &\|(\mathbb{P}\boldsymbol{u}, \boldsymbol{l}, \boldsymbol{\omega})\|_X + \|\mathcal{A}_{FS}(\mathbb{P}\boldsymbol{u}, \boldsymbol{l}, \boldsymbol{\omega})\|_X \\ &= \|(\mathbb{P}\boldsymbol{u}, \boldsymbol{l}, \boldsymbol{\omega})\|_X + \|A_q\mathbb{P}\boldsymbol{u} - A_q\mathbb{P}S(\boldsymbol{l}, \boldsymbol{\omega})\|_{\boldsymbol{L}^q(\Omega_F(0))} + \|K^{-1}C_1\mathbb{P}\boldsymbol{u} + K^{-1}C_2(\boldsymbol{l}, \boldsymbol{\omega})\|_{\mathbb{C}^6} \\ &\leq C\left(\|\mathbb{P}\boldsymbol{u}\|_{\boldsymbol{W}^{2,q}(\Omega_F(0))} + \|\boldsymbol{l}\|_{\mathbb{C}^3} + \|\boldsymbol{\omega}\|_{\mathbb{C}^3}\right). \end{aligned}$$

To prove the reverse inequality, we write

$$\begin{aligned} &\|\mathbb{P}\boldsymbol{u}\|_{\boldsymbol{W}^{2,q}(\Omega_F(0))} + \|\boldsymbol{l}\|_{\mathbb{C}^3} + \|\boldsymbol{\omega}\|_{\mathbb{C}^3} \\ &\leq \frac{1}{C_1}\|A_q\mathbb{P}\boldsymbol{u} - A_q\mathbb{P}S(\boldsymbol{l}, \boldsymbol{\omega})\|_{\boldsymbol{L}^q(\Omega_F(0))} + \|\mathbb{P}S(\boldsymbol{l}, \boldsymbol{\omega})\|_{\boldsymbol{W}^{2,q}(\Omega_F(0))} + \|\boldsymbol{l}\|_{\mathbb{C}^3} + \|\boldsymbol{\omega}\|_{\mathbb{C}^3} \\ &\leq \frac{1}{C_1}\|A_q\mathbb{P}\boldsymbol{u} - A_q\mathbb{P}S(\boldsymbol{l}, \boldsymbol{\omega})\|_{\boldsymbol{L}^q(\Omega_F(0))} + \|\boldsymbol{l}\|_{\mathbb{C}^3} + \|\boldsymbol{\omega}\|_{\mathbb{C}^3}. \end{aligned}$$

This completes the proof. ∎

Next we show the $\mathcal{R}$-boundedness of the resolvent operator of $\mathcal{A}_{FS}$ which will give us the maximal $L^p - L^q$-regularity for the linear problem (4.1) with $\boldsymbol{h} = \boldsymbol{0}$.

Theorem 4.6. *Let $q \in (1, \infty)$ and $\alpha \geq 0$ be as in (2.4). There exists $\theta > 0$ such that $\Sigma_{\pi/2+\theta} \subset \rho(\mathcal{A}_{FS})$ and*

$$\mathcal{R}\{\lambda(\lambda I_9 - \mathcal{A}_{FS})^{-1} : \lambda \in \Sigma_{\pi/2+\theta}\} < \infty. \tag{4.10}$$

In other words, $\tilde{\mathcal{A}}_{FS}$ is $\mathcal{R}$-sectorial.

Proof. The proof follows the similar argument as in [32, Theorem 3.11]. We first write $\mathcal{A}_{FS} = \tilde{\mathcal{A}}_{FS} + B_{FS}$ where

$$\tilde{\mathcal{A}}_{FS} = \begin{pmatrix} A_q & -A_q\mathbb{P}S \\ 0 & 0 \end{pmatrix}, \quad B_{FS} = \begin{pmatrix} 0 & 0 \\ K^{-1}C_1 & K^{-1}C_2 \end{pmatrix}.$$

Next we prove that $\tilde{\mathcal{A}}_{FS}$ with $\mathcal{D}(\tilde{\mathcal{A}}_{FS}) = \mathcal{D}(\mathcal{A}_{FS})$ is $\mathcal{R}$-sectorial on X. Note that we can write, using the identity $-(\lambda I_3 - A_q)^{-1} A_q \mathbb{P} S = -\lambda(\lambda I_3 - A_q)^{-1}\mathbb{P}S + \mathbb{P}S$,

$$\lambda(\lambda I_9 - \tilde{\mathcal{A}}_{FS})^{-1} = \begin{pmatrix} \lambda(\lambda I_3 - A_q)^{-1} & -\lambda(\lambda I_3 - A_q)^{-1}\mathbb{P}S + \mathbb{P}S \\ 0 & I \end{pmatrix}.$$

Since the Stokes operator A_q is $\mathcal{R}$-sectorial in $\boldsymbol{L}^q(\Omega_F(0))$ (see [4]) and using the properties 3. and 4. of Proposition 3.4, we get

$$\mathcal{R}\{-\lambda(\lambda I_3 - A_q)^{-1}\mathbb{P}S + \mathbb{P}S\} \leq \mathcal{R}\{-\lambda(\lambda I_3 - A_q)^{-1}\}\mathcal{R}(\mathbb{P}S) + \mathcal{R}(\mathbb{P}S) < \infty.$$

Observe that the $\mathcal{R}$-boundedness of $\mathbb{P}S$ follows from the definition easily, only using the continuity of $\mathbb{P}S$. Therefore the desired result follows.

Finally we show that B_{FS} is a small perturbation of $\tilde{\mathcal{A}}_{FS}$ which yields the $\mathcal{R}$-sectoriality of B_{FS}. This concludes the proof.

To do so, first let us show that $B_{FS} \in \mathcal{L}(\mathcal{D}(\mathcal{A}_{FS}), \mathbb{C}^9)$. By Lemma 4.5, for any $(\mathbb{P}\boldsymbol{u}, \boldsymbol{l}, \boldsymbol{\omega}) \in \mathcal{D}(\mathcal{A}_{FS})$, we have $(\mathbb{P}\boldsymbol{u}, \boldsymbol{l}, \boldsymbol{\omega}) \in \boldsymbol{W}^{2,q}(\Omega_F(0)) \times \mathbb{C}^3 \times \mathbb{C}^3$. Therefore, by trace theorem, $(\mathbb{D}(\mathbb{P}\boldsymbol{u}))\boldsymbol{n} \in \boldsymbol{W}^{1-1/q,q}(\partial\Omega_S(0))$ and $\int_{\partial\Omega_S(0)} (\mathbb{D}(\mathbb{P}\boldsymbol{u}))\boldsymbol{n}\,\mathrm{d}s \in \mathbb{C}^3$. On the other hand, $\Delta\mathbb{P}\boldsymbol{u} \in \boldsymbol{L}^q(\Omega_F(0))$ and $\operatorname{div}\,\Delta\mathbb{P}\boldsymbol{u} = 0$ which implies $\Delta\mathbb{P}\boldsymbol{u}\cdot\boldsymbol{n} \in \boldsymbol{W}^{-1/q,q}(\partial\Omega_S(0))$. Also the following condition is satisfied (due to the divergence free condition)

$$\langle \Delta\mathbb{P}\boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\boldsymbol{W}^{-1/q,q}\times\boldsymbol{W}^{1/q,q'}} = 0.$$

Thus $N(\Delta\mathbb{P}\boldsymbol{u}\cdot\boldsymbol{n}) \in \boldsymbol{W}^{1,q}(\Omega_F(0))$ and $\int_{\partial\Omega_S(0)} N(\Delta\mathbb{P}\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}\;\mathrm{d}s \in \mathbb{C}^3$. Other terms of C_1 can be checked in the same way. Similarly, for the matrix C_2, notice that $S(\boldsymbol{l}, \boldsymbol{\omega}) \in \boldsymbol{W}^{2,q}(\Omega_F(0))$ and $\mathbb{P}S(\boldsymbol{l}, \boldsymbol{\omega}) \in \boldsymbol{W}^{2,q}(\Omega_F(0))$ since $\mathbb{P}\boldsymbol{u} \in \boldsymbol{W}^{2,q}(\Omega_F(0))$ and hence, $(I_3 - \mathbb{P})S(\boldsymbol{l}, \boldsymbol{\omega}) \in \boldsymbol{W}^{2,q}(\Omega_F(0))$. therefore, $\mathbb{D}((I_3 - \mathbb{P})S(\boldsymbol{l}, \boldsymbol{\omega})) \in \boldsymbol{W}^{1-1/q,q}(\partial\Omega_S(0))$ and $\int_{\partial\Omega_S(0)} \mathbb{D}((I_3 - \mathbb{P})S(\boldsymbol{l}, \boldsymbol{\omega}))\boldsymbol{n}\;\mathrm{d}s \in \mathbb{C}^3$. The other term of C_2 can be treated in the same way. Thus, we deduce that the operator

$$B_{FS} : \mathcal{D}(B_{FS}) = \mathcal{D}(\mathcal{A}_{FS}) \to \mathbb{C}^9$$

is a bounded linear operator, due to the continuity of the trace operator and the elliptic regularity results. This concludes that B_{FS} is a finite rank operator and hence compact. Therefore we can say $(B_{FS}, \mathcal{D}(B_{FS}))$ is $\tilde{\mathcal{A}}_{FS}$-compact by the definition [19, Chapter III, Definition 2.15]. Then, from [19, Chapter III, Lemma 2.16], we get that B_{FS} is $\tilde{\mathcal{A}}_{FS}$-bounded with $\tilde{\mathcal{A}}_{FS}$-bound is 0, that is, for all $\delta > 0$, there exists $C(\delta) > 0$ such that

$$\|B_{FS}\, z\| \leq \delta\|\tilde{\mathcal{A}}_{FS}\, z\| + C(\delta)\|z\| \qquad \forall z \in \mathcal{D}(\tilde{\mathcal{A}}_{FS}).$$

Finally, applying the perturbation Theorem 3.7, we obtain that $\tilde{\mathcal{A}}_{FS} + B_{FS}$ is $\mathcal{R}$-sectorial. ■

The above result together with Theorem 3.6 and Proposition 3.8 yields the following maximal regularity for the system (4.1).

Theorem 4.7. *Let $\Omega_F(0)$ be a bounded domain of class $\mathcal{C}^{2,1}$, $p, q \in (1,\infty)$ and $\alpha \geq 0$ be as in (2.4). Also assume that $(\boldsymbol{u}_0, \boldsymbol{l}_0, \boldsymbol{\omega}_0) \in B^{2-2/p}_{q,p}(\Omega_F(0)) \times \mathbb{R}^3 \times \mathbb{R}^3$ satisfies the compatibility condition (2.5). Then for any $\boldsymbol{f} \in L^p(0,\infty; \boldsymbol{L}^q(\Omega_F(0)))$, $\boldsymbol{g}_1 \in L^p(0,\infty;\mathbb{R}^3)$ and $\boldsymbol{g}_2 \in L^p(0,\infty;\mathbb{R}^3)$, problem (4.1) with $\boldsymbol{h} = \boldsymbol{0}$ admits a unique solution*

$$\boldsymbol{u} \in W^{2,1}_{q,p}(Q^\infty_F), \pi \in L^p(0,\infty; W^{1,q}(\Omega_F(0))), (\boldsymbol{l}, \boldsymbol{\omega}) \in W^{1,p}(0,\infty;\mathbb{R}^6)$$

which satisfies the estimate

$$\begin{aligned}&\|\boldsymbol{u}\|_{W^{2,1}_{q,p}(Q^\infty_F)} + \|\pi\|_{L^p(0,\infty;W^{1,q}(\Omega_F(0)))} + \|\boldsymbol{l}\|_{W^{1,p}(0,\infty;\mathbb{R}^3)} + \|\boldsymbol{\omega}\|_{W^{1,p}(0,\infty;\mathbb{R}^3)}\\ &\leq C\left(\|\boldsymbol{f}\|_{L^p(0,\infty;\boldsymbol{L}^q(\Omega_F(0)))} + \|(\boldsymbol{g}_1,\boldsymbol{g}_2)\|_{L^p(0,\infty;\mathbb{R}^6)} + \|\boldsymbol{u}_0\|_{B^{2-2/p}_{q,p}(\Omega_F(0))} + \|(\boldsymbol{l}_0,\boldsymbol{\omega}_0)\|_{\mathbb{R}^6}\right)\end{aligned}$$

where the constant $C > 0$ depends only on α, p, q and $\Omega_S(0)$.

4.2 Exponential stability

Next we show that the operator $\mathcal{A}_{FS}$ generates an exponentially stable semigroup.

Theorem 4.8. *Let $p, q \in (1,\infty)$ and $\alpha \geq 0$ be as in (2.4). The operator $\mathcal{A}_{FS}$ generates an exponentially stable semigroup $(e^{t\mathcal{A}_{FS}})_{t\geq 0}$ on X. In other words, there exist constants $\eta_0 > 0$ and $C > 0$ such that*

$$\|e^{t\mathcal{A}_{FS}}(\boldsymbol{u}_0,\boldsymbol{l}_0,\boldsymbol{\omega}_0)^T\|_X \leq Ce^{-\eta_0 t}\|(\boldsymbol{u}_0,\boldsymbol{l}_0,\boldsymbol{\omega}_0)^T\|_X.$$

Proof. First note that the entire right half place $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\}$ is contained in the resolvent set of $\mathcal{A}_{FS}$. Indeed, from Theorem 4.6, we have that

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\} \setminus \{0\} \in \rho(\mathcal{A}_{FS})$$

and $0 \in \rho(\mathcal{A}_{FS})$ is shown in the next theorem. Therefore, since the resolvent set is an open set, we get actually, for some $\eta > 0$,

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq -\eta\} \subset \rho(\mathcal{A}_{FS}).$$

Now as $\mathcal{A}_{FS}$ generates an analytic semigroup, Proposition 2.9 in [7, Part II, Chapter, pp 120] says that the corresponding semigroup is of negative type. Then Corollary 2.2 (i) in [7, pp 93] provides the exponential stability. ∎

Theorem 4.9. *Let $p, q \in (1,\infty)$, $\alpha \geq 0$ be as in (2.4) and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$. Then, for any $(\boldsymbol{f},\boldsymbol{g}_1,\boldsymbol{g}_2) \in X$, the resolvent system (4.2) admits a unique solution satisfying the following estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,q}(\Omega_F(0))} + \|\pi\|_{W^{1,q}(\Omega_F(0))} + \|\boldsymbol{l}\|_{\mathbb{C}^3} + \|\boldsymbol{\omega}\|_{\mathbb{C}^3} \leq C\|(\boldsymbol{f},\boldsymbol{g}_1,\boldsymbol{g}_2)\|_X. \tag{4.11}$$

Proof. From Theorem 4.6, we know that there exists $\tilde{\lambda} > 0$ such that $(\tilde{\lambda}I - \mathcal{A}_{FS})$ is invertible. Thus (4.2) can be written as (since it is equivalent to (4.9) by Proposition 4.4),

$$\begin{aligned}&\begin{pmatrix}\mathbb{P}\boldsymbol{u}\\ \boldsymbol{l}\\ \boldsymbol{\omega}\end{pmatrix} = \left[I + (\lambda-\tilde{\lambda})(\tilde{\lambda}I - \mathcal{A}_{FS})^{-1}\right]^{-1}(\tilde{\lambda}I-\mathcal{A}_{FS})^{-1}\begin{pmatrix}\mathbb{P}\boldsymbol{f}\\ \tilde{\boldsymbol{g}}_1\\ \tilde{\boldsymbol{g}}_2\end{pmatrix}\\ &(I-\mathbb{P})\boldsymbol{u} = (I-\mathbb{P})S(\boldsymbol{l},\boldsymbol{\omega})\\ &\pi = N(\Delta\mathbb{P}\boldsymbol{u}\cdot\boldsymbol{n}) - \lambda N_S((\boldsymbol{l}+\boldsymbol{\omega}\times\boldsymbol{y})\cdot\boldsymbol{n}).\end{aligned} \tag{4.12}$$

In fact $(\tilde{\lambda}I - \mathcal{A}_{FS})$ has a continuous inverse, therefore $(\tilde{\lambda}I - \mathcal{A}_{FS})^{-1}$ is a compact operator. Hence by Fredholm alternative theorem, the existence and uniqueness of solution of the above system is equivalent. So it is enough to show the uniqueness of (4.12) only. Then the estimate (4.11) follows easily.

Let $(\boldsymbol{u}, \pi, \boldsymbol{l}, \boldsymbol{\omega}) \in \boldsymbol{W}^{2,q}(\Omega_F(0)) \times W^{1,q}(\Omega_F(0)) \times \mathbb{C}^3 \times \mathbb{C}^3$ satisfies the homogeneous system

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{0}, \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega_F(0), \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad 2\left[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \partial\Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = (\boldsymbol{l} + \boldsymbol{\omega} \times \boldsymbol{y}) \cdot \boldsymbol{n} & \text{on } \partial\Omega_S(0), \\ 2\left[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \alpha(\boldsymbol{l} + \boldsymbol{\omega} \times \boldsymbol{y})_{\boldsymbol{\tau}} & \text{on } \partial\Omega_S(0), \\ \lambda m \boldsymbol{l} = - \displaystyle\int_{\partial\Omega_S(0)} \sigma(\boldsymbol{u}, \pi)\boldsymbol{n}, \\ \lambda J(0)\boldsymbol{\omega} = - \displaystyle\int_{\partial\Omega_S(0)} \boldsymbol{y} \times \sigma(\boldsymbol{u}, \pi)\boldsymbol{n}. \end{cases} \tag{4.13}$$

We first show that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,2}(\Omega_F(0)) \times W^{1,2}(\Omega_F(0))$. If $q \geq 2$, it is obvious. If $q \in (1,2)$, we rewrite (4.13) as

$$\begin{aligned} &(\tilde{\lambda} I - \mathcal{A}_{FS}) \begin{pmatrix} \mathbb{P}\boldsymbol{u} \\ \boldsymbol{l} \\ \boldsymbol{\omega} \end{pmatrix} = (\lambda - \tilde{\lambda}) \begin{pmatrix} \mathbb{P}\boldsymbol{u} \\ \boldsymbol{l} \\ \boldsymbol{\omega} \end{pmatrix} \\ &(I - \mathbb{P})\boldsymbol{u} = (I - \mathbb{P}) S(\boldsymbol{l}, \boldsymbol{\omega}) \\ &\pi = N(\Delta \mathbb{P}\boldsymbol{u} \cdot \boldsymbol{n}) - \lambda N_S((\boldsymbol{l} + \boldsymbol{\omega} \times \boldsymbol{y}) \cdot \boldsymbol{n}). \end{aligned} \tag{4.14}$$

But as $\boldsymbol{W}^{2,q}(\Omega_F(0)) \subset \boldsymbol{L}^2(\Omega_F(0))$ and $(\tilde{\lambda} I - \mathcal{A}_{FS})$ is invertible, we obtain $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,2}(\Omega_F(0)) \times W^{1,2}(\Omega_F(0))$.

