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Abstract

We identify a class *maximal* dissipative solutions to models of compressible viscous fluids that maximize the energy dissipation rate. Then we show that any maximal dissipative solution approaches an equilibrium state for large times.

Keywords: Non-Newtonian fluid, compressible fluid, dissipative solution, long–time behavior
MSC: 35Q35, 35B40, 35D99

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1 Problem formulation

We consider a mathematical model of a compressible viscous fluid occupying a bounded physical domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. The state of the fluid at a given time $t \geq 0$ and a spatial position $x \in \Omega$ is characterized by the mass density $\varrho = \varrho(t, x)$ and the bulk velocity $\mathbf{u} = \mathbf{u}(t, x)$ satisfying the following system of partial differential equations:

$$\begin{aligned} \partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) &= 0, \\ \partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) &= \mathrm{div}_x \mathbb{S}, \end{aligned} \tag{1.1}$$

where p is the pressure and $\mathbb{S}$ the viscous stress tensor. The viscous stress is related to the symmetric velocity gradient

$$\mathbb{D}_x \mathbf{u} = \frac{\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^t}{2}$$

through a general *rheological law*

$$\mathbb{S} \in \partial F(\mathbb{D}_x), \tag{1.2}$$

where ∂F is the subdifferential of a convex potential F. In view of Fenchel–Young identity, the relation (1.2) can be written in an "implicit" form

$$\mathbb{S} : \mathbb{D}_x \mathbf{u} = F(\mathbb{D}_x \mathbf{u}) + F^*(\mathbb{S}), \tag{1.3}$$

where F^* is the conjugate of F. Finally, we consider the no–slip boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{1.4}$$

together with the initial conditions

$$\varrho(0, \cdot) = \varrho_0, \ \ \varrho \mathbf{u}(0, \cdot) = \mathbf{m}_0. \tag{1.5}$$

Smooth solutions of (1.1)–(1.5) satisfy the total energy balance

$$\int_\Omega E(\varrho, \mathbf{m})(\tau, \cdot) \ \mathrm{d}x + \int_0^\tau \int_\Omega \left(F(\mathbb{D}_x \mathbf{u}) + F^*(\mathbb{S}) \right) \ \mathrm{d}x \, \mathrm{d}t = \int_\Omega E(\varrho_0, \mathbf{m}_0) \ \mathrm{d}x \tag{1.6}$$

for any $\tau \geq 0$, where E is the total energy,

$$E = \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho), \ \ \mathbf{m} \equiv \varrho \mathbf{u}, \ \ P'(\varrho)\varrho - P(\varrho) = p(\varrho).$$

In addition, in view of (1.1), (1.4), the total mass of the fluid is a conserved quantity,

$$M = \int_\Omega \varrho(\tau, \cdot) \ \mathrm{d}x = \int_\Omega \varrho_0 \ \mathrm{d}x \tag{1.7}$$

for any $\tau \geq 0$. In accordance with the Second law of thermodynamics, the dissipation potential F must satisfy

$$F(\mathbb{D}_x \mathbf{u}) + F^*(\mathbb{S}) \geq 0;$$

whence the equilibrium (time independent) states $[\tilde{\varrho}, \tilde{\mathbf{u}}]$ satisfy

$$F(\mathbb{D}_x \tilde{\mathbf{u}}) + F^*(\widetilde{\mathbb{S}}) = 0, \ \tilde{\mathbf{u}}|_{\partial\Omega} = 0, \ \int_\Omega \tilde{\varrho} \ \mathrm{d}x = M. \tag{1.8}$$

For *real* fluids, the dissipation is always present therefore (1.8) implies

$$\tilde{\mathbf{u}} = 0, \ \widetilde{\mathbb{S}} = 0;$$

whence, in accordance with (1.1)

$$\partial_t \tilde{\varrho} = 0, \ \nabla_x p(\tilde{\varrho}) = 0.$$

Thus if the pressure is a strictly monotone (increasing) function of ϱ, we may infer that

$$\tilde{\varrho}(x) = \overline{\varrho}, \text{ where } \overline{\varrho} > 0 \text{ is a constant, } \overline{\varrho}|\Omega| = M. \tag{1.9}$$

In view of (1.9), it is convenient to fix the pressure potential P in the energy,

$$E\left(\varrho, \mathbf{m} \Big| \overline{\varrho}\right) = \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - P'(\overline{\varrho})(\varrho - \overline{\varrho}) - P(\overline{\varrho}).$$

The energy being a convex function of $[\varrho, \mathbf{m}]$, the quantity $E\left(\varrho, \mathbf{m} \Big| \overline{\varrho}\right)$ can be interpreted as the Bregman distance between $[\varrho, \mathbf{m}]$ and the equilibrium state $[\overline{\varrho}, 0]$.