Now multiplying the 1st equation of (4.13) by $\overline{\boldsymbol{u}}$ and integrating by parts, we get

$$\begin{aligned} 0 &= \lambda \int_{\Omega_F(0)} |\boldsymbol{u}|^2 + 2 \int_{\Omega_F(0)} |\mathbb{D}\boldsymbol{u}|^2 - 2 \int_{\partial\Omega_F(0)} (\mathbb{D}\boldsymbol{u})\boldsymbol{n} \cdot \overline{\boldsymbol{u}} + \int_{\partial\Omega_S(0)} \pi \, \overline{\boldsymbol{u}} \cdot \boldsymbol{n} \\ &= \lambda \int_{\Omega_F(0)} |\boldsymbol{u}|^2 + 2 \int_{\Omega_F(0)} |\mathbb{D}\boldsymbol{u}|^2 - \int_{\partial\Omega_S(0)} (2(\mathbb{D}\boldsymbol{u})\boldsymbol{n} - \pi \boldsymbol{n}) \cdot \overline{\boldsymbol{u}} - 2 \int_{\partial\Omega} [(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} \cdot \overline{\boldsymbol{u}}_{\boldsymbol{\tau}} \\ &= \lambda \int_{\Omega_F(0)} |\boldsymbol{u}|^2 + 2 \int_{\Omega_F(0)} |\mathbb{D}\boldsymbol{u}|^2 - \int_{\partial\Omega_S(0)} \sigma(\boldsymbol{u}, \pi)\boldsymbol{n} \cdot \overline{\boldsymbol{u}} + \int_{\partial\Omega} \alpha |\boldsymbol{u}_{\boldsymbol{\tau}}|^2. \end{aligned}$$

Note that

$$\begin{aligned} &\int_{\partial\Omega_S(0)} \sigma(\boldsymbol{u}, \pi)\boldsymbol{n} \cdot \overline{\boldsymbol{u}} \\ &= \int_{\partial\Omega_S(0)} \sigma(\boldsymbol{u}, \pi)\boldsymbol{n} \cdot \big(\overline{\boldsymbol{u}} - (\overline{\boldsymbol{l}} + \overline{\boldsymbol{\omega}} \times \boldsymbol{y})\big) + \int_{\partial\Omega_S(0)} \sigma(\boldsymbol{u}, \pi)\boldsymbol{n} \cdot (\overline{\boldsymbol{l}} + \overline{\boldsymbol{\omega}} \times \boldsymbol{y}) \\ &= \int_{\partial\Omega_S(0)} [\sigma(\boldsymbol{u}, \pi)\boldsymbol{n}]_{\boldsymbol{\tau}} \cdot \big(\overline{\boldsymbol{u}} - (\overline{\boldsymbol{l}} + \overline{\boldsymbol{\omega}} \times \boldsymbol{y})\big)_{\boldsymbol{\tau}} + \overline{\boldsymbol{l}} \cdot \int_{\partial\Omega_S(0)} \sigma(\boldsymbol{u}, \pi)\boldsymbol{n} + \overline{\boldsymbol{\omega}} \cdot \int_{\partial\Omega_S(0)} \sigma(\boldsymbol{u}, \pi)\boldsymbol{n} \times \boldsymbol{y} \\ &= - \int_{\partial\Omega_S(0)} \alpha |\boldsymbol{u}_{\boldsymbol{\tau}} - (\overline{\boldsymbol{l}} + \overline{\boldsymbol{\omega}} \times \boldsymbol{y})_{\boldsymbol{\tau}}|^2 - \lambda m |\boldsymbol{l}|^2 - \lambda J(0)\boldsymbol{\omega} \cdot \overline{\boldsymbol{\omega}} \end{aligned}$$

where we used the 4th and 5th equations of (4.13), multiplied by $\bar{\boldsymbol{l}}, \overline{\boldsymbol{\omega}}$ respectively. Thus we obtain the energy equality

$$\lambda \int\limits_{\Omega_F(0)} |\boldsymbol{u}|^2 + 2 \int\limits_{\Omega_F(0)} |\mathbb{D}\boldsymbol{u}|^2 + \int\limits_{\partial\Omega_S(0)} \alpha |\boldsymbol{u_\tau} - (\bar{\boldsymbol{l}} + \overline{\boldsymbol{\omega}} \times \boldsymbol{y})_{\boldsymbol{\tau}}|^2 + \lambda m |\boldsymbol{l}|^2 + \lambda J(0)\boldsymbol{\omega} \cdot \overline{\boldsymbol{\omega}} + \int\limits_{\partial\Omega} \alpha |\boldsymbol{u_\tau}|^2 = 0.$$

Taking the real part of the above equation yields,

$$\operatorname{Re}\lambda \int\limits_{\Omega_F(0)} |\boldsymbol{u}|^2 + 2 \int\limits_{\Omega_F(0)} |\mathbb{D}\boldsymbol{u}|^2 + \int\limits_{\partial\Omega_S(0)} \alpha |\boldsymbol{u_\tau} - (\bar{\boldsymbol{l}} + \overline{\boldsymbol{\omega}} \times \boldsymbol{y})_{\boldsymbol{\tau}}|^2 + \operatorname{Re}\lambda \; m |\boldsymbol{l}|^2$$
$$+ \operatorname{Re}\lambda \; J(0)\boldsymbol{\omega} \cdot \overline{\boldsymbol{\omega}} + \int\limits_{\partial\Omega} \alpha |\boldsymbol{u_\tau}|^2 = 0.$$

But as $\operatorname{Re}\lambda \geq 0$ and $\alpha > 0$ (and also, $J(0)a \cdot \bar{a} > 0 \; \forall a \in \mathbb{R}^3$), we obtain $\boldsymbol{l} = \boldsymbol{0}$ and

$$2 \int\limits_{\Omega_F(0)} |\mathbb{D}\boldsymbol{u}|^2 + \int\limits_{\partial\Omega} \alpha |\boldsymbol{u_\tau}|^2 = 0.$$

This implies, along with the fact that $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ (cf. [1, Proposition 3.7]), that $\boldsymbol{u} = \boldsymbol{0}$ in $\Omega_F(0)$. Finally, from the boundary condition $2\left[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u_\tau} = \alpha(\boldsymbol{l} + \boldsymbol{\omega} \times \boldsymbol{y})_{\boldsymbol{\tau}}$ on $\partial\Omega_S(0)$, we deduce $\boldsymbol{\omega} = \boldsymbol{0}$. Remember, $(\boldsymbol{\omega} \times \boldsymbol{y}) \cdot \boldsymbol{n} = 0$ holds always. ■

We now prove the maximal $L^p - L^q$ regularity for the system (4.1) with non-vanishing divergence condition which is required to treat the full non-linear system as described in the beginning of Section 4. Let us introduce the notation:

$$W^{2,1}_{q,p}(Q^\infty_F) := L^p(0,\infty; \boldsymbol{W}^{2,q}(\Omega_F(0))) \cap W^{1,p}(0,\infty; \boldsymbol{L}^q(\Omega_F(0)))$$

with the graph norm

$$\|\boldsymbol{v}\|_{W^{2,1}_{q,p}(Q^\infty_F)} := \|\boldsymbol{v}\|_{L^p(0,\infty;\boldsymbol{W}^{2,q}(\Omega_F(0)))} + \|\boldsymbol{v}\|_{W^{1,p}(0,\infty;\boldsymbol{L}^q(\Omega_F(0)))}.$$

We prove the following theorem.

Theorem 4.10. *Let $p, q \in (1,\infty)$ and $\alpha \geq 0$ be as in (2.4). Let $\eta \in (0, \eta_0)$ where η_0 is the constant introduced in Theorem 4.8 and $(\boldsymbol{l}_0, \boldsymbol{\omega}_0, \boldsymbol{u}_0) \in \mathbb{R}^3 \times \mathbb{R}^3 \times B^{2(1-1/p)}_{q,p}(\Omega_F(0))$ satisfying the compatibility conditions (2.5). Then for any $e^{\eta t}\boldsymbol{f} \in L^p(0,\infty; \boldsymbol{L}^q(\Omega_F(0))), e^{\eta t}\boldsymbol{h} \in W^{2,1}_{q,p}(Q^\infty_F), e^{\eta t}\boldsymbol{g}_1 \in L^p(0,\infty;\mathbb{R}^3)$ and $e^{\eta t}\boldsymbol{g}_2 \in L^p(0,\infty,\mathbb{R}^3)$ satisfying*

$$\operatorname{div} \boldsymbol{h}|_{t=0} = \operatorname{div} \boldsymbol{u}_0 \quad \text{on} \;\; \Omega_F(0) \quad \text{and} \quad \boldsymbol{h} \cdot \boldsymbol{n}|_{\partial\Omega_F(0)} = \boldsymbol{0},$$

the system (4.1) admits a unique strong solution

$$e^{\eta t}\boldsymbol{u} \in W^{2,1}_{q,p}(Q^\infty_F), \quad e^{\eta t}\pi \in L^p(0,\infty; W^{1,q}(\Omega_F(0)))$$
$$e^{\eta t}\boldsymbol{l} \in W^{1,p}(0,\infty;\mathbb{R}^3), \quad e^{\eta t}\boldsymbol{\omega} \in W^{1,p}(0,\infty;\mathbb{R}^3).$$

Moreover, there exists a constant $C_L > 0$, depending only on α, p, q and $\Omega_S(0)$ such that

$$\begin{aligned}
&\|e^{\eta t}\boldsymbol{u}\|_{W^{2,1}_{q,p}(Q^\infty_F)} + \|e^{\eta t}\pi\|_{L^p(0,\infty;W^{1,q}(\Omega_F(0)))} + \|e^{\eta t}\boldsymbol{l}\|_{L^p(0,\infty;\mathbb{R}^3)} + \|e^{\eta t}\boldsymbol{\omega}\|_{L^p(0,\infty;\mathbb{R}^3)} \\
&\leq C_L \Big(\|\boldsymbol{u}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega_F(0))} + \|\boldsymbol{l}_0\|_{\mathbb{R}^3} + \|\boldsymbol{\omega}_0\|_{\mathbb{R}^3} + \|e^{\eta t}\boldsymbol{f}\|_{L^p(0,\infty;\boldsymbol{L}^q(\Omega_F(0)))} \\
&\qquad + \|e^{\eta t}\boldsymbol{h}\|_{W^{2,1}_{q,p}(Q^\infty_F)} + \|e^{\eta t}\boldsymbol{g}_1\|_{L^p(0,\infty;\mathbb{R}^3)} + \|e^{\eta t}\boldsymbol{g}_2\|_{L^p(0,\infty;\mathbb{R}^3)} \Big).
\end{aligned} \tag{4.15}$$

Proof. We first consider the case $\eta = 0$. Let us set $\boldsymbol{v} := \boldsymbol{u} - \boldsymbol{h}$. Then $(\boldsymbol{v}, \pi, \boldsymbol{l}, \boldsymbol{\omega})$ satisfies the following system

$$
\begin{cases}
\boldsymbol{u}_t - \operatorname{div} \sigma(\boldsymbol{u}, \pi) = \boldsymbol{F} & \text{in } \Omega_F(0) \times (0,T), \\
\operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega_F(0) \times (0,T), \\
\boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad 2\left[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \partial\Omega \times (0,T), \\
\boldsymbol{u} \cdot \boldsymbol{n} = (\boldsymbol{l} + \boldsymbol{\omega} \times \boldsymbol{y}) \cdot \boldsymbol{n} & \text{on } \partial\Omega_S(0) \times (0,T), \\
2\left[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \alpha(\boldsymbol{l} + \boldsymbol{\omega} \times \boldsymbol{y})_{\boldsymbol{\tau}} & \text{on } \partial\Omega_S(0) \times (0,T), \\
m\boldsymbol{l}' = -\displaystyle\int_{\partial\Omega_S(0)} \sigma(\boldsymbol{u}, \pi)\boldsymbol{n} + \boldsymbol{G}_1, & t \in (0,T), \\
J(0)\boldsymbol{\omega}' = -\displaystyle\int_{\partial\Omega_S(0)} \boldsymbol{y} \times \sigma(\boldsymbol{u}, \pi)\boldsymbol{n} + \boldsymbol{G}_2, & t \in (0,T), \\
\boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in } \Omega_F(0), \\
\boldsymbol{l}(0) = \boldsymbol{l}_0, \quad \boldsymbol{\omega}(0) = \boldsymbol{\omega}_0
\end{cases}
\tag{4.16}
$$

where

$$
\boldsymbol{F} = \boldsymbol{f} - \partial_t \boldsymbol{h} + \Delta \boldsymbol{h}, \quad \boldsymbol{G}_1 = \boldsymbol{g}_1 - \int_{\partial\Omega_S(0)} (\mathbb{D}\boldsymbol{h})\boldsymbol{n}, \quad \boldsymbol{G}_2 = \boldsymbol{g}_2 - \int_{\partial\Omega_S(0)} \boldsymbol{y} \times (\mathbb{D}\boldsymbol{h})\boldsymbol{n}.
$$

Under the hypothesis of the theorem, we have that $(\boldsymbol{F}, \boldsymbol{G}_1, \boldsymbol{G}_2) \in L^p(0,\infty; X)$ and

$$
\begin{aligned}
\|(\boldsymbol{F}, \boldsymbol{G}_1, \boldsymbol{G}_2)\|_{L^p(0,\infty;X)} \leq C \Big(\|\boldsymbol{f}\|_{L^p(0,\infty;\boldsymbol{L}^q(\Omega_F(0)))} + \|\boldsymbol{h}\|_{W^{2,1}_{q,p}(Q_F^\infty)} \\
+ \|\boldsymbol{g}_1\|_{L^p(0,\infty;\mathbb{R}^3)} + \|\boldsymbol{g}_2\|_{L^p(0,\infty;\mathbb{R}^3)} \Big).
\end{aligned}
\tag{4.17}
$$

Also, we have $(\boldsymbol{u}_0, \boldsymbol{l}_0, \boldsymbol{\omega}_0) \in (X, \mathcal{D}(\mathcal{A}_{FS}))_{1-1/p,p}$ from the assumptions of the theorem. Hence, by Theorem 4.7, the system (4.17) admits a unique solution $(\boldsymbol{v}, \boldsymbol{l}, \boldsymbol{\omega}) \in L^p(0,\infty; \mathcal{D}(\mathcal{A}_{FS})) \cap W^{1,p}(0,\infty; X)$ which also satisfies

$$
\begin{aligned}
&\|(\boldsymbol{v}, \boldsymbol{l}, \boldsymbol{\omega})\|_{L^p(0,\infty;\mathcal{D}(\mathcal{A}_{FS}))} + \|(\boldsymbol{v}, \boldsymbol{l}, \boldsymbol{\omega})\|_{W^{1,p}(0,\infty;X)} \\
&\leq C \Big(\|(\boldsymbol{u}_0, \boldsymbol{l}_0, \boldsymbol{\omega}_0)\|_{(X,\mathcal{D}(\mathcal{A}_{FS}))_{1-1/p,p}} + \|(\boldsymbol{F}, \boldsymbol{G}_1, \boldsymbol{G}_2)\|_{L^p(0,\infty;X)} \Big).
\end{aligned}
\tag{4.18}
$$

Thus, $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{h}, \pi, \boldsymbol{l}, \boldsymbol{\omega}$ is the unique solution of (4.1). The estimate (4.15) follows combining (4.18) and (4.17).

Finally the result for $\eta > 0$ can be deduced from the previous case, simply multiplying all the functions by $e^{\eta t}$ and noting that $\mathcal{A}_{FS} + \eta I$ also generates an C^0-semigroup of negative type for all $\eta \in (0, \eta_0)$. ∎

5 Non-linear problem

In order to handle the full non-linear coupled system, we require to make the fluid domain time independent. Therefore we use the change of variable, as used in [15, section 2], which coincide with $Q(t)\boldsymbol{y} + \boldsymbol{h}(t)$ in a neighbourhood of the rigid body and is equal to the identity far from the rigid body, to rewrite the coupled system in a fixed spatial domain. For the convenience of the reader we summarized the construction and basic properties of the change of variable in the Appendix.

5.1 Estimates on the non-linear terms

In the first part of this subsection, we show estimates on the transforms X and Y in terms of $\tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}}$. Then we can study the Lipschitz properties of $\boldsymbol{F}_0, \boldsymbol{H}, \boldsymbol{F}_1, \boldsymbol{F}_2$. For $p \in (1,\infty)$, let p' denote the conjugate of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

We also introduce the set

$$S_\gamma := \{(\tilde{\boldsymbol{u}}, \tilde{\pi}, \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}}) : \|(\tilde{\boldsymbol{u}}, \tilde{\pi}, \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}})\|_S \leq \gamma\}$$

where

$$\begin{aligned}\|(\tilde{\boldsymbol{u}}, \tilde{\pi}, \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}})\|_S :=& \|e^{\eta(\cdot)}\tilde{\boldsymbol{u}}\|_{L^p(0,\infty;\boldsymbol{W}^{2,q}(\Omega_F(0)))} + \|e^{\eta(\cdot)}\tilde{\boldsymbol{u}}\|_{W^{1,p}(0,\infty;\boldsymbol{L}^q(\Omega_F(0)))} \\ &+ \|e^{\eta(\cdot)}\tilde{\pi}\|_{L^p(0,\infty;W^{1,q}(\Omega_F(0)))} + \|e^{\eta(\cdot)}\tilde{\boldsymbol{l}}\|_{W^{1,p}(0,\infty;\mathbb{R}^3)} + \|e^{\eta(\cdot)}\tilde{\boldsymbol{\omega}}\|_{W^{1,p}(0,\infty;\mathbb{R}^3)}.\end{aligned}$$

Proposition 5.1. *Let $p, q \in (1,\infty)$. There exists constants $\gamma_0 \in (0,1)$ and $C > 0$, depending only on p, q, η and $\Omega_F(0)$ such that for every $\gamma \in (0,\gamma_0)$ and every $(\tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}})$ with $\|e^{\eta(\cdot)}\tilde{\boldsymbol{l}}\|_{W^{1,p}(0,\infty;\mathbb{R}^3)} + \|e^{\eta(\cdot)}\tilde{\boldsymbol{\omega}}\|_{W^{1,p}(0,\infty;\mathbb{R}^3)} \leq \gamma$,*

$$\|Q - I_3\|_{L^\infty(0,\infty;\mathbb{R}^{3\times 3})} \leq C\,\gamma; \tag{5.1}$$

$$\|J_X - I_3\|_{L^\infty(0,\infty;C^2(\overline{\Omega})} \leq C\,\gamma; \tag{5.2}$$

$$\|J_Y - I_3\|_{L^\infty(0,\infty;C^2(\overline{\Omega})} \leq C\,\gamma; \tag{5.3}$$

$$\|\frac{\partial^2 Y}{\partial x_j \partial x_k}\|_{L^\infty(0,\infty;C^2(\overline{\Omega})} \leq C\,\gamma; \tag{5.4}$$

$$\|\partial_t X\|_{L^\infty(0,\infty;\Omega_F(0))} \leq C\,\gamma \tag{5.5}$$

$$\|J_Y Q - I_3\|_{L^\infty(0,\infty;C^2(\overline{\Omega}))} \leq C\,\gamma. \tag{5.6}$$

Proof. First we show the existence of a constant $\gamma_0 \in (0,1)$ such that for every $\gamma \in (0,\gamma_0)$ and for every $(\tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}})$ with $\|e^{\eta(\cdot)}\tilde{\boldsymbol{l}}\|_{W^{1,p}(0,\infty;\mathbb{R}^3)} + \|e^{\eta(\cdot)}\tilde{\boldsymbol{\omega}}\|_{W^{1,p}(0,\infty;\mathbb{R}^3)} \leq \gamma$, the condition (7.1) is verified.

The matrix Q being an orthogonal linear transformation satisfies $Q \in SO(3)$ and thus $|Q(t)| = 1$ for all $t \geq 0$. Here $|A|$ denotes the Frobenius norm (or, Euclidean norm) for any matrix A. Since Q satisfies the problem (7.6), we can write

$$Q(t)\boldsymbol{a} = \boldsymbol{a} + \int_0^t e^{-\eta s} e^{\eta s} Q(s)\,(\tilde{\omega}(s) \times \boldsymbol{a})\ \mathrm{d}s$$

which gives

$$|Q(t)\boldsymbol{a} - \boldsymbol{a}| \leq \int_0^t e^{-\eta s} e^{\eta s} |\tilde{\omega}(s) \times \boldsymbol{a}|\ \mathrm{d}s.$$

Therefore, we can estimate the operator norm

$$\|Q(t) - I_3\| = \sup_{0 \neq \boldsymbol{a} \in \mathbb{R}^3} \frac{|Q(t)\boldsymbol{a} - \boldsymbol{a}|}{|\boldsymbol{a}|} \leq \int_0^t e^{-\eta s} e^{\eta s} |\tilde{\omega}(s)|\ \mathrm{d}s.$$

But as the operator norm and the Frobenius norm is equivalent on the matrix space, we can have

$$\|Q(t)-I_3\|_{\mathbb{R}^{3\times3}} \leq \sqrt{3}\|Q(t)-I_3\| \leq \sqrt{3}\int_0^t e^{-\eta s}e^{\eta s}|\tilde{\omega}(s)|\,\mathrm{d}s.$$

Consequently,

$$\begin{aligned}\|Q-I_3\|_{L^\infty(0,\infty;\mathbb{R}^{3\times3})} &\leq \sqrt{3}\int_0^\infty e^{-\eta s}e^{\eta s}|\tilde{\omega}(s)|\,\mathrm{d}s\\ &\leq \sqrt{3}\left(\int_0^\infty e^{-p'\eta s}\,\mathrm{d}s\right)^{1/p'}\|e^{\eta(\cdot)}\tilde{\boldsymbol{\omega}}\|_{L^p(0,\infty;\mathbb{R}^3)} \leq \left(\frac{1}{p'\eta}\right)^{1/p'}\gamma.\end{aligned}$$

Similarly, we can write from (7.5)$_3$,

$$\begin{aligned}\|\boldsymbol{h}\|_{L^\infty(0,\infty;\mathbb{R}^3)} &\leq \int_0^\infty e^{-\eta s}e^{\eta s}|Q(s)||\tilde{\boldsymbol{l}}(s)|\,\mathrm{d}s\\ &\leq \left(\int_0^\infty e^{-p'\eta s}\,\mathrm{d}s\right)^{1/p'}\|e^{\eta(\cdot)}\tilde{\boldsymbol{l}}\|_{L^p(0,\infty;\mathbb{R}^3)} \leq \left(\frac{1}{p'\eta}\right)^{1/p'}\gamma.\end{aligned}$$

Combining the above two inequalities give

$$\|Q-I_3\|_{L^\infty(0,\infty;\mathbb{R}^{3\times3})}\mathrm{diam}(\Omega_S(0)) + \|\boldsymbol{h}\|_{L^\infty(0,\infty;\mathbb{R}^3)} \leq \left(\frac{1}{p'\eta}\right)^{1/p'}\gamma\left(1+\mathrm{diam}(\Omega_S(0))\right).$$

Let us define

$$\gamma_0 = \min 1, \frac{\beta}{2C_{p,\eta}\left(1+\mathrm{diam}(\Omega_S(0))\right)} \qquad \text{where } C_{p,\eta} = \left(\frac{1}{p'\eta}\right)^{1/p'}. \tag{5.7}$$

With this choice of γ_0, we satisfy the condition (7.1).

Next we prove some regularity of X, Y, J_X, J_Y. The mapping X, solution of the differential equation (7.3) can be written as

$$X(\boldsymbol{y},t) = \boldsymbol{y} + \int_0^t \Lambda(X(\boldsymbol{y},s),s)\,\mathrm{d}s.$$

Differentiating it with respect to $\boldsymbol{y}$, we obtain,

$$J_X(\boldsymbol{y},t) = I_3 + \int_0^t \nabla\Lambda(X(\boldsymbol{y},s),s)J_X(\boldsymbol{y},s)\,\mathrm{d}s.$$

Note that from the definition of Λ, we can write, for all $\boldsymbol{x}\in\Omega_S(t)$,

$$\nabla\Lambda(\boldsymbol{x},t) = \begin{pmatrix} 0 & -w_3(t) & w_2(t)\\ w_3(t) & 0 & -w_1(t)\\ -w_2(t) & w_1(t) & 0\end{pmatrix}.$$

Also, $\nabla\Lambda = 0$ for all $\boldsymbol{x}$ with $\mathrm{dist}(\boldsymbol{x}, \partial\Omega) < \beta/8$. Otherwise,

$$\|\nabla\Lambda(\boldsymbol{x},t)\|_{C^2(\overline{\Omega})} \leq C\left(|\tilde{\boldsymbol{\omega}}(t)| + |\tilde{\boldsymbol{l}}(t)|\right)$$

where the constant C depends on $\Omega_F(0)$. So we have,

$$\begin{aligned}
&\|J_X(\cdot,t)\|_{C^2(\overline{\Omega})} \\
&\leq 1 + C\int_0^t e^{-\eta s}e^{\eta s}\left(|\tilde{\boldsymbol{\omega}}(s)| + |\tilde{\boldsymbol{l}}(s)|\right)\|J_X(\cdot,s)\|_{C^2(\overline{\Omega})}\,\mathrm{d}s \\
&\leq 1 + C\left(\|e^{\eta(\cdot)}\tilde{\boldsymbol{\omega}}\|_{L^\infty(0,\infty;\mathbb{R}^3)} + \|e^{\eta(\cdot)}\tilde{\boldsymbol{l}}\|_{L^\infty(0,\infty;\mathbb{R}^3)}\right)\int_0^t e^{-\eta s}\|J_X(\cdot,s)\|_{C^2(\overline{\Omega})}\,\mathrm{d}s \\
&\leq 1 + C\int_0^t e^{-\eta s}\|J_X(\cdot,s)\|_{C^2(\overline{\Omega})}\,\mathrm{d}s.
\end{aligned}$$

Now the Gronwall's inequality yields,

$$\|J_X(\cdot,t)\|_{C^2(\overline{\Omega})} \leq \exp\left(C\int_0^t e^{-\eta s}\,\mathrm{d}s\right) \leq e^{C/\eta} \quad \forall\, t \in (0,\infty).$$

With this estimate at hand, we obtain

$$\|J_X - I_3\|_{L^\infty(0,\infty;C^2(\overline{\Omega}))} \leq C\int_0^\infty e^{-\eta s}e^{\eta s}\left(|\tilde{\boldsymbol{\omega}}(s)| + |\tilde{\boldsymbol{l}}(s)|\right)\,\mathrm{d}s \leq C\,\gamma.$$

Also from the following relation, since $\det J_X = 1$,

$$(\mathrm{cof}\ J_X)^T = (\det J_X)J_X^{-1} = J_X^{-1},$$

we can deduce,

$$\|\mathrm{cof}\ J_X\|_{L^\infty(0,\infty;C^2(\overline{\Omega}))} \leq C$$

which follows from Lemma 5.2. This implies also,

$$\|J_Y\|_{L^\infty(0,\infty;C^2(\overline{\Omega}))} \leq C \tag{5.8}$$

since $J_Y = J_X^{-1}$. Using the above estimate and (5.2), we further get

$$\|J_Y - I_3\|_{L^\infty(0,\infty;C^2(\overline{\Omega}))} \leq \|J_Y\|_{L^\infty(0,\infty;C^2(\overline{\Omega}))}\|J_X - I_3\|_{L^\infty(0,\infty;C^2(\overline{\Omega}))} \leq C\,\gamma.$$

To deduce the regularity of $\frac{\partial^2 Y}{\partial x_j \partial x_k}$, we write

$$\frac{\partial}{\partial x_j}J_Y = \frac{\partial}{\partial y_i}\left(\mathrm{cof}\ J_X\right)\frac{\partial Y_i}{\partial x_j}$$

from which it follows, along with the definition (7.2) of Λ, since cof J_X involves second derivative of Λ,

$$\|\frac{\partial}{\partial x_j} J_Y\|_{L^\infty(0,\infty;C^2(\overline{\Omega}))} \leq C \sup_{t\in(0,\infty)} \left(|\tilde{\boldsymbol{\omega}}(t)| + |\tilde{\boldsymbol{l}}(t)|\right) \leq C\ \gamma.$$

Similarly, we obtain the following estimates

$$\begin{aligned}
\|\partial_t X\|_{L^\infty(0,\infty;\Omega_F(0))} &= \|\Lambda\|_{L^\infty(0,\infty;\Omega_F(0))} \\
&\leq C \sup_{t\in(0,\infty)} \left(e^{-\eta t} e^{\eta t} (|\tilde{\boldsymbol{\omega}}(t)| + |\tilde{\boldsymbol{l}}(t)|)\right) \\
&\leq C \left(\|e^{\eta(\cdot)}\tilde{\boldsymbol{\omega}}\|_{L^\infty(0,\infty,\mathbb{R}^3)} + \|e^{\eta(\cdot)}\tilde{\boldsymbol{l}}\|_{L^\infty(0,\infty;\mathbb{R}^3)}\right) \leq C\ \gamma;
\end{aligned}$$

$$\begin{aligned}
\|\partial_t J_Y\|_{L^\infty(0,\infty;\Omega_F(0))} &= \|J_X^{-1}\ \partial_t J_X\ J_X^{-1}\|_{L^\infty(0,\infty;\Omega_F(0))} = \|J_Y \nabla\Lambda\ J_Y\|_{L^\infty(0,\infty;\Omega_F(0))} \leq C\ \gamma; \\
\|J_Y Q - I_3\|_{L^\infty(0,\infty;C^2(\overline{\Omega}))} &\leq \|J_Y - Q^T\|_{L^\infty(0,\infty;C^2(\overline{\Omega}))} \\
&\leq \|J_Y - I_3\|_{L^\infty(0,\infty;C^2(\overline{\Omega}))} + \|Q - I_3\|_{L^\infty(0,\infty;\mathbb{R}^{3\times 3})} \leq C\ \gamma.
\end{aligned}$$

■

Lemma 5.2. *If $\boldsymbol{f}(\boldsymbol{x},t)$ belongs to $L^\infty(0,\infty;C^2(\Omega))$ with $\boldsymbol{f}(\boldsymbol{x},t) \geq m > 0$ in $\Omega\times(0,\infty)$ then $1/\boldsymbol{f}$ belongs to $L^\infty(0,\infty;C^2(\Omega))$,*

$$\|1/\boldsymbol{f}\|_{L^\infty(0,\infty;C^2(\Omega))} \leq C\|\boldsymbol{f}\|_{L^\infty(0,\infty;C^2(\Omega))}. \tag{5.9}$$

Proof. Let $G \in C^\infty(\mathbb{R})$, non-negative such that $G(0) = 0$ and $G(r) = 1/r$ for $|r| \geq m$. Since the derivative of G is bounded and $G(0) = 0$, by the Mean value theorem, we have

$$|G(s)| \leq Ms \quad \forall\ s \in \mathbb{R}.$$

Thus $|G(\boldsymbol{f}(\boldsymbol{x},t))| \leq M|\boldsymbol{f}(\boldsymbol{x},t)|$ for every $\boldsymbol{x} \in \Omega$ which implies (5.9). ■

The following general embedding of $W^{2,1}_{q,p}(Q^\infty_F)$ is needed to cope with the gradient terms. It mainly relies on the mixed derivative theorem, followed by Sobolev embedding.

Lemma 5.3. [16, *Lemma* 4.2] *Let $\Omega_F(0)$ be a $\mathcal{C}^{1,1}$ domain with compact boundary, $p,q \in (1,\infty)$, $\theta \in (0,1)$ and $T > 0$. Also assume that $s = 0$ or $s = 1$ and $k,m \in (1,\infty)$ obeys $\frac{2-s}{2} + \frac{3}{2m} - \frac{3}{2q} \geq \frac{1}{p} - \frac{1}{q}$. Then*

$$W^{2,1}_{q,p}(Q^\infty_F) \hookrightarrow W^{\theta,p}(0,T;W^{2-2\theta,q}(\Omega_F(0))) \hookrightarrow L^k(0,T;W^{s,m}(\Omega_F(0))).$$

Now we are in the position to estimate the non-linear terms.