Our goal is to study the long–time behavior of solutions to the problem (1.1)–(1.5), in particular, we show that any individual trajectory approaches a single equilibrium determined uniquely by the total mass of the fluid. To the best of our knowledge, the problem of *global existence* for the problem (1.1)–(1.5) is largely open even in the class of weak (distributional) solutions; the only exception being the Navier–Stokes system, where both F and F^* are quadratic, and the global existence of weak solutions was shown by Lions [16] and extended in [10], and the problem with linear pressure and exponentially growing viscosity coefficients studied by Mamontov [18], [19].

In the light of the afore mentioned difficulties with global solvability, we consider the problem (1.1)–(1.5) in the framework of *dissipative solutions* introduced in [1]. The leading idea is to replace the viscous stress $\mathbb{S}$ by $\mathbb{S}_{\text{eff}} = \mathbb{S} - \mathfrak{R}$, with an extra stress $\mathfrak{R}$ called Reynolds stress,

$$\mathfrak{R}(\tau) \in \mathcal{M}^+(\overline{\Omega}; \mathbb{R}^{d\times d}_{\text{sym}}), \ \tau > 0,$$

where $\mathcal{M}^+(\overline{\Omega}; \mathbb{R}^{d\times d}_{\text{sym}})$ is the set of positively semi–definite matrix–valued measures in $\overline{\Omega}$.

The dissipative solutions satisfy

$$\begin{gathered} \partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \\ \partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}_{\text{eff}}, \end{gathered} \tag{1.10}$$

together with the energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_\Omega E\left(\varrho,\mathbf{m}\Big|\overline{\varrho}\right)(\tau,\cdot)\ \mathrm{d}x + D\int_{\overline{\Omega}}\mathrm{d}\ \mathrm{tr}[\mathfrak{R}](\tau)\right] + \int_\Omega \Big(F(\mathbb{D}_x\mathbf{u}) + F^*(\mathbb{S}_{\rm eff}+\mathfrak{R})\Big)\ \mathrm{d}x\,\mathrm{d}t \le 0,$$
$$\left[\int_\Omega E\left(\varrho,\mathbf{m}\Big|\overline{\varrho}\right)\ \mathrm{d}x + D\int_{\overline{\Omega}}\mathrm{d}\ \mathrm{tr}[\mathfrak{R}]\right] \le \int_\Omega E\left(\varrho_0,\mathbf{m}_0\Big|\overline{\varrho}\right)\ \mathrm{d}x, \tag{1.11}$$

where $D > 0$ is a constant determined solely by the structural properties of F and p, see Section 2 for details.

Although the class of dissipative solutions is apparently larger than that of conventional weak (distributional) solutions, they still enjoy the following properties:

- **Existence.** The dissipative solutions exist globally–in–time for any finite energy initial data,

$$\varrho \in C_{\rm weak,loc}([0,\infty); L^\gamma(\Omega)),\ \ \mathbf{m} \equiv \varrho\mathbf{u} \in C_{\rm weak,loc}([0,\infty); L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^d)) \text{ for some } \gamma > 1,$$

  see [1, Theorem 3.8].

- **Compatibility.** Any dissipative solution $[\varrho,\mathbf{u}]$, $\varrho > 0$, that is continuously differentiable is in fact a classical solution of the problem, in particular

$$\mathfrak{R} = 0,\ \ \mathbb{S}_{\rm eff} = \mathbb{S},\ \ \mathbb{S} \in \partial F(\mathbb{D}_x\mathbf{u}),$$

  see [1, Theorem 4.1].

- **Weak–strong uniqueness.** A dissipative solution coincides with the strong solution emanating from the same initial data as long as the latter solution exists, see [1, Theorem 6.3].

The terminology "dissipative solution" was first used by Lions [15] in the context of the Euler system, where the equations are simply replaced by the associated relative energy inequality. Brenier [4] proposed an alternative approach to construct generalized solutions of the Euler system via maximization of a concave functional. Our concept of dissipative solution is closer to the measure–valued solution in the spirit of DiPerna's pioneering work [7], see also the monograph Málek et al. [17] and the references cited therein. The key observation is that the oscillation and concentration defects can be conveniently unified giving rise to a single positively definite Reynolds stress, the trace of which is controlled by the energy dissipation defect.