Proposition 5.4. *Let assume $p,q \in (1,\infty)$ satisfying the condition $\frac{1}{p} + \frac{3}{2q} \leq \frac{3}{2}$. There exists constants $\gamma_0 \in (0,1)$ and $C_N > 0$, depending only on p,q,η and $\Omega_F(0)$ such that for every $\gamma \in (0,\gamma_0)$ and for every $(\tilde{\boldsymbol{u}},\tilde{\pi},\tilde{\boldsymbol{l}},\tilde{\boldsymbol{\omega}}) \in S_\gamma$, we have*

$$\begin{aligned}
\|e^{\eta(\cdot)}\boldsymbol{F}_0\|_{L^p(0,\infty;\boldsymbol{L}^q(\Omega_F(0)))} &+ \|e^{\eta(\cdot)}\boldsymbol{H}\|_{W^{2,1}_{q,p}(Q^\infty_F)} \\
&+ \|e^{\eta(\cdot)}\boldsymbol{F}_1\|_{L^p(0,\infty;\mathbb{R}^3)} + \|e^{\eta(\cdot)}\boldsymbol{F}_2\|_{L^p(0,\infty;\mathbb{R}^3)} \leq C_N\gamma^2.
\end{aligned} \tag{5.10}$$

Moreover, there exists a constant $C_{\text{lip}} > 0$, depending only on p, q, η and $\Omega_F(0)$ such that for every $(\tilde{\boldsymbol{u}}^i, \tilde{\pi}^i, \tilde{\boldsymbol{l}}^i, \tilde{\boldsymbol{\omega}}^i) \in S_\gamma, i = 1, 2,$

$$\begin{aligned}
&\|e^{\eta(\cdot)}\boldsymbol{F}_0(\tilde{\boldsymbol{u}}^1, \tilde{\pi}^1, \tilde{\boldsymbol{l}}^1, \tilde{\boldsymbol{\omega}}^1) - \boldsymbol{F}_0(\tilde{\boldsymbol{u}}^2, \tilde{\pi}^2, \tilde{\boldsymbol{l}}^2, \tilde{\boldsymbol{\omega}}^2)\|_{L^p(0,\infty;\boldsymbol{L}^q(\Omega_F(0)))} \\
&+ \|e^{\eta(\cdot)}\boldsymbol{H}(\tilde{\boldsymbol{u}}^1, \tilde{\boldsymbol{l}}^1, \tilde{\boldsymbol{\omega}}^1) - e^{\eta(\cdot)}\boldsymbol{H}(\tilde{\boldsymbol{u}}^2, \tilde{\boldsymbol{l}}^2, \tilde{\boldsymbol{\omega}}^2)\|_{W^{2,1}_{q,p}(Q^\infty_F)} \\
&+ \|e^{\eta(\cdot)}\boldsymbol{F}_1(\tilde{\boldsymbol{l}}^1, \tilde{\boldsymbol{\omega}}^1) - e^{\eta(\cdot)}\boldsymbol{F}_1(\tilde{\boldsymbol{l}}^2, \tilde{\boldsymbol{\omega}}^2)\|_{L^p(0,\infty;\mathbb{R}^3)} + \|e^{\eta(\cdot)}\boldsymbol{F}_2(\tilde{\boldsymbol{\omega}}^1) - e^{\eta(\cdot)}\boldsymbol{F}_2(\tilde{\boldsymbol{\omega}}^2)\|_{L^p(0,\infty;\mathbb{R}^3)} \\
&\leq C_{\text{lip}}\gamma\ \|(\tilde{\boldsymbol{u}}^1, \tilde{\pi}^1, \tilde{\boldsymbol{l}}^1, \tilde{\boldsymbol{\omega}}^1) - (\tilde{\boldsymbol{u}}^2, \tilde{\pi}^2, \tilde{\boldsymbol{l}}^2, \tilde{\boldsymbol{\omega}}^2)\|_S.
\end{aligned} \tag{5.11}$$

Proof. **Estimate of $\boldsymbol{F}_0$:**

$$\|e^{\eta(\cdot)}\boldsymbol{F}_0\|_{L^p(0,\infty;\boldsymbol{L}^q(\Omega_F(0)))} \leq C\ \gamma^2.$$

• Estimate of the first three terms of $\boldsymbol{F}_0$. With the help of the estimates (5.1), (5.8) and (5.5), we get

$$\begin{aligned}
&\|e^{\eta(\cdot)}\left(((I_3 - Q)\partial_t\tilde{\boldsymbol{u}})_i - (Q(\tilde{\boldsymbol{\omega}} \times \tilde{\boldsymbol{u}}))_i - (\partial_t X \cdot J_Y^T\nabla)(Q\tilde{\boldsymbol{u}})_i\right)\|_{L^p(0,\infty;\boldsymbol{L}^q(\Omega_F(0)))} \\
&\leq C\big(\|I_3 - Q\|_{L^\infty(0,\infty;\mathbb{R}^{3\times3})} + \|\tilde{\boldsymbol{\omega}}\|_{L^\infty(0,\infty;\mathbb{R}^3)} \\
&\qquad\qquad + \|\partial_t X\|_{L^\infty(0,\infty;\Omega_F(0))}\|J_Y\|_{L^\infty(0,\infty;C^2(\overline{\Omega}))}\big)\|e^{\eta(\cdot)}\tilde{\boldsymbol{u}}\|_{W^{2,1}_{q,p}(Q^\infty_F)} \\
&\leq C\ \gamma^2.
\end{aligned}$$

• Estimate of the fourth term of $\boldsymbol{F}_0$. With the estimates (5.8) and the fact that $|Q(t)| \leq 1$ for all $t \geq 0$, we have

$$\begin{aligned}
&\|e^{\eta(\cdot)}\left((Q\tilde{\boldsymbol{u}}) \cdot (J_Y^T\nabla)\right)(Q\tilde{\boldsymbol{u}})_i\|_{L^p(0,\infty;\boldsymbol{L}^q(\Omega_F(0)))} \\
&\leq C\|e^{\eta(\cdot)}\tilde{\boldsymbol{u}}\|_{L^{3p}(0,\infty;\boldsymbol{L}^{3q}(\Omega_F(0)))}\|\nabla\tilde{\boldsymbol{u}}\|_{L^{3p/2}(0,\infty;\boldsymbol{L}^{3q/2}(\Omega_F(0)))}.
\end{aligned}$$

But due to the condition $\frac{1}{p} + \frac{3}{2q} \leq \frac{3}{2}$, the following embeddings

$$W^{2,1}_{q,p}(Q^\infty_F) \hookrightarrow L^{3p}(0,\infty;\boldsymbol{L}^{3q}(\Omega_F(0))) \quad \text{and} \quad W^{2,1}_{q,p}(Q^\infty_F) \hookrightarrow L^{3p/2}(0,\infty;\boldsymbol{L}^{3q/2}(\Omega_F(0)))$$

hold from lemma 5.3. Thus we obtain

$$\|e^{\eta(\cdot)}\left((Q\tilde{\boldsymbol{u}}) \cdot (J_Y^T\nabla)\right)(Q\tilde{\boldsymbol{u}})_i\|_{L^p(0,\infty;\boldsymbol{L}^q(\Omega_F(0)))} \leq C\ \gamma^2.$$

• Estimate of the fifth term of $\boldsymbol{F}_0$. It follows from (5.4) immediately that

$$\begin{aligned}
&\|e^{\eta(\cdot)}\sum_{m,l,j}\frac{\partial(Q\tilde{\boldsymbol{u}})_i}{\partial y_l}\frac{\partial Y_m}{\partial x_j}\frac{\partial}{\partial y_m}\left(\frac{\partial Y_l}{\partial x_j}\right)\|_{L^p(0,\infty;\boldsymbol{L}^q(\Omega_F(0)))} \\
&\leq C\|\frac{\partial^2 Y}{\partial x_j\partial x_k}\|_{L^\infty(0,\infty;C^2(\overline{\Omega}))}\|e^{\eta(\cdot)}\tilde{\boldsymbol{u}}\|_{W^{2,1}_{q,p}(Q^\infty_F)} \leq C\ \gamma^2.
\end{aligned}$$

• Estimate of the sixth and seventh term of $\boldsymbol{F}_0$. We can re-write the two terms as,

$$\begin{aligned}
&\sum_{m,l,j}\frac{\partial^2(Q\tilde{\boldsymbol{u}})_i}{\partial y_m\partial y_l}\frac{\partial Y_l}{\partial x_j}\frac{\partial Y_m}{\partial x_j} - \Delta\tilde{\boldsymbol{u}}_i \\
&= \sum_{m,l,j}\frac{\partial^2(Q\tilde{\boldsymbol{u}})_i}{\partial y_m\partial y_l}\left(\frac{\partial Y_l}{\partial x_j} - \delta_{lj}\right)\frac{\partial Y_m}{\partial x_j} + \sum_{m,l}\frac{\partial^2(Q\tilde{\boldsymbol{u}})_i}{\partial y_m\partial y_l}\left(\frac{\partial Y_m}{\partial x_l} - \delta_{ml}\right) + ((Q - I_3)\Delta\tilde{\boldsymbol{u}})_i
\end{aligned}$$

Therefore, it follows from (5.1), (5.8) and (5.3) that

$$\|e^{\eta(\cdot)}\left(\sum_{m,l,j}\frac{\partial^2(Q\tilde{\boldsymbol{u}})_i}{\partial y_m\partial y_l}\frac{\partial Y_l}{\partial x_j}\frac{\partial Y_m}{\partial x_j}-\Delta\tilde{\boldsymbol{u}}_i\right)\|_{L^p(0,\infty;\boldsymbol{L}^q(\Omega_F(0)))}$$
$$\leq C\left(\|J_Y-I_3\|_{L^\infty(0,\infty;C^2(\overline{\Omega}))}+\|Q-I_3\|_{L^\infty(0,\infty;\mathbb{R}^{3\times 3})}\right)\|e^{\eta(\cdot)}\tilde{\boldsymbol{u}}\|_{W^{2,1}_{q,p}(Q_F^\infty)}\leq C\,\gamma^2.$$

• Estimate of the last term of $\boldsymbol{F}_0$. Finally, the following estimate follows from (5.3),

$$\|e^{\eta(\cdot)}\left((I_3-J_Y^T)\nabla\tilde{\pi}\right)_i\|_{L^p(0,\infty;\boldsymbol{L}^q(\Omega_F(0)))}$$
$$\leq\|I_3-J_Y\|_{L^\infty(0,\infty;C^2(\overline{\Omega}))}\|e^{\eta(\cdot)}\nabla\tilde{\pi}\|_{L^p(0,\infty;L^q(\Omega_F(0)))}\leq C\,\gamma^2.$$

Estimate of $\boldsymbol{H}$:

$$\|e^{\eta(\cdot)}\boldsymbol{H}\|_{W^{2,1}_{q,p}(Q_F^\infty)}\leq C\,\gamma^2.$$

We obtain from (5.6),

$$\|e^{\eta(\cdot)}(I_3-J_YQ)\tilde{\boldsymbol{u}}\|_{W^{2,1}_{q,p}(Q_F^\infty)}\leq\|I_3-J_YQ\|_{L^\infty(0,\infty;C^2(\overline{\Omega}))}\|e^{\eta(\cdot)}\tilde{\boldsymbol{u}}\|_{W^{2,1}_{q,p}(Q_F^\infty)}\leq C\,\gamma^2.$$

Estimate of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$: From the expressions of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$, it is obvious to see

$$\|e^{\eta(\cdot)}\boldsymbol{F}_1\|_{L^p(0,\infty;\mathbb{R}^3)}+\|e^{\eta(\cdot)}\boldsymbol{F}_2\|_{L^p(0,\infty;\mathbb{R}^3)}\leq C\,\gamma^2.$$

This completes the proof of the estimate (5.10).

The Lipschitz property can be proved in the same way. ■

Theorem 5.5. *Let $\Omega_F(0)$ be a bounded domain of class $\mathcal{C}^{2,1}$, $p,q\in(1,\infty)$ satisfy the condition $\frac{1}{p}+\frac{3}{2q}\leq\frac{3}{2}$ and $\alpha\geq 0$ be as in (2.4). Let $\eta\in(0,\eta_0)$ where η_0 is the constant introduced in Theorem 4.8. Then there exist a constant $\tilde{\gamma}>0$ depending only on p,q,η and $\Omega_F(0)$ such that for all $\gamma\in(0,\tilde{\gamma})$ and for all $(\boldsymbol{u}_0,\boldsymbol{l}_0,\boldsymbol{\omega}_0)\in B^{2(1-1/p)}_{q,p}(\Omega_F(0))\times\mathbb{R}^3\times\mathbb{R}^3$ satisfying the compatibility condition (2.5) and*

$$\|\boldsymbol{u}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega_F(0))}+\|\boldsymbol{l}_0\|_{\mathbb{R}^3}+\|\boldsymbol{\omega}_0\|_{\mathbb{R}^3}\leq\frac{\gamma}{2C_L},\tag{5.12}$$

where C_L is the continuity constant appeared in Theorem 4.10, the system (7.7)-(7.11) admits a unique strong solution $(\tilde{\boldsymbol{u}},\tilde{\pi},\tilde{\boldsymbol{l}},\tilde{\boldsymbol{\omega}})$ such that

$$\|(\tilde{\boldsymbol{u}},\tilde{\pi},\tilde{\boldsymbol{l}},\tilde{\boldsymbol{\omega}})\|_S\leq\gamma.$$

Proof. Let us define

$$\tilde{\gamma}=\min\{\gamma_0,\frac{1}{2C_LC_N},\frac{1}{2C_LC_{\text{lip}}}\}$$

where γ_0 is defined as in (5.7) and C_L, C_N, C_{lip} are the constants appearing in Theorem 4.10, Proposition 5.4 and Proposition . Let $\gamma\in(0,\tilde{\gamma})$. We will show that the mapping

$$\mathcal{N}:(\boldsymbol{v},\varphi,\boldsymbol{\kappa},\boldsymbol{\tau})\mapsto(\tilde{\boldsymbol{u}},\tilde{\pi},\tilde{\boldsymbol{l}},\tilde{\boldsymbol{\omega}})$$

which maps $(\boldsymbol{v},\varphi,\boldsymbol{\kappa},\boldsymbol{\tau})\in S_\gamma$ to the solution $(\tilde{\boldsymbol{u}},\tilde{\pi},\tilde{\boldsymbol{l}},\tilde{\boldsymbol{\omega}})$ of the linear problem (7.7) with right hand sides $\boldsymbol{F}_0(\boldsymbol{v},\varphi,\boldsymbol{\kappa},\boldsymbol{\tau}),\boldsymbol{H}(\boldsymbol{v},\boldsymbol{\kappa},\boldsymbol{\tau}),\boldsymbol{F}_1(\boldsymbol{\kappa},\boldsymbol{\tau}),\boldsymbol{F}_2(\boldsymbol{\tau})$, is a contraction in S_γ. The fixed point of $\mathcal{N}$ then satisfies (7.7)-(7.11).

First we prove that the image of $\mathcal{N}$ is contained in S_γ. We can apply Theorem 4.10 to estimate the solution $(\tilde{\boldsymbol{u}},\tilde{\pi},\tilde{\boldsymbol{l}},\tilde{\boldsymbol{\omega}})$ in terms of the given data and then (5.10), (5.12) and the definition of $\tilde{\gamma}$ to estimate the given data further to obtain

$$\begin{aligned}\|\mathcal{N}(\boldsymbol{v},\varphi,\boldsymbol{\kappa},\boldsymbol{\tau})\|_S &\leq C_L\Big(\|\boldsymbol{u}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega_F(0))}+\|\boldsymbol{l}_0\|_{\mathbb{R}^3}+\|\boldsymbol{\omega}_0\|_{\mathbb{R}^3}+\|\boldsymbol{F}_0\|_{L^p(0,\infty;\boldsymbol{L}^q(\Omega_F(0)))}\\ &\qquad +\|\boldsymbol{H}\|_{W^{2,1}_{q,p}(Q_F^\infty)}+\|\boldsymbol{F}_1\|_{L^p(0,\infty;\mathbb{R}^3)}+\|\boldsymbol{F}_2\|_{L^p(0,\infty;\mathbb{R}^3)}\Big)\\ &\leq \gamma.\end{aligned}$$

Thus $\mathcal{N}$ is the mapping from S_γ to itself for all $\gamma\in(0,\tilde{\gamma})$. Next, to prove that $\mathcal{N}$ is a contraction, let $(\boldsymbol{v}^i,\varphi^i,\boldsymbol{\kappa}^i,\boldsymbol{\tau}^i)\in S_\gamma, i=1,2$ and we use the index $i\in\{1,2\}$ on a function to denote that it is associated to $(\boldsymbol{v}^i,\varphi^i,\boldsymbol{\kappa}^i,\boldsymbol{\tau}^i)$; For example, $\boldsymbol{F}_0^1$ means $\boldsymbol{F}_0(\boldsymbol{v}^1,\varphi^1,\boldsymbol{\kappa}^1,\boldsymbol{\tau}^1)$, $(\boldsymbol{F}_1)^2$ means $\boldsymbol{F}_1(\boldsymbol{v}^2,\varphi^2,\boldsymbol{\kappa}^2,\boldsymbol{\tau}^2)$ and so on. We then estimate similarly, using Theorem 4.10 and (5.11) ,

$$\begin{aligned}&\|\mathcal{N}(\boldsymbol{v}^1,\varphi^1,\boldsymbol{\kappa}^1,\boldsymbol{\tau}^1)-\mathcal{N}(\boldsymbol{v}^2,\varphi^2,\boldsymbol{\kappa}^2,\boldsymbol{\tau}^2)\|_S\\ &\leq C_L\Big(\|\boldsymbol{F}_0^1-\boldsymbol{F}_0^2\|_{L^p(0,\infty;\boldsymbol{L}^q(\Omega_F(0)))}+\|\boldsymbol{H}_1-\boldsymbol{H}_2\|_{W^{2,1}_{q,p}(Q_F^\infty)}+\|(\boldsymbol{F}_1^1-\boldsymbol{F}_1^2\|_{L^p(0,\infty;\mathbb{R}^3)}\\ &\qquad +\|\boldsymbol{F}_2^1-\boldsymbol{F}_2^2\|_{L^p(0,\infty;\mathbb{R}^3)}\Big)\\ &\leq C_LC_{\text{lip}}\gamma\|(\boldsymbol{v}_1,\varphi_1,\boldsymbol{\kappa}_1,\boldsymbol{\tau}_1)-(\boldsymbol{v}_2,\varphi_2,\boldsymbol{\kappa}_2,\boldsymbol{\tau}_2)\|_S.\end{aligned}$$

Now the definition of $\tilde{\gamma}$ concludes the proof. ∎

Proof of Theorem 2.1. The solution to the original problem (2.1) can be obtained from the corresponding backward change of coordinates and variables, given in (7.5) which preserves regularity. Moreover, the solution $(\boldsymbol{u},\pi,\boldsymbol{l},\boldsymbol{\omega})$ to the original problem must be unique as a consequence of the uniqueness of the fixed point.

Since $\gamma<\tilde{\gamma}$, condition (7.1) is verified and $X(\cdot,t)$ is a well-defined C^1-diffeomorphism from $\Omega_F(0)$ to $\Omega_F(t)$ for every $t\in[0,\infty)$. Therefore there exists a unique $Y(\cdot,t)$ as defined in Lemma 7.2. For all $t\in[0,\infty)$ and $\boldsymbol{x}\in\Omega_F(t)$, setting

$$\begin{aligned}\boldsymbol{u}(\boldsymbol{x},t)&=Q(t)\tilde{\boldsymbol{u}}(Y(\boldsymbol{x},t),t), &\quad \pi(\boldsymbol{x},t)&=\tilde{\pi}(Y(\boldsymbol{x},t),t),\\ \boldsymbol{l}(t)&=Q(t)\tilde{\boldsymbol{l}}(t), &\quad \boldsymbol{\omega}(t)&=Q(t)\tilde{\boldsymbol{\omega}}(t),\end{aligned}$$

the new variables $(\boldsymbol{u},\pi,\boldsymbol{l},\boldsymbol{\omega})$ satisfy the original system (2.1) with the estimate (2.6). Note that all the derivatives of the solution $(\boldsymbol{u},\pi,\boldsymbol{l},\boldsymbol{\omega})$ are combinations of $(\tilde{\boldsymbol{u}},\tilde{\pi},\tilde{\boldsymbol{l}},\tilde{\boldsymbol{\omega}})$ multiplied at most by X and its derivatives which are smooth enough to get the prescribed regularity. ∎

6 Non-Newtonian case

In this section, we discuss the non-Newtonian case. The main difference and the difficulty here is that we need to replace the Laplacian in the fluid equation by a quasi-linear operator arising from the Generalized stress tensor T. Observe that we can write this new term as,

$$
\begin{aligned}
\left[\operatorname{div}\left(\mu(|\mathbb{D}\boldsymbol{u}|^2)\mathbb{D}\boldsymbol{u}\right)\right]_i &= \sum_{j=1}^{3} \partial_j \left(\mu(|\mathbb{D}\boldsymbol{u}|^2)\mathbb{D}_{ij}\boldsymbol{u}\right) \\
&= \frac{1}{2}\mu(|\mathbb{D}\boldsymbol{u}|^2) \sum_{j=1}^{3} \left(\partial_{ij}^2 u_j + \partial_j^2 u_i\right) + \mu'(|\mathbb{D}\boldsymbol{u}|^2) \sum_{j,k,l=1}^{3} 2\mathbb{D}_{ij}\boldsymbol{u}\ \mathbb{D}_{kl}\boldsymbol{u}\ \partial_j \mathbb{D}_{kl}\boldsymbol{u} \\
&= \frac{1}{2}\mu(|\mathbb{D}\boldsymbol{u}|^2) \sum_{j=1}^{3} \left(\partial_{ij}^2 u_j + \partial_j^2 u_i\right) + 2\mu'(|\mathbb{D}\boldsymbol{u}|^2) \sum_{j,k,l=1}^{3} \mathbb{D}_{ij}\boldsymbol{u}\ \mathbb{D}_{kl}\boldsymbol{u}\ \partial_j \partial_k u_l \\
&= \sum_{j,k,l=1}^{3} a_{ij}^{kl}(\boldsymbol{u}) \partial_j \partial_k u_l
\end{aligned}
$$

where

$$a_{ij}^{kl}(\boldsymbol{u}) := \frac{1}{2}\mu(|\mathbb{D}\boldsymbol{u}|^2)\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + 2\mu'(|\mathbb{D}\boldsymbol{u}|^2)\mathbb{D}_{ij}\boldsymbol{u}\ \mathbb{D}_{kl}\boldsymbol{u}.$$

Note that the coefficients $a_{ij}^{kl}(\boldsymbol{u})$ are real. Consider the differential operator

$$A(\boldsymbol{u}) := \sum_{j,k=1}^{3} a_{ij}^{kl}(\boldsymbol{u}) \partial_j \partial_k$$

which is defined precisely as

$$\left(A(\boldsymbol{u})\boldsymbol{w}\right)_i := \sum_{j,k,l=1}^{3} a_{ij}^{kl}(\boldsymbol{u}) \partial_j \partial_k w_l.$$

As done in the Newtonian case, we transfer the system of equations (1.2) defined on an unknown moving domain to a fixed domain by coordinate transformation. We use the exact same change of variables as in (7.5) where the transformed generalized stress tensor is defined by

$$\tilde{T}(\tilde{\boldsymbol{u}}, \tilde{\pi}) = Q^{-1}(t)\ T(Q(t)\tilde{\boldsymbol{u}}(\boldsymbol{y}, t), \tilde{\pi}(\boldsymbol{y}, t)\ Q(t).$$

To transform the term $A(\boldsymbol{u})\boldsymbol{u}$, we calculate

$$2\mathbb{D}_{ij}\boldsymbol{u} = \partial_i u_j + \partial_j u_i = \sum_{k=1}^{3} \frac{(Q\tilde{\boldsymbol{u}})_j}{\partial y_k} \frac{\partial Y_k}{\partial x_i} + \sum_{l=1}^{3} \frac{(Q\tilde{\boldsymbol{u}})_i}{\partial y_l} \frac{\partial Y_l}{\partial x_j} =: 2\tilde{\mathbb{D}}_{ij}\tilde{\boldsymbol{u}}$$

and use the obvious notation $\tilde{\mathbb{D}}\boldsymbol{w} = \left(\tilde{\mathbb{D}}_{ij}\boldsymbol{w}\right)_{ij}$ to denote the transformed symmetric part of the gradient. Therefore, the transformed quasi-linear fluid operator can be written as

$$\left(\mathcal{A}(\tilde{\boldsymbol{u}})\boldsymbol{w}\right)_i = \tilde{a}_{ij}^{klm}(\tilde{\boldsymbol{u}}) \partial_j \partial_k w_l$$

where

$$\tilde{a}_{ij}^{klm}(\tilde{\boldsymbol{u}}) := \frac{1}{2}\mu(|\tilde{\mathbb{D}}\tilde{\boldsymbol{u}}|^2)\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + 2\mu'(|\tilde{\mathbb{D}}\tilde{\boldsymbol{u}}|^2)\tilde{\mathbb{D}}_{ij}\tilde{\boldsymbol{u}}\ \tilde{\mathbb{D}}_{kl}\tilde{\boldsymbol{u}}.$$

On the fixed domain, the transformed system then becomes,

$$\begin{cases} \tilde{\boldsymbol{u}}_t - \mathcal{A}(\tilde{\boldsymbol{u}})\tilde{\boldsymbol{u}} + \nabla\tilde{\pi} = \boldsymbol{F}_0(\tilde{\boldsymbol{u}}, \tilde{\pi}, \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}}) & \text{in } \Omega_F(0)\times(0,T), \\ \text{div } \tilde{\boldsymbol{u}} = \text{div } \boldsymbol{H}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}}) & \text{in } \Omega_F(0)\times(0,T), \\ \tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{n}} = 0, \quad \left[\tilde{T}(\tilde{\boldsymbol{u}},\tilde{\pi})\tilde{\boldsymbol{n}}\right]_{\boldsymbol{\tau}} + \alpha\tilde{\boldsymbol{u}}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \partial\Omega\times(0,T), \\ \tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{n}} = \tilde{\boldsymbol{u}}_S\cdot\tilde{\boldsymbol{n}}, \quad \left[\tilde{T}(\tilde{\boldsymbol{u}},\tilde{\pi})\tilde{\boldsymbol{n}}\right]_{\boldsymbol{\tau}} + \alpha\tilde{\boldsymbol{u}}_{\boldsymbol{\tau}} = \alpha\tilde{\boldsymbol{u}}_{S\boldsymbol{\tau}} & \text{on } \partial\Omega_S(0)\times(0,T), \\ m\tilde{\boldsymbol{l}}' = -\int\limits_{\partial\Omega_S(0)} \tilde{T}(\tilde{\boldsymbol{u}},\tilde{\pi})\tilde{\boldsymbol{n}} + \boldsymbol{F}_1(\tilde{\boldsymbol{l}},\tilde{\boldsymbol{\omega}}), & t\in(0,T), \\ \tilde{J}\tilde{\boldsymbol{\omega}}' = -\int\limits_{\partial\Omega_S(0)} y\times\tilde{T}(\tilde{\boldsymbol{u}},\tilde{\pi})\tilde{\boldsymbol{n}} + \boldsymbol{F}_2(\tilde{\boldsymbol{\omega}}), & t\in(0,T), \\ \tilde{\boldsymbol{u}}(0) = \boldsymbol{u}_0 & \text{in } \Omega_F(0), \\ \tilde{\boldsymbol{h}}'(0) = \boldsymbol{l}_0, \quad \tilde{\boldsymbol{\omega}}(0) = \boldsymbol{\omega}_0 \end{cases} \tag{6.1}$$

where $\boldsymbol{F}_0, \boldsymbol{H}, \boldsymbol{F}_1, \boldsymbol{F}_2$ are defined in (7.8) - (7.11).

Next we linearize the above system and prove the maximal regularity of the linear problem. We linearize $\mathcal{A}$ by the operator A_*, defined as

$$A_*\tilde{\boldsymbol{u}} := A(\boldsymbol{u}^*)\tilde{\boldsymbol{u}}$$

which fixes the coefficients in the original operator A to a reference solution $\boldsymbol{u}^*$ of the problem

$$\begin{cases} \partial_t\boldsymbol{u}^* - \Delta\boldsymbol{u}^* + \nabla\pi^* = \boldsymbol{0} & \text{in } \Omega_F(0)\times(0,T), \\ \text{div } \boldsymbol{u}^* = 0 & \text{in } \Omega_F(0)\times(0,T), \\ \boldsymbol{u}^*\cdot\tilde{\boldsymbol{n}} = 0, \quad [\sigma(\boldsymbol{u}^*,\pi^*)\tilde{\boldsymbol{n}}]_{\boldsymbol{\tau}} + \alpha\boldsymbol{u}^*_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \partial\Omega\times(0,T), \\ \boldsymbol{u}^*\cdot\tilde{\boldsymbol{n}} = (\boldsymbol{l}^* + \boldsymbol{\omega}^*\times\boldsymbol{y})\cdot\tilde{\boldsymbol{n}} & \text{on } \partial\Omega_S(0)\times(0,T), \\ [\sigma(\boldsymbol{u}^*,\pi^*)\tilde{\boldsymbol{n}}]_{\boldsymbol{\tau}} + \alpha\boldsymbol{u}^*_{\boldsymbol{\tau}} = \alpha(\boldsymbol{l}^* + \boldsymbol{\omega}^*\times\boldsymbol{y})_{\boldsymbol{\tau}} & \text{on } \partial\Omega_S(0)\times(0,T), \\ m(\boldsymbol{l}^*)' = -\int\limits_{\partial\Omega_S(0)} \sigma(\boldsymbol{u}^*,\pi^*)\tilde{\boldsymbol{n}}, & t\in(0,T), \\ J(0)(\boldsymbol{\omega}^*)' = -\int\limits_{\partial\Omega_S(0)} \boldsymbol{y}\times\sigma(\boldsymbol{u}^*,\pi^*)\tilde{\boldsymbol{n}}, & t\in(0,T), \\ \boldsymbol{u}^*(0) = \boldsymbol{u}_0 & \text{in } \Omega_F(0), \\ \boldsymbol{l}^*(0) = \boldsymbol{l}_0, \quad \boldsymbol{\omega}^*(0) = \boldsymbol{\omega}_0. \end{cases} \tag{6.2}$$

The existence of $(\boldsymbol{u}^*, \pi^*, \boldsymbol{l}^*, \boldsymbol{\omega}^*)$ satisfying (6.2) follows from Theorem 4.7. We then define

$(\hat{\boldsymbol{u}}, \hat{\pi}, \hat{\boldsymbol{l}}, \hat{\boldsymbol{\omega}}) := (\tilde{\boldsymbol{u}} - \boldsymbol{u}^*, \tilde{\pi} - \pi^*, \tilde{\boldsymbol{l}} - \boldsymbol{l}^*, \tilde{\boldsymbol{\omega}} - \boldsymbol{\omega}^*)$ to rewrite (6.1) into the equivalent system

$$\begin{cases} \partial_t \hat{\boldsymbol{u}} - A_* \hat{\boldsymbol{u}} + \nabla \hat{\pi} = \boldsymbol{G}_0(\hat{\boldsymbol{u}}, \hat{\pi}, \hat{\boldsymbol{l}}, \hat{\boldsymbol{\omega}}) & \text{in } \Omega_F(0) \times (0,T), \\ \operatorname{div} \hat{\boldsymbol{u}} = \operatorname{div} \boldsymbol{H}(\hat{\boldsymbol{u}}, \hat{\boldsymbol{l}}, \hat{\boldsymbol{\omega}}) & \text{in } \Omega_F(0) \times (0,T), \\ \boldsymbol{u}^* \cdot \tilde{\boldsymbol{n}} = 0, \quad [\sigma(\hat{\boldsymbol{u}}, \hat{\pi})\tilde{\boldsymbol{n}}]_{\boldsymbol{\tau}} + \alpha \hat{\boldsymbol{u}}_{\boldsymbol{\tau}} = \boldsymbol{H}_1(\hat{\boldsymbol{u}}, \hat{\pi}, \hat{\boldsymbol{l}}, \hat{\boldsymbol{\omega}}) & \text{on } \partial\Omega \times (0,T), \\ \hat{\boldsymbol{u}} \cdot \tilde{\boldsymbol{n}} = (\hat{\boldsymbol{l}} + \hat{\boldsymbol{\omega}} \times \boldsymbol{y}) \cdot \tilde{\boldsymbol{n}} & \text{on } \partial\Omega_S(0) \times (0,T), \\ [\sigma(\hat{\boldsymbol{u}}, \hat{\pi})\tilde{\boldsymbol{n}}]_{\boldsymbol{\tau}} + \alpha \hat{\boldsymbol{u}}_{\boldsymbol{\tau}} = \alpha(\hat{\boldsymbol{l}} + \hat{\boldsymbol{\omega}} \times \boldsymbol{y})_{\boldsymbol{\tau}} + \boldsymbol{H}_1(\hat{\boldsymbol{u}}, \hat{\pi}, \hat{\boldsymbol{l}}, \hat{\boldsymbol{\omega}}) & \text{on } \partial\Omega_S(0) \times (0,T), \\ m\,\hat{\boldsymbol{l}}' = -\displaystyle\int_{\partial\Omega_S(0)} \sigma(\hat{\boldsymbol{u}}, \hat{\pi})\tilde{\boldsymbol{n}} + \boldsymbol{G}_1(\hat{\boldsymbol{u}}, \hat{\pi}, \hat{\boldsymbol{l}}, \hat{\boldsymbol{\omega}}), & t \in (0,T), \\ J(0)\,\hat{\boldsymbol{\omega}}' = -\displaystyle\int_{\partial\Omega_S(0)} \boldsymbol{y} \times \sigma(\hat{\boldsymbol{u}}, \hat{\pi})\tilde{\boldsymbol{n}} + \boldsymbol{G}_2(\hat{\boldsymbol{u}}, \hat{\pi}, \hat{\boldsymbol{l}}, \hat{\boldsymbol{\omega}}), & t \in (0,T), \\ \hat{\boldsymbol{u}}(0) = \boldsymbol{0} & \text{in } \Omega_F(0), \\ \hat{\boldsymbol{l}}(0) = \boldsymbol{0}, \quad \hat{\boldsymbol{\omega}}(0) = \boldsymbol{0} \end{cases} \tag{6.3}$$

where

$$\begin{aligned} \boldsymbol{G}_0(\hat{\boldsymbol{u}}, \hat{\pi}, \hat{\boldsymbol{l}}, \hat{\boldsymbol{\omega}}) &:= \boldsymbol{F}_0(\hat{\boldsymbol{u}}, \hat{\pi}, \hat{\boldsymbol{l}}, \hat{\boldsymbol{\omega}}) - \Delta \boldsymbol{u}^* - Q(\boldsymbol{u}^*, \hat{\boldsymbol{u}}), \\ Q(\boldsymbol{u}^*, \hat{\boldsymbol{u}}) &:= A_* \hat{\boldsymbol{u}} - \mathcal{A}(\boldsymbol{u}^* + \hat{\boldsymbol{u}})(\boldsymbol{u}^* + \hat{\boldsymbol{u}}), \\ \boldsymbol{G}_1(\hat{\boldsymbol{u}}, \hat{\pi}, \hat{\boldsymbol{l}}, \hat{\boldsymbol{\omega}}) &:= \int_{\partial\Omega_S(0)} (\sigma - \tilde{T})(\hat{\boldsymbol{u}}, \hat{\pi})\tilde{\boldsymbol{n}} + \int_{\partial\Omega_S(0)} (\sigma - \tilde{T})(\boldsymbol{u}^*, \pi^*)\tilde{\boldsymbol{n}} - m(\boldsymbol{\omega}^* + \hat{\boldsymbol{\omega}}) \times (\boldsymbol{l}^* + \hat{\boldsymbol{l}}), \\ \boldsymbol{G}_2(\hat{\boldsymbol{u}}, \hat{\pi}, \hat{\boldsymbol{l}}, \hat{\boldsymbol{\omega}}) &:= \int_{\partial\Omega_S(0)} \boldsymbol{y} \times (\sigma - \tilde{T})(\hat{\boldsymbol{u}}, \hat{\pi})\tilde{\boldsymbol{n}} + \int_{\partial\Omega_S(0)} \boldsymbol{y} \times (\sigma - \tilde{T})(\boldsymbol{u}^*, \pi^*)\tilde{\boldsymbol{n}} - \tilde{J}(\boldsymbol{\omega}^* + \hat{\boldsymbol{\omega}}) \times (\boldsymbol{\omega}^* + \hat{\boldsymbol{\omega}}), \\ \boldsymbol{H}_1(\hat{\boldsymbol{u}}, \hat{\pi}, \hat{\boldsymbol{l}}, \hat{\boldsymbol{\omega}}) &:= \Big[(\sigma - \tilde{T})(\hat{\boldsymbol{u}}, \hat{\pi})\tilde{\boldsymbol{n}} + (\sigma - \tilde{T})(\boldsymbol{u}^*, \pi^*)\tilde{\boldsymbol{n}}\Big]_{\boldsymbol{\tau}} \end{aligned} \tag{6.4}$$

and $\boldsymbol{F}_0, \boldsymbol{H}$ are the same as in (7.8). Fixing $\boldsymbol{G}_0, \boldsymbol{H}, \boldsymbol{H}_1, \boldsymbol{G}_1, \boldsymbol{G}_2$ yields the linearization of (6.1) that we would like to study. The main result concerning the linearized problem is the maximal regularity, stated in the following theorem.