Anticipating that dissipative solutions are possibly not uniquely determined by the initial data $[\varrho_0,\mathbf{m}_0]$, we identify a smaller class of *maximal* dissipative solutions – the dissipative solutions with a maximal rate of energy dissipation. We show that maximal dissipative solutions exist for any finite energy initial data, and, in addition, they enjoy the following remarkable property:

$$\|\mathfrak{R}(\tau)\|_{\mathcal{M}(\overline{\Omega};\mathbb{R}^{d\times d}_{\rm sym})} \to 0 \text{ as } \tau \to \infty. \tag{1.12}$$

In other words, the maximal dissipative solutions behave like the conventional weak solutions in the long run.

Finally, imposing some technical hypotheses on F and p we show that any maximal dissipative solution $[\varrho, \mathbf{m}]$ approaches an equilibrium state for large times:

$$\mathbf{m} = \varrho \mathbf{u}(\tau, \cdot) \to 0 \text{ in } L^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^d), \ \varrho(\tau, \cdot) \to \overline{\varrho} \text{ in } L^{\gamma}(\Omega) \text{ as } \tau \to \infty. \tag{1.13}$$

The paper is organized as follows. In Section 2, we recall the concept of dissipative solution and introduce the class of solutions with maximal energy dissipation. In Section 3, we study the long–time behavior of maximal solutions. In particular, we show (1.12), see Theorem 3.1. In Sections 4, we introduce additional hypotheses to be imposed on F and p as well as on the dissipative solution in order to prove (1.13). Then we show a general result on convergence for a special class of dissipative solution, see Theorem 4.2. Finally, in Section 5 we show unconditional convergence to equilibrium for the dissipative solutions imposing only extra restrictions on p and F, see Theorem 5.1. The paper is concluded by a short discussion concerning possible extensions to driven systems in Section 6.

2 Dissipative solutions

We start by recalling the basic restrictions on the structural properties of F and p introduced in [1].

The pressure $p = p(\varrho)$, with the associated pressure potential $P(\varrho)$,

$$P'(\varrho)\varrho - P(\varrho) = p(\varrho),$$

satisfy

$$\begin{aligned} & p \in C[0,\infty) \cap C^2(0,\infty), \ p(0) = 0, \ p'(\varrho) > 0 \text{ for } \varrho > 0, \ P(0) = 0, \\ & P - \underline{a}p, \ \overline{a}p - P \text{ are convex functions for certain constants } \underline{a} > 0, \ \overline{a} > 0. \end{aligned} \tag{2.1}$$

Note that the standard isentropic pressure $p(\varrho) = a\varrho^\gamma$ satisfies (2.1) with

$$\underline{a} = \overline{a} = \frac{1}{\gamma - 1}.$$

Without loss of generality, we may fix

$$\underline{a} = \sup \left\{ a > 0 \ \middle| \ P - ap \text{ is convex} \right\}, \ \overline{a} = \inf \left\{ a > 0 \ \middle| \ ap - P \text{ is convex} \right\}.$$

As shown in [1, Section 2.1.1], we have

$$P(\varrho) \geq a\varrho^\gamma \text{ for certain } a > 0, \ \gamma = 1 + \frac{1}{\overline{a}}, \text{ and all } \varrho \geq 1. \tag{2.2}$$

The dissipative potential satisfies

$$F : \mathbb{R}^{d\times d}_{\rm sym} \to [0,\infty) \text{ is a (proper) convex function, } F(0) = 0. \tag{2.3}$$