Theorem 6.1. *Let $\Omega_F(0)$ be a bounded domain of class $\mathcal{C}^{2,1}$, $p > 5$ and $\alpha \geq 0$ be as in (2.4). Also assume that*

$$g_0 \in L^p(0,T; \boldsymbol{L}^p(\Omega_F(0))), \chi \in W^{2,1}_{p,p}(Q_F^T), \boldsymbol{l} \in L^p(0,T;\mathbb{R}^3), \boldsymbol{\omega} \in L^p(0,T;\mathbb{R}^3)$$

and

$$h_1 \in W^{\frac{1}{2}-\frac{1}{2p},p}(0,T; \boldsymbol{L}^p(\partial\Omega_F(0))) \cap L^p(0,T; \boldsymbol{W}^{1-\frac{1}{p}}(\partial\Omega_F(0)))$$

where

$$\operatorname{div} \chi|_{t=0} = 0, h_1 \cdot \tilde{\boldsymbol{n}} = 0 \text{ on } \partial\Omega_F(0).$$

Then the problem

$$
\begin{cases}
\partial_t \boldsymbol{u} - A_* \boldsymbol{u} + \nabla \pi = g_0 & \text{in } \Omega_F(0) \times (0,T), \\
\text{div } \boldsymbol{u} = \text{div } \chi & \text{in } \Omega_F(0) \times (0,T), \\
\boldsymbol{u} \cdot \tilde{\boldsymbol{n}} = 0, \quad [\sigma(\boldsymbol{u},\pi)\tilde{\boldsymbol{n}}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = h_1 & \text{on } \partial\Omega \times (0,T), \\
\boldsymbol{u} \cdot \tilde{\boldsymbol{n}} = (\boldsymbol{l} + \boldsymbol{\omega} \times \boldsymbol{y}) \cdot \tilde{\boldsymbol{n}} & \text{on } \partial\Omega_S(0) \times (0,T), \\
[\sigma(\boldsymbol{u},\pi)\tilde{\boldsymbol{n}}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \alpha(\boldsymbol{l} + \boldsymbol{\omega} \times \boldsymbol{y})_{\boldsymbol{\tau}} + h_1 & \text{on } \partial\Omega_S(0) \times (0,T), \\
m\boldsymbol{l}' = - \int\limits_{\partial\Omega_S(0)} \sigma(\boldsymbol{u},\pi)\tilde{\boldsymbol{n}} + g_1, & t \in (0,T), \\
J(0)\boldsymbol{\omega}' = - \int\limits_{\partial\Omega_S(0)} \boldsymbol{y} \times \sigma(\boldsymbol{u},\pi)\tilde{\boldsymbol{n}} + g_2, & t \in (0,T), \\
\boldsymbol{u}(0) = \boldsymbol{0} & \text{in } \Omega_F(0), \\
\boldsymbol{l}(0) = \boldsymbol{0}, \quad \boldsymbol{\omega}(0) = \boldsymbol{0}
\end{cases}
\tag{6.5}
$$

has a unique solution $\boldsymbol{u} \in W^{2,1}_{q,p}(Q^\infty_F), \pi \in L^p(0,T;W^{1,p}(\Omega_F(0))),\ (\boldsymbol{l},\boldsymbol{\omega}) \in W^{1,p}(0,T;\mathbb{R}^6)$ *satisfying the estimate*

$$
\begin{aligned}
&\|\boldsymbol{u}\|_{W^{2,1}_{p,p}(Q^\infty_F)} + \|\pi\|_{L^p(0,\infty;W^{1,p}(\Omega_F(0)))} + \|\boldsymbol{l}\|_{L^p(0,\infty;\mathbb{R}^3)} + \|\boldsymbol{\omega}\|_{L^p(0,\infty;\mathbb{R}^3)} \\
&\leq C \Big(\|g_0\|_{L^p(0,\infty;\boldsymbol{L}^q(\Omega_F(0)))} + \|\chi\|_{W^{2,1}_{q,p}(Q^\infty_F)} + \|(g_1,g_2)\|_{L^p(0,\infty;\mathbb{R}^6)} \\
&\qquad + \|h_1\|_{W^{\frac{1}{2}-\frac{1}{2p},p}(0,T;\boldsymbol{L}^p(\partial\Omega_F(0))) \cap L^p(0,T;\boldsymbol{W}^{1-\frac{1}{p}}(\partial\Omega_F(0)))} \Big).
\end{aligned}
\tag{6.6}
$$

Unlike in the Newtonian case, we reduce the inhomogeneous divergence condition to the divergence-free problem and then treat the system. To show the maximal regularity property, we split the fluid and the solid equations and with the help of the maximal regularity of the *generalized Stokes operator* (which is proved in [40]), we rewrite suitably the forces acting on the rigid body as in [22]. In the following subsection, we combine the relevant results concerning the linear fluid-structure problem corresponding to the non-Newtonian fluid.

6.1 Maximal regularity of the Linearized system

We obtain divergence free and homogeneous boundary conditions for the problem by subtracting the solution $(\mathring{\boldsymbol{u}}, \mathring{\pi})$ from $(\boldsymbol{u}, \pi)$.

Proposition 6.2. [8, *Theorem* 4.1]. *Let* $p > 5$ *and* $\alpha \geq 0$ *be as in (2.4). Further assume that*

$$
g_0 \in L^p(0,T;\boldsymbol{L}^p(\Omega_F(0))), \chi \in W^{2,1}_{p,p}(Q^T_F), \boldsymbol{l} \in L^p(0,T;\mathbb{R}^3), \boldsymbol{\omega} \in L^p(0,T;\mathbb{R}^3)
$$

and

$$
h_1 \in W^{\frac{1}{2}-\frac{1}{2p},p}(0,T;\boldsymbol{L}^p(\partial\Omega_F(0))) \cap L^p(0,T;\boldsymbol{W}^{1-\frac{1}{p}}(\partial\Omega_F(0))), \boldsymbol{u}_0 \in W^{2-2/p,p}(\Omega_F(0))
$$

where

$$
\text{div } \chi|_{t=0} = 0,\ (\boldsymbol{l} + \boldsymbol{\omega} \times \boldsymbol{y}) \cdot \tilde{\boldsymbol{n}}|_{t=0} = \boldsymbol{0} \text{ on } \partial\Omega_S(0),\ h_1 \cdot \tilde{\boldsymbol{n}} = 0 \text{ on } \partial\Omega_F(0)
$$

and

$$\mathbb{P}h_1 + \alpha\mathbb{P}(\boldsymbol{l} + \boldsymbol{\omega}\times\boldsymbol{y})_{\boldsymbol{\tau}}\mathbb{1}_{\partial\Omega_S(0)} = \mathbf{0}.$$

Then there exists a unique strong solution

$$\mathring{\boldsymbol{u}} \in L^p(0,T;\boldsymbol{W}^{2,p}(\Omega_F(0)))\cap W^{1,p}(0,T;\boldsymbol{L}^p(\Omega_F(0))), \mathring{\pi}\in L^p(0,T;W^{1,p}(\Omega_F(0)))$$

of the following Stokes problem

$$\begin{cases} \mathring{\boldsymbol{u}}_t - A_*\mathring{\boldsymbol{u}} + \nabla\mathring{\pi} = g_0 & \text{in } \Omega_F(0),\\ \operatorname{div}\ \mathring{\boldsymbol{u}} = \operatorname{div}\ \chi & \text{in } \Omega_F(0),\\ \mathring{\boldsymbol{u}}\cdot\tilde{\boldsymbol{n}} = 0, \quad 2\left[(\mathbb{D}\mathring{\boldsymbol{u}})\tilde{\boldsymbol{n}}\right]_{\boldsymbol{\tau}} + \alpha\mathring{\boldsymbol{u}}_{\boldsymbol{\tau}} = h_1 & \text{on } \partial\Omega,\\ \mathring{\boldsymbol{u}}\cdot\tilde{\boldsymbol{n}} = (\boldsymbol{l}+\boldsymbol{\omega}\times\boldsymbol{y})\cdot\tilde{\boldsymbol{n}} & \text{on } \partial\Omega_S(0),\\ 2\left[(\mathbb{D}\mathring{\boldsymbol{u}})\tilde{\boldsymbol{n}}\right]_{\boldsymbol{\tau}} + \alpha\mathring{\boldsymbol{u}}_{\boldsymbol{\tau}} = \alpha(\boldsymbol{l}+\boldsymbol{\omega}\times\boldsymbol{y})_{\boldsymbol{\tau}} + h_1 & \text{on } \partial\Omega_S(0),\\ \mathring{\boldsymbol{u}}(0) = \mathbf{0}. \end{cases} \tag{6.7}$$

The solution depends continuously on the data in the corresponding spaces.

Note that the proof of the above result is done in [8, Theorem 4.1] for the full slip condition $\alpha = 0$. The case when $\alpha > 0$ is a function, the additional term $\alpha\boldsymbol{u}_{\boldsymbol{\tau}}$ being a lower order perturbation does not affect the analysis of well-posedness and regularity and can be derived by the same analysis. Indeed, with the help of [40, Proposition 3.3.9] , we obtain the maximal regularity (or equivalently, bounded imaginary powers) of the perturbed Stokes operator with full Navier boundary condition from the Stokes operator with $\alpha = 0$.

Writing $(\hat{\boldsymbol{u}}, \hat{\pi}, \hat{\boldsymbol{l}}, \hat{\boldsymbol{\omega}}) = (\boldsymbol{u} - \mathring{\boldsymbol{u}}, \pi - \mathring{\pi}, \boldsymbol{l}, \boldsymbol{\omega})$, (6.5) becomes equivalent to the system

$$\begin{cases} \partial_t\hat{\boldsymbol{u}} - A_*\hat{\boldsymbol{u}} + \nabla\hat{\pi} = \mathbf{0} & \text{in } \Omega_F(0)\times(0,T),\\ \operatorname{div}\ \hat{\boldsymbol{u}} = 0 & \text{in } \Omega_F(0)\times(0,T),\\ \hat{\boldsymbol{u}}\cdot\tilde{\boldsymbol{n}} = 0, \quad [\sigma(\hat{\boldsymbol{u}},\hat{\pi})\tilde{\boldsymbol{n}}]_{\boldsymbol{\tau}} + \alpha\hat{\boldsymbol{u}}_{\boldsymbol{\tau}} = \mathbf{0} & \text{on } \partial\Omega\times(0,T),\\ \hat{\boldsymbol{u}}\cdot\tilde{\boldsymbol{n}} = 0, \quad [\sigma(\hat{\boldsymbol{u}},\hat{\pi})\tilde{\boldsymbol{n}}]_{\boldsymbol{\tau}} + \alpha\hat{\boldsymbol{u}}_{\boldsymbol{\tau}} = \mathbf{0} & \text{on } \partial\Omega_S(0)\times(0,T),\\ m\hat{\boldsymbol{l}}' = -\displaystyle\int_{\partial\Omega_S(0)}\sigma(\hat{\boldsymbol{u}},\hat{\pi})\tilde{\boldsymbol{n}} - \int_{\partial\Omega_S(0)}\sigma(\mathring{\boldsymbol{u}},\mathring{\pi})\tilde{\boldsymbol{n}} + g_1, & t\in(0,T),\\ J(0)\hat{\boldsymbol{\omega}}' = -\displaystyle\int_{\partial\Omega_S(0)}\boldsymbol{y}\times\sigma(\hat{\boldsymbol{u}},\hat{\pi})\tilde{\boldsymbol{n}} - \int_{\partial\Omega_S(0)}\boldsymbol{y}\times\sigma(\mathring{\boldsymbol{u}},\mathring{\pi})\tilde{\boldsymbol{n}} + g_2, & t\in(0,T),\\ \hat{\boldsymbol{u}}(0) = \mathbf{0} & \text{in } \Omega_F(0),\\ \hat{\boldsymbol{l}}(0) = \mathbf{0}, \quad \hat{\boldsymbol{\omega}}(0) = \mathbf{0}. \end{cases} \tag{6.8}$$

Let us define the operator, for all $0 < \varepsilon < 1 - \frac{1}{p}$,

$$\mathcal{J} : W^{\varepsilon+1/p,p}(\Omega_F(0);\mathbb{R}^{3\times3}) \to \mathbb{R}^6$$
$$h \mapsto \begin{pmatrix} \int_{\partial\Omega_S(0)} h\tilde{\boldsymbol{n}} \\ \int_{\partial\Omega_S(0)} \boldsymbol{y}\times h\tilde{\boldsymbol{n}} \end{pmatrix}.$$

It follows from the boundedness of the trace operator that

$$\|\mathcal{J}h\| \leq C\|h\|_{W^{\varepsilon+1/p,p}(\Omega_F(0))}.$$

Now the fifth, sixth and eighth equations for the rigid motion in the system (6.8) can be written as

$$\begin{cases} \mathbb{I}\begin{pmatrix} \hat{\boldsymbol{l}}' \\ \hat{\boldsymbol{\omega}}' \end{pmatrix} = -\mathcal{J}(\sigma(\hat{\boldsymbol{u}},\hat{\pi})) - \mathcal{J}(\sigma(\mathring{\boldsymbol{u}},\mathring{\pi})) + \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \\ (\hat{\boldsymbol{l}}(0),\hat{\boldsymbol{\omega}}(0)) = (\boldsymbol{0},\boldsymbol{0}). \end{cases}$$

where $\mathbb{I}$ is the constant momentum matrix as before. This allows us to rewrite the above set of equations in the form

$$\begin{pmatrix} \hat{\boldsymbol{l}} \\ \hat{\boldsymbol{\omega}} \end{pmatrix} = \mathcal{R}\begin{pmatrix} \hat{\boldsymbol{l}} \\ \hat{\boldsymbol{\omega}} \end{pmatrix} + \hat{g}$$

where $\mathcal{R} : W_0^{1,p}(0,T;\mathbb{R}^6) \to W_0^{1,p}(0,T;\mathbb{R}^6)$ is given by

$$\mathcal{R}\begin{pmatrix} \hat{\boldsymbol{l}} \\ \hat{\boldsymbol{\omega}} \end{pmatrix}(t) := -\int_0^t \mathbb{I}^{-1}\mathcal{J}(\sigma(\hat{\boldsymbol{u}},\hat{\pi}))$$

and

$$\hat{g}(t) := \int_0^t \mathbb{I}^{-1}\left[-\mathcal{J}(\sigma(\mathring{\boldsymbol{u}},\mathring{\pi})) + \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\right].$$

The following lemma says that for sufficiently small $T > 0$, there exists a unique $(\hat{\boldsymbol{l}},\hat{\boldsymbol{\omega}}) \in W_0^{1,p}(0,T;\mathbb{R}^6)$ satisfying

$$\begin{pmatrix} \hat{\boldsymbol{l}} \\ \hat{\boldsymbol{\omega}} \end{pmatrix} = (I_6 - \mathcal{R})^{-1}\hat{g}.$$

Proof of this lemma is exactly similar to the one, done in [23] which is based on the estimate of $\mathcal{J}$ and the maximal regularity $(\boldsymbol{u},\pi)$ solving the fluid part of the system (6.8), thus we are not repeating it. Note that we have the presence of slip in the boundary condition which is different from [23]; Hence we need to use the maximal regularity property of the generalized Stokes operator A_* with slip condition (as mentioned just after the Proposition 6.2).

Lemma 6.3. *The map $\mathcal{R}$ is bounded and $\|\mathcal{R}\|_{W_0^{1,p}(0,T;\mathbb{R}^6)} \le 1$ for sufficiently small $T > 0$. Moreover, $\hat{g} \in W_0^{1,p}(0,T;\mathbb{R}^6)$.*

From the above lemma, apart from the existence of $(\hat{\boldsymbol{l}},\hat{\boldsymbol{\omega}})$, we obtain the following estimate:

$$\begin{aligned} &\|\hat{\boldsymbol{l}}\|_{W^{1,p}(0,T;\mathbb{R}^3)} + \|\hat{\boldsymbol{\omega}}\|_{W^{1,p}(0,T;\mathbb{R}^3)} \\ &\le C\left(1 - \|\mathcal{R}\|_{\mathcal{L}(W_0^{1,p}(0,T),W_0^{1,p}(0,T))}\right)\Big(\|g_0\|_{L^p(0,\infty;\boldsymbol{L}^q(\Omega_F(0)))} + \|\chi\|_{W^{2,1}_{q,p}(Q_F^T)} + \|(g_1,g_2)\|_{L^p(0,\infty;\mathbb{R}^6)} \\ &\qquad + \|h_1\|_{W^{\frac{1}{2}-\frac{1}{2p},p}(0,T;\boldsymbol{L}^p(\partial\Omega_F(0)))\cap L^p(0,T;\boldsymbol{W}^{1-\frac{1}{p}}(\partial\Omega_F(0)))}\Big). \end{aligned}$$

Plugging $\hat{\boldsymbol{l}},\hat{\boldsymbol{\omega}}$ into (6.8) yields a solution

$$\begin{aligned} &\boldsymbol{u} = \hat{\boldsymbol{u}} + \mathring{\boldsymbol{u}} \in W^{2,1}_{p,p}(Q_F^T), \quad && \pi = \hat{\pi} + \mathring{\pi} \in L^p(0,T;W^{1,p}(\Omega_F(0))), \\ &\boldsymbol{l} = \hat{\boldsymbol{l}} \in W^{1,p}(0,T;\mathbb{R}^3), && \boldsymbol{\omega} = \hat{\boldsymbol{\omega}} \in W^{1,p}(0,T;\mathbb{R}^3) \end{aligned}$$

of (6.5) satisfying the estimate (6.6).

6.2 Fixed point argument

Theorem 6.1 now allows us to solve (6.3) via a contraction mapping argument. We introduce as in the Section 5.1,

$$S_\gamma := \{(\tilde{\boldsymbol{u}}, \tilde{\pi}, \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}}) : \|(\tilde{\boldsymbol{u}}, \tilde{\pi}, \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}})\|_S \leq \gamma\}$$

with

$$\|(\tilde{\boldsymbol{u}}, \tilde{\pi}, \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}})\|_S := \|\tilde{\boldsymbol{u}}\|_{W^{2,1}_{p,p}(Q_F^\infty)} + \|\tilde{\pi}\|_{L^p(0,\infty;W^{1,q}(\Omega_F(0)))} + \|\tilde{\boldsymbol{l}}\|_{W^{1,p}(0,\infty;\mathbb{R}^3)} + \|\tilde{\boldsymbol{\omega}}\|_{W^{1,p}(0,\infty;\mathbb{R}^3)}$$

as the underlying set in the natural function spaces. Let

$$\mathcal{N} : \begin{pmatrix} \tilde{\boldsymbol{u}} \\ \tilde{\pi} \\ \tilde{\boldsymbol{l}} \\ \tilde{\boldsymbol{\omega}} \end{pmatrix} \mapsto \begin{pmatrix} \boldsymbol{G}_0(\tilde{\boldsymbol{u}}, \tilde{\pi}, \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}}) \\ \boldsymbol{H}(\tilde{\boldsymbol{u}}, \tilde{\pi}, \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}}) \\ \boldsymbol{H}_1(\tilde{\boldsymbol{u}}, \tilde{\pi}, \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}}) \\ \boldsymbol{G}_1(\tilde{\boldsymbol{u}}, \tilde{\pi}, \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}}) \\ \boldsymbol{G}_2(\tilde{\boldsymbol{u}}, \tilde{\pi}, \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}}) \end{pmatrix} \mapsto \begin{pmatrix} \boldsymbol{u} \\ \pi \\ \boldsymbol{l} \\ \boldsymbol{\omega} \end{pmatrix}$$

be the function which maps $(\tilde{\boldsymbol{u}}, \tilde{\pi}, \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}}) \in S_\gamma$ to $(\boldsymbol{G}_0, \boldsymbol{H}, \boldsymbol{H}_1, \boldsymbol{G}_1, \boldsymbol{G}_2)$ which are defined in (6.4), and then to the solution of the linear problem with fixed right hand sides, using Theorem 6.1. For sufficiently small $\gamma > 0$, we show that the Banach fixed point theorem can be applied to the map $\mathcal{N}$.

Theorem 6.4. *For T and γ sufficiently small, the function $\mathcal{N}$ maps S_γ into itself and it is contractive.*

Proof. First we show that the image of $\mathcal{N}$ is contained in S_γ. Let us assume that $(\tilde{\boldsymbol{u}}, \tilde{\pi}, \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}}) \in S_\gamma$ and that $(\boldsymbol{u}^*, \pi^*, \boldsymbol{l}^*, \boldsymbol{\omega}^*)$ are given by (6.2). We want to show the following estimate

$$\|\mathcal{N}(\tilde{\boldsymbol{u}}, \tilde{\pi}, \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}})\|_S \leq C(T,\gamma)\|(\tilde{\boldsymbol{u}}, \tilde{\pi}, \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}})\|_S \tag{6.9}$$

where $C(T,\gamma) \to 0$ as $T, \gamma \to 0$. Due to the maximal regularity of the linear problem, it follows directly from the following estimate

$$\begin{aligned} &\|\boldsymbol{G}_0\|_{L^p(0,T;\boldsymbol{L}^p(\Omega_F(0)))} + \|\boldsymbol{H}\|_{W^{2,1}_{p,p}(Q_F^\infty)} + \|\boldsymbol{H}_1\| + \|\boldsymbol{G}_1\|_{L^p(0,T;\mathbb{R}^3)} + \|\boldsymbol{G}_2\|_{L^p(0,T;\mathbb{R}^3)} \\ &\leq C(T,\gamma)\|(\tilde{\boldsymbol{u}}, \tilde{\pi}, \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}})\|_S. \end{aligned}$$

Estimate for $\boldsymbol{H}$ is deduced in (5.10). In order to show the estimate for $\boldsymbol{G}_0$, it suffices to consider the term

$$Q(\boldsymbol{u}^*, \tilde{\boldsymbol{u}}) := A_*\tilde{\boldsymbol{u}} - \mathcal{A}(\boldsymbol{u}^* + \tilde{\boldsymbol{u}})(\boldsymbol{u}^* + \tilde{\boldsymbol{u}})$$

which is new compared to the Newtonian case. Estimating Q follows the same argument as in [23, pp 1431-34] which involves essentially writing it as a difference of suitable forms, hence we skip it. We obtain

$$\|Q(\boldsymbol{u}^*, \tilde{\boldsymbol{u}})\|_{L^p(0,T;\boldsymbol{L}^p(\Omega_F(0)))} \leq C\left(\gamma^2 + \|\boldsymbol{u}^*\|_{W^{2,1}_{p,p}(Q_F^\infty)} + T\right)$$

which yields

$$
\begin{aligned}
&\|\boldsymbol{G}_0(\tilde{\boldsymbol{u}},\tilde{\pi},\tilde{\boldsymbol{l}},\tilde{\boldsymbol{\omega}})\|_{L^p(0,T;\boldsymbol{L}^p(\Omega_F(0))}\\
&\leq \|\boldsymbol{F}_0(\tilde{\boldsymbol{u}},\tilde{\pi},\tilde{\boldsymbol{l}},\tilde{\boldsymbol{\omega}})\|_{L^p(0,T;\boldsymbol{L}^p(\Omega_F(0))}+\|\boldsymbol{u}^*\|_{W^{2,1}_{p,p}(Q_F^\infty)}+\|Q(\boldsymbol{u}^*,\tilde{\boldsymbol{u}})\|_{L^p(0,T;\boldsymbol{L}^p(\Omega_F(0))}\\
&\leq C(\gamma^2+\|\boldsymbol{u}^*\|_{W^{2,1}_{p,p}(Q_F^\infty)}+T).
\end{aligned}
$$

Also, by definition,

$$
\begin{aligned}
&\|\boldsymbol{G}_1(\tilde{\boldsymbol{u}},\tilde{\pi},\tilde{\boldsymbol{l}},\tilde{\boldsymbol{\omega}})\|_{L^p(0,T)}\\
&\leq m\|(\boldsymbol{\omega}^*+\hat{\boldsymbol{\omega}})\times(\boldsymbol{l}^*+\hat{\boldsymbol{l}})\|_{L^p(0,T)}+\Big\|\int\limits_{\partial\Omega_S(0)}(\sigma-\tilde{\sigma})(\tilde{\boldsymbol{u}},\tilde{\pi})\tilde{\boldsymbol{n}}\Big\|+\Big\|\int\limits_{\partial\Omega_S(0)}(\tilde{\sigma}-\tilde{T})(\tilde{\boldsymbol{u}},\tilde{\pi})\tilde{\boldsymbol{n}}\Big\|_{L^p(0,T)}\\
&\quad+\Big\|\int\limits_{\partial\Omega_S(0)}(\sigma-\tilde{\sigma})(\boldsymbol{u}^*,\pi^*)\tilde{\boldsymbol{n}}\Big\|_{L^p(0,T)}+\Big\|\int\limits_{\partial\Omega_S(0)}(\tilde{\sigma}-\tilde{T})(\boldsymbol{u}^*,\pi^*)\tilde{\boldsymbol{n}}\Big\|_{L^p(0,T)}\\
&\leq C\gamma^2+C\|\mathcal{J}(\sigma-\tilde{\sigma})(\tilde{\boldsymbol{u}},\tilde{\pi})\|_{L^p(0,T)}+C\|\mathcal{J}(Q^T(2-\mu(|\mathbb{D}\tilde{\boldsymbol{u}}|^2))\mathbb{D}(Q\tilde{\boldsymbol{u}})Q)\|_{L^p(0,T)}\\
&\qquad+C\|\mathcal{J}(\sigma-\tilde{\sigma})(\boldsymbol{u}^*,\pi^*)\|_{L^p(0,T)}+C\|\mathcal{J}((2\mathbb{D}\boldsymbol{u}^*-Q^T\mu(|\mathbb{D}\boldsymbol{u}^*|^2))\mathbb{D}(Q\boldsymbol{u}^*)Q)\|_{L^p(0,T)}\\
&\leq C\gamma^2+C\|\mathbb{D}(Q(\boldsymbol{u}^*+\tilde{\boldsymbol{u}}))\|_{L^p(0,T;C(\overline{\Omega_F(0)}))}\\
&\leq C\gamma^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\|\boldsymbol{G}_2(\tilde{\boldsymbol{u}},\tilde{\pi},\tilde{\boldsymbol{l}},\tilde{\boldsymbol{\omega}})\|_{L^p(0,T)}\\
&\leq C\Big\|\mathcal{J}(\sigma-\tilde{\sigma})(\tilde{\boldsymbol{u}},\tilde{\pi})\Big\|_{L^p(0,T)}+C\Big\|\mathcal{J}(\tilde{\sigma}-\tilde{T})(\tilde{\boldsymbol{u}},\tilde{\pi})\Big\|_{L^p(0,T)}+C\Big\|\mathcal{J}(\sigma-\tilde{\sigma})(\boldsymbol{u}^*,\pi^*)\Big\|_{L^p(0,T)}\\
&\quad+C\Big\|\mathcal{J}(\tilde{\sigma}-\tilde{T})(\boldsymbol{u}^*,\pi^*)\Big\|_{L^p(0,T)}+C\|(\boldsymbol{\omega}^*+\hat{\boldsymbol{\omega}})\times(\boldsymbol{\omega}^*+\hat{\boldsymbol{\omega}})\|_{L^p(0,T)}\\
&\leq C\gamma^2
\end{aligned}
$$

and for $\boldsymbol{H}_1(\tilde{\boldsymbol{u}},\tilde{\pi},\tilde{\boldsymbol{l}},\tilde{\boldsymbol{\omega}})$. Thus we obtain (6.9).

Next we prove that $\mathcal{N}$ is contractive. For that, let $(\tilde{\boldsymbol{u}}^i,\tilde{\pi}^i,\tilde{\boldsymbol{l}}^i,\tilde{\boldsymbol{\omega}}^i)\in S_\gamma, i\in\{1,2\}$. As done in Theorem 5.5, we estimate the differences of the functions $\boldsymbol{G}_0,\boldsymbol{H},\boldsymbol{H}_1,\boldsymbol{G}_1,\boldsymbol{G}_2$ corresponding to $(\tilde{\boldsymbol{u}}^i,\tilde{\pi}^i,\tilde{\boldsymbol{l}}^i,\tilde{\boldsymbol{\omega}}^i)$. This Lipschitz estimates can be shown again exactly using the same argument as in [23, pp 1435-37]. Concerning the estimates on two extra terms $\boldsymbol{H}$ and $\boldsymbol{H}_1$, $\boldsymbol{H}$ is already treated in (5.11) and $\boldsymbol{H}_1$ can also be estimated in the similar way as for $\boldsymbol{G}_1$. This completes the proof. The fixed point of $\mathcal{N}$ is the solution of (6.3)-(6.4). ■

Proposition 6.5. *Let $p>5$, Ω be a bounded domain of class $\mathcal{C}^{2,1}$ and $\alpha\geq 0$ satisfies (2.4). Also assume that $(\boldsymbol{u}_0,\boldsymbol{l}_0,\boldsymbol{\omega}_0)\in W^{2-2/p,p}(\Omega_F(0))\times\mathbb{R}^3\times\mathbb{R}^3$ satisfying the compatibility condition (2.5). Then there exists $T_0>0$ such that the problem (1.2) admits a unique strong solution on $[0,T_0)$*

$$
\begin{aligned}
&\boldsymbol{u}\in L^p(0,T;\boldsymbol{W}^{2,p}(\Omega_F(\cdot)))\cap W^{1,p}(0,T;\boldsymbol{L}^p(\Omega_F(\cdot))),\\
&\pi\in L^p(0,T;W^{1,p}(\Omega_F(\cdot))),\boldsymbol{l}\in W^{1,p}(0,T;\mathbb{R}^3),\boldsymbol{\omega}\in W^{1,p}(0,T;\mathbb{R}^3).
\end{aligned}
$$

Moreover, we can choose T_0 such that one of the following alternatives holds true:
(a) $T_0=\infty$;
(b) the function $t\mapsto\|\boldsymbol{u}(t)\|$ is not bounded in $[0,T_0)$.

Proof. Theorem 6.4 gives a unique strong solution $(\hat{\boldsymbol{u}}, \hat{\pi}, \hat{\boldsymbol{l}}, \hat{\boldsymbol{\omega}})$ to problem (6.3)-(6.4). The solution to the original problem (1.2) can be obtained by adding the reference solution $(\boldsymbol{u}^*, \pi^*, \boldsymbol{l}^*, \boldsymbol{\omega}^*)$ and doing the backward coordinate transform, as in the proof of Theorem 2.1.

That one of the alternatives (a) or (b) holds true, can be proved in the classical way, see for example [15, Section 3.3]. ■

Proof of Theorem 2.3. The solution of the problem (1.2) is global in time, provided the given data are small, can be proved using the same argument as in [15, Section 4.3]. ■

6.3 Non-linear slip condition

In this final subsection, we discuss a more generalized boundary condition where the velocity of the fluid flow satisfies a wall-law:

$$[T(\boldsymbol{u}, \pi)\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha|\boldsymbol{u}|\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0}.$$

Consider the system (1.2) with the above boundary condition, namely

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla_x)\boldsymbol{u} = \operatorname{div} T(\boldsymbol{u}, \pi) & \text{in } \Omega_F(t)\times(0,T), \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega_F(t)\times(0,T), \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad [T(\boldsymbol{u}, \pi)\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha|\boldsymbol{u}|\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \partial\Omega\times(0,T), \\ \boldsymbol{u}\cdot\boldsymbol{n} = \boldsymbol{u}_S\cdot\boldsymbol{n} & \text{on } \partial\Omega_S(t)\times(0,T), \\ [T(\boldsymbol{u}, \pi)\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha|\boldsymbol{u}|\boldsymbol{u}_{\boldsymbol{\tau}} = \alpha|\boldsymbol{u}_S|\boldsymbol{u}_{S\boldsymbol{\tau}} & \text{on } \partial\Omega_S(t)\times(0,T), \\ m\boldsymbol{l}'(t) = -\displaystyle\int_{\partial\Omega_S(t)} T(\boldsymbol{u}, \pi)\boldsymbol{n}, & t\in(0,T), \\ (J\boldsymbol{\omega})'(t) = -\displaystyle\int_{\partial\Omega_S(t)} (\boldsymbol{x}-\boldsymbol{h}(t))\times T(\boldsymbol{u}, \pi)\boldsymbol{n}, & t\in(0,T), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in } \Omega_F(0), \\ \boldsymbol{l}(0) = \boldsymbol{l}_0, \quad \boldsymbol{\omega}(0) = \boldsymbol{\omega}_0. \end{cases} \tag{6.10}$$

To prove the well-posedness of (6.10), the idea is first to linearize the boundary condition as

$$[T(\boldsymbol{u}, \pi)\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha\boldsymbol{u}_{\boldsymbol{\tau}} = \alpha(1-|\tilde{\boldsymbol{u}}|)\tilde{\boldsymbol{u}}_{\boldsymbol{\tau}}$$

which falls under the non-Newtonian case, for given $\tilde{\boldsymbol{u}}$. Then one may show this is a contraction map which finally establishes the existence result Theorem 2.5 with the help of the Banach fixed point theorem. To prove the contraction, it exactly follows from Theorem 6.4 and the fact that the boundary condition can be written as,

$$\alpha(1-|\tilde{\boldsymbol{u}}^1|)\tilde{\boldsymbol{u}}^1_{\boldsymbol{\tau}} - \alpha(1-|\tilde{\boldsymbol{u}}^2|)\tilde{\boldsymbol{u}}^2_{\boldsymbol{\tau}} = \alpha(\tilde{\boldsymbol{u}}^1-\tilde{\boldsymbol{u}}^2)_{\boldsymbol{\tau}} - \alpha|\tilde{\boldsymbol{u}}^1|(\tilde{\boldsymbol{u}}^1-\tilde{\boldsymbol{u}}^2)_{\boldsymbol{\tau}} - \alpha(|\tilde{\boldsymbol{u}}^1|-|\tilde{\boldsymbol{u}}^2|)\tilde{\boldsymbol{u}}^2_{\boldsymbol{\tau}}.$$

In the same spirit, we may also prove the similar result as Theorem 2.1 with the nonlinear slip condition.

7 Appendix: change of variables

In this Section we summarize main facts about the change of variables used to transform the problem to the fixed reference domain. Let us first assume that

$$\|\boldsymbol{h}\|_{L^\infty(0,\infty;\mathbb{R}^3)} + \|Q - I_3\|_{L^\infty(0,\infty;\mathbb{R}^3)}\mathrm{diam}(\Omega_S(0)) \leq \frac{\beta}{2}. \tag{7.1}$$

This implies $\mathrm{dist}(\Omega_S(t), \partial\Omega) \geq \beta/2$ for all $t \in [0,\infty)$. For all $\mu > 0$, we denote,

$$\Omega_\mu = \{\boldsymbol{x} \in \Omega : \mathrm{dist}(\boldsymbol{x}, \partial\Omega) > \mu\}.$$

Now we consider a a cut-off function $\psi \in C^\infty(\mathbb{R}^3, \mathbb{R})$ with compact support contained in $\Omega_{\beta/8}$ and equal to 1 in $\overline{\Omega}_{\beta/4}$. Let us also introduce the functions $\boldsymbol{w} : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3$ as

$$\boldsymbol{w}(\boldsymbol{x},t) = \boldsymbol{l}(t) \times (\boldsymbol{x} - \boldsymbol{h}(t)) + \frac{|\boldsymbol{x} - \boldsymbol{h}(t)|^2}{2}\boldsymbol{\omega}(t)$$

and $\Lambda : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3$ defined as

$$\Lambda(\boldsymbol{x},t) = \psi(\boldsymbol{x})\left(\boldsymbol{l}(t) + \boldsymbol{\omega}(t) \times (\boldsymbol{x} - \boldsymbol{h}(t))\right) + \begin{pmatrix} \frac{\partial\psi}{\partial x_2}(\boldsymbol{x})w_3(\boldsymbol{x},t) - \frac{\partial\psi}{\partial x_3}(\boldsymbol{x})w_2(\boldsymbol{x},t) \\ \frac{\partial\psi}{\partial x_3}(\boldsymbol{x})w_1(\boldsymbol{x},t) - \frac{\partial\psi}{\partial x_1}(\boldsymbol{x})w_3(\boldsymbol{x},t) \\ \frac{\partial\psi}{\partial x_1}(\boldsymbol{x})w_2(\boldsymbol{x},t) - \frac{\partial\psi}{\partial x_2}(\boldsymbol{x})w_1(\boldsymbol{x},t) \end{pmatrix}. \tag{7.2}$$

With these definitions, Λ satisfies the following lemma (cf. [15, Lemma 2.1]):

Lemma 7.1. *Let $\boldsymbol{w}$ and Λ be defined as above. Then, we have*
(1) $\Lambda = 0$ *outside* $\Omega_{\beta/8}$.
(2) $\mathrm{div}\ \Lambda = 0$ *in* $\mathbb{R}^3 \times [0,T]$.
(3) $\Lambda(\boldsymbol{x},t) = \boldsymbol{l}(t) + \boldsymbol{\omega}(t) \times (\boldsymbol{x} - \boldsymbol{h}(t))$ *for all* $\boldsymbol{x} \in \Omega_S(t)$ *and* $t \in [0,T]$.
(4) $\Lambda \in C(\mathbb{R}^3 \times [0,T], \mathbb{R}^3)$. *Moreover, for all* $t \in [0,T], \Lambda(\cdot,t)$ *is a* C^∞ *function and for all* $\boldsymbol{x} \in \mathbb{R}^3$, $\Lambda(\boldsymbol{x},\cdot) \in H^1([0,T],\mathbb{R}^3)$.

Next consider X be the flow associated to Λ, satisfying the differential equation

$$\begin{aligned} \frac{\partial X}{\partial t}(\boldsymbol{y},t) &= \Lambda(X(\boldsymbol{y},t),t), \quad t > 0 \\ X(\boldsymbol{y},0) &= \boldsymbol{y} \in \mathbb{R}^3. \end{aligned} \tag{7.3}$$

We have the following result, proved in [15, Lemma 2.2].

Lemma 7.2. *For all $\boldsymbol{y} \in \mathbb{R}^3$, the initial value problem (7.3) admits a unique solution $X(\boldsymbol{y},\cdot)$: $[0,T] \to \mathbb{R}^3$ which is a C^1 function in $[0,T]$. Moreover, we have the following properties,*
(1) *For all $t \in [0,T]$, the mapping $\boldsymbol{y} \mapsto X(\boldsymbol{y},t)$ is a C^∞-diffeomorphism from $\mathbb{R}^3$ onto itself and from $\Omega_F(0)$ onto $\Omega_F(t)$.*
(2) *Denote by $Y(\cdot,t)$ the inverse of $X(\cdot,t)$. Then, for all $\boldsymbol{x} \in \mathbb{R}^3$, the mapping $t \mapsto Y(\boldsymbol{x},t)$ is a C^1 function in $[0,T]$.*
(3) *For all $\boldsymbol{y} \in \mathbb{R}^3$ and for all $t \in [0,T]$, the determinant of the jacobian matrix J_X of $X(\cdot,t)$ is equal to 1, that is,*

$$\det\ J_X(\boldsymbol{y},t) = 1.$$

From here onwards, J_X and J_Y denote the jacobian matrix of X and Y respectively, that is,

$$J_X = \left(\frac{\partial X_i}{\partial y_j}\right)_{ij} \quad \text{and } J_Y = \left(\frac{\partial Y_i}{\partial x_j}\right)_{ij}.$$

Note that, for each $\boldsymbol{y} \in \Omega_S(0)$, the function $X(\boldsymbol{y},t) = \boldsymbol{h}(t) + Q(t)\boldsymbol{y}, t \geq 0$ is the solution of (7.3), which is easy to verify. This implies, on $\overline{\Omega_S(0)}$,

$$J_X = Q \quad \text{and consequently,} \quad J_Y = Q^T. \tag{7.4}$$

Similarly, on $\partial\Omega$, $X(\boldsymbol{y},t) = \boldsymbol{y}, t \geq 0$ which yields $J_X = I_3 = J_Y$.

Let us now define the functions: for $(\boldsymbol{y},t) \in \Omega_F(0) \times (0,\infty)$,

$$\begin{cases} \tilde{\boldsymbol{u}}(\boldsymbol{y},t) = Q^{-1}(t)\ \boldsymbol{u}(X(\boldsymbol{y},t),t), \\ \tilde{\pi}(\boldsymbol{y},t) = \pi(X(\boldsymbol{y},t),t), \\ \tilde{\boldsymbol{l}}(t) = Q^{-1}(t)\ \boldsymbol{l}(t), \\ \tilde{\boldsymbol{\omega}}(t) = Q^{-1}(t)\ \boldsymbol{\omega}(t) \\ \tilde{J} = Q^{-1}(t)J(t)Q(t) \\ \tilde{\boldsymbol{n}}(\boldsymbol{y},t) = Q^{-1}(t)\boldsymbol{n}(X(\boldsymbol{y},t),t). \end{cases} \tag{7.5}$$

Notice that $\tilde{\boldsymbol{n}}$ becomes the outward normal at $\Omega_F(0)$. Also, from (1.1) and $(7.5)_4$, it easily follows that

$$\dot{Q}(t)\boldsymbol{a} = Q(t)(\tilde{\boldsymbol{\omega}} \times \boldsymbol{a}) \quad \forall \boldsymbol{a} \in \mathbb{R}^3. \tag{7.6}$$

In these new variables, the time derivative is transformed into

$$\partial_t u_i = (\dot{Q}\tilde{\boldsymbol{u}})_i + (Q\partial_t\tilde{\boldsymbol{u}})_i + (\partial_t X \cdot J_Y^T\nabla)(Q\tilde{\boldsymbol{u}})_i = (Q(\tilde{\omega} \times \tilde{\boldsymbol{u}}))_i + (Q\partial_t\tilde{\boldsymbol{u}})_i + (\partial_t X \cdot J_Y^T\nabla)(Q\tilde{\boldsymbol{u}})_i,$$

the convection term is transformed into

$$(\boldsymbol{u} \cdot \nabla_x)u_i = \left((Q\tilde{\boldsymbol{u}}) \cdot (J_Y^T\nabla_y)\right)(Q\tilde{\boldsymbol{u}})_i,$$

the diffusion term is transformed into

$$\Delta_x u_i = \sum_{m,l,j} \frac{\partial(Q\tilde{\boldsymbol{u}})_i}{\partial y_l}\frac{\partial Y_m}{\partial x_j}\frac{\partial}{\partial y_m}\left(\frac{\partial Y_l}{\partial x_j}\right) + \sum_{m,l,j}\frac{\partial^2(Q\tilde{\boldsymbol{u}})_i}{\partial y_m\partial y_l}\frac{\partial Y_l}{\partial x_j}\frac{\partial Y_m}{\partial x_j},$$

and the pressure is transformed to,

$$(\nabla\pi)_i = (J_Y^T\nabla_y\tilde{\pi})_i.$$

Furthermore, we obtain

$$\text{div } \boldsymbol{u} = \nabla_y\tilde{\boldsymbol{u}} : (J_YQ)^T$$

which can also be written as, by Piola's identity (cf. [14, pp 39], [20, Ch. 8.1.4.b]),

$$\nabla_y\tilde{\boldsymbol{u}} : (J_YQ)^T = \text{div}_y\left((J_YQ)\tilde{\boldsymbol{u}}\right)$$

since $J_Y Q = \mathrm{cof}(QJ_X) = \mathrm{cof}\nabla_y(QX)$ because of $\det J_X = 1$. Concerning the boundary condition, we calculate the symmetric gradient,

$$(\nabla_x \boldsymbol{u})_{ij} = \partial_j u_i = \sum_{l=1}^{3} \frac{\partial (Q\tilde{\boldsymbol{u}})_i}{\partial y_l}\frac{\partial Y_l}{\partial x_j} = \sum_{l,k=1}^{3} Q_{ik}\frac{\partial \tilde{\boldsymbol{u}}_k}{\partial y_l}\frac{\partial Y_l}{\partial x_j} = (Q\nabla_y \tilde{\boldsymbol{u}} J_Y)_{ij}.$$

This shows that at the interface $\partial\Omega_S(0)$, because of (7.4), $\nabla_x \boldsymbol{u} = Q\,\nabla_y \tilde{\boldsymbol{u}}\, Q^T$ and hence, $(\nabla_x \boldsymbol{u})^T = Q(\nabla_y \tilde{\boldsymbol{u}})^T Q^T$ which gives,

$$\mathbb{D}_x \boldsymbol{u} = Q\,\mathbb{D}_y \tilde{\boldsymbol{u}}\, Q^T$$

and consequently,

$$\sigma(\boldsymbol{u},\pi) = Q\sigma(\tilde{\boldsymbol{u}},\tilde{\pi})Q^T.$$

Therefore, the slip boundary condition becomes,

$$[\sigma(\tilde{\boldsymbol{u}},\tilde{\pi})\tilde{\boldsymbol{n}}]_{\boldsymbol{\tau}} + \alpha\tilde{\boldsymbol{u}}_{\boldsymbol{\tau}} = \alpha\left(\tilde{\boldsymbol{l}} + \tilde{\boldsymbol{\omega}} \times \boldsymbol{y}\right)_{\boldsymbol{\tau}} \qquad \text{on } \ \partial\Omega_S(0)$$

and similarly at $\partial\Omega$. It can be shown as in [27, Theorem 2.5] that the fluid part of the original problem (2.1) admits a strong solution $(\boldsymbol{u},\pi)$ if and only if there exists a corresponding solution $(\tilde{\boldsymbol{u}},\tilde{\pi}) \in W^{2,1}_{q,p}(Q^{\infty}_F) \times L^p(0,\infty; W^{1,q}(\Omega_F(0)))$ to the fluid part of the transformed problem (7.7).

Next, we write the equations for rigid body. From $(7.5)_3$, we find that

$$m\boldsymbol{l}'(t) = m(\dot{Q}\tilde{\boldsymbol{l}} + Q\tilde{\boldsymbol{l}}') = mQ(\tilde{\boldsymbol{\omega}} \times \tilde{\boldsymbol{l}}) + mQ\tilde{\boldsymbol{l}}'.$$

Moreover, we have

$$\int\limits_{\partial\Omega_S(t)} \sigma(\boldsymbol{u},\pi)\boldsymbol{n} = Q \int\limits_{\partial\Omega_S(0)} \sigma(\tilde{\boldsymbol{u}},\tilde{\pi})\tilde{n}$$

and

$$\int\limits_{\partial\Omega_S(t)} (\boldsymbol{x} - \boldsymbol{h}(t)) \times \sigma(\boldsymbol{u},\pi)\boldsymbol{n} = Q \int\limits_{\partial\Omega_S(0)} \boldsymbol{y} \times \sigma(\tilde{\boldsymbol{u}},\tilde{\pi})\tilde{n}.$$

Therefore, the equation of linear momentum becomes,

$$m\tilde{\boldsymbol{l}}' + m\tilde{\boldsymbol{\omega}} \times \tilde{\boldsymbol{l}} = -\int\limits_{\partial\Omega_S(0)} \sigma(\tilde{\boldsymbol{u}},\tilde{\pi})\tilde{\boldsymbol{n}}.$$

Similarly, using the following identity, for any special orthogonal matrix $M \in SO(3)$,

$$Ma \times Mb = M(a \times b) \quad \forall a, b \in \mathbb{R}^3,$$

the equation of angular momentum becomes,

$$\tilde{J}\tilde{\boldsymbol{\omega}}'(t) - \tilde{J}\tilde{\boldsymbol{\omega}} \times \tilde{\boldsymbol{\omega}} = -\int\limits_{\partial\Omega_S(0)} \boldsymbol{y} \times \sigma(\tilde{\boldsymbol{u}},\tilde{\pi})\tilde{\boldsymbol{n}}.$$

Note that, $\tilde{J}$ is independent of time, since

$$\tilde{J}a \cdot b = \int\limits_{\partial\Omega_S(0)} \rho_S(\boldsymbol{y})(a \times \boldsymbol{y}) \cdot (b \times \boldsymbol{y}) \mathrm{d}y \quad \forall a, b \in \mathbb{R}^3.$$

Therefore, on the cylindrical domain $\Omega_F(0) \times (0, T)$, the coupled system for the Newtonian fluid (2.1) transforms into,

$$\begin{cases} \tilde{\boldsymbol{u}}_t - \Delta \tilde{\boldsymbol{u}} + \nabla \tilde{\pi} = \boldsymbol{F}_0(\tilde{\boldsymbol{u}}, \tilde{\pi}, \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}}) & \text{in } \Omega_F(0) \times (0,T), \\ \operatorname{div} \tilde{\boldsymbol{u}} = \mathcal{G}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}}) = \operatorname{div} \boldsymbol{H}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}}) & \text{in } \Omega_F(0) \times (0,T), \\ \tilde{\boldsymbol{u}} \cdot \tilde{\boldsymbol{n}} = 0, \quad [\sigma(\tilde{\boldsymbol{u}}, \tilde{\pi})\tilde{\boldsymbol{n}}]_{\boldsymbol{\tau}} + \alpha \tilde{\boldsymbol{u}}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \partial\Omega \times (0,T), \\ \tilde{\boldsymbol{u}} \cdot \tilde{\boldsymbol{n}} = \tilde{\boldsymbol{u}}_S \cdot \tilde{\boldsymbol{n}}, \quad [\sigma(\tilde{\boldsymbol{u}}, \tilde{\pi})\tilde{\boldsymbol{n}}]_{\boldsymbol{\tau}} + \alpha \tilde{\boldsymbol{u}}_{\boldsymbol{\tau}} = \alpha \tilde{\boldsymbol{u}}_{S\boldsymbol{\tau}} & \text{on } \partial\Omega_S(0) \times (0,T), \\ m\tilde{\boldsymbol{l}}' = -\int\limits_{\partial\Omega_S(0)} \sigma(\tilde{\boldsymbol{u}}, \tilde{\pi})\tilde{\boldsymbol{n}} + \boldsymbol{F}_1(\tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}}), & t \in (0,T), \\ \tilde{J}\tilde{\boldsymbol{\omega}}' = -\int\limits_{\partial\Omega_S(0)} \boldsymbol{y} \times \sigma(\tilde{\boldsymbol{u}}, \tilde{\pi})\tilde{\boldsymbol{n}} + \boldsymbol{F}_2(\tilde{\boldsymbol{\omega}}), & t \in (0,T), \\ \tilde{\boldsymbol{u}}(0) = \boldsymbol{u}_0 & \text{in } \Omega_F(0), \\ \tilde{\boldsymbol{l}}(0) = \boldsymbol{l}_0, \quad \tilde{\boldsymbol{\omega}}(0) = \boldsymbol{\omega}_0 \end{cases} \tag{7.7}$$

where

$$\tilde{\boldsymbol{u}}_S := \tilde{\boldsymbol{l}} + \tilde{\boldsymbol{\omega}} \times \boldsymbol{y};$$

$$\begin{aligned} (\boldsymbol{F}_0)_i(\tilde{\boldsymbol{u}}, \tilde{\pi}, \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}}) := {} & ((I_3 - Q)\partial_t \tilde{\boldsymbol{u}})_i - (Q(\tilde{\boldsymbol{\omega}} \times \tilde{\boldsymbol{u}}))_i - (\partial_t X \cdot J_Y^T \nabla)(Q\tilde{\boldsymbol{u}})_i - \left((Q\tilde{\boldsymbol{u}}) \cdot (J_Y^T \nabla)\right)(Q\tilde{\boldsymbol{u}})_i \\ & + \sum_{m,l,j} \frac{\partial (Q\tilde{\boldsymbol{u}})_i}{\partial y_l} \frac{\partial Y_m}{\partial x_j} \frac{\partial}{\partial y_m} \left(\frac{\partial Y_l}{\partial x_j} \right) + \sum_{m,l,j} \frac{\partial^2 (Q\tilde{\boldsymbol{u}})_i}{\partial y_m \partial y_l} \frac{\partial Y_l}{\partial x_j} \frac{\partial Y_m}{\partial x_j} - \Delta \tilde{\boldsymbol{u}}_i \\ & + \left((I_3 - J_Y^T)\nabla\tilde{\pi}\right)_i; \end{aligned} \tag{7.8}$$

$$\mathcal{G}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}}) := \nabla \tilde{\boldsymbol{u}} : (I_3 - (J_Y Q)^T) = \operatorname{div} \boldsymbol{H} \quad \text{with} \quad \boldsymbol{H}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{\omega}}) := (I_3 - J_Y Q)\tilde{\boldsymbol{u}}; \tag{7.9}$$

$$\boldsymbol{F}_1(\tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}}) := -m\tilde{\boldsymbol{\omega}} \times \tilde{\boldsymbol{l}}; \tag{7.10}$$

$$\boldsymbol{F}_2(\tilde{\boldsymbol{\omega}}) := \tilde{J}\tilde{\boldsymbol{\omega}} \times \tilde{\boldsymbol{\omega}}. \tag{7.11}$$

Note that a solution $(\tilde{\boldsymbol{u}}, \tilde{\pi}, \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{\omega}})$ to (7.7) yields a solution $(\boldsymbol{u}, \pi, \boldsymbol{l}, \boldsymbol{\omega})$ to (2.1) by (7.5).

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