Moreover, for any $R > 0$ there exists a (Young) function A_R satisfying

- $A : [0,\infty) \to [0,\infty)$ convex,
- A increasing,
- $A(0) = 0$,
- $a_1 A(z) \le A(2z) \le a_2 A(z)$ for any $z \in [0,\infty)$, where $a_1 > 2$, $a_2 < \infty$,

such that

$$F(\mathbb{D} + \mathbb{Q}) - F(\mathbb{D}) - \mathbb{S} : \mathbb{Q} \ge A_R \left(\left| \mathbb{Q} - \frac{1}{d} {\rm tr}[\mathbb{Q}]\mathbb{I} \right| \right) \tag{2.4}$$

for all $\mathbb{D}, \mathbb{S}, \mathbb{Q} \in R^{d\times d}_{\rm sym}$ such that

$$|\mathbb{D}| \le R, \ \mathbb{S} \in \partial F(\mathbb{D}).$$

As shown in [1, Section 2.1.2], it follows from (2.4) that there exist $\mu > 0$ and $q > 1$ such that

$$F(\mathbb{D}) \ge \mu \left| \mathbb{D} - \frac{1}{d} {\rm tr}[\mathbb{D}]\mathbb{I} \right|^q \text{ for all } |\mathbb{D}| > 1. \tag{2.5}$$

We are ready to introduce the concept of *dissipative solution* to the problem (1.1)–(1.5).

Definition 2.1 (Dissipative solution)**.** Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded Lipschitz domain. The quantity $[\varrho, \mathbf{u}]$ is called *dissipative solution* of the problem (1.1)–(1.5) in $[0,\infty) \times \Omega$ if:

-
$$\varrho \ge 0, \varrho \in C_{\rm weak,loc}([0,\infty); L^\gamma(\Omega)),$$
$$\mathbf{u} \in L^q([0,\infty); W^{1,q}_0(\Omega; \mathbb{R}^d)), \ \mathbf{m} \equiv \varrho\mathbf{u} \in C_{\rm weak,loc}([0,\infty); L^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^d));$$

- the integral identity

$$\left[\int_\Omega \varrho\varphi \ {\rm d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \Big[\varrho\partial_t\varphi + \varrho\mathbf{u} \cdot \nabla_x\varphi \Big] \ {\rm d}x \, {\rm d}t, \ \ \varrho(0,\cdot) = \varrho_0, \tag{2.6}$$

  holds for any $\tau \ge 0$, and any test function $\varphi \in C^1_{\rm loc}([0,\infty) \times \overline{\Omega})$;

- there exist
$$\mathbb{S} \in L^1_{\rm loc}([0,T) \times \Omega; \mathbb{R}^{d\times d}_{\rm sym}),\ \ \mathfrak{R} \in L^\infty(0,T; \mathcal{M}^+(\overline{\Omega}; \mathbb{R}^{d\times d}_{\rm sym})),$$
such that the integral identity
$$\begin{aligned}\left[\int_\Omega \varrho \mathbf{u} \cdot \boldsymbol{\varphi} \ {\rm d}x \right]_{t=0}^{t=\tau} &= \int_0^\tau \int_\Omega \Big[\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \boldsymbol{\varphi} + p(\varrho) {\rm div}_x \boldsymbol{\varphi} - \mathbb{S} : \nabla_x \boldsymbol{\varphi} \Big] \ {\rm d}x \\ &+ \int_0^\tau \int_\Omega \nabla_x \boldsymbol{\varphi} : {\rm d}\ \mathfrak{R}(t) \ {\rm d}t,\ \ \varrho \mathbf{u}(0,\cdot) = \mathbf{m}_0,\end{aligned} \tag{2.7}$$
holds for any $\tau \geq 0$ and any test function $\boldsymbol{\varphi} \in C^1_c([0,\infty) \times \Omega; \mathbb{R}^d)$;
- the energy inequality
$$\begin{aligned}\int_\Omega E\left(\varrho, \mathbf{m} \Big| \overline{\varrho}\right)(\tau,\cdot) \ {\rm d}x + D \int_{\overline{\Omega}} {\rm d}\ {\rm tr}[\mathfrak{R}](\tau) + \int_0^\tau \int_\Omega \Big(F(\mathbb{D}_x \mathbf{u}) + F^*(\mathbb{S}) \Big) \ {\rm d}x\, {\rm d}t \\ \leq \int_\Omega E\left(\varrho_0, \mathbf{m}_0 \Big| \overline{\varrho}\right) \ {\rm d}x\end{aligned} \tag{2.8}$$
holds for a.e. $\tau \geq 0$, where
$$D = \min\left\{ \frac{1}{2}; \frac{a}{d} \right\}.$$

The dissipative solutions have been introduced in [1], specifically Definition 2.1 and Remarks 2.2, 2.3. The constant D was computed explicitly as pointed out in [1, Remark 2.3]. In (2.8), the kinetic energy is defined as a convex l.s.c. function of $(\varrho, \mathbf{m}) \in \mathbb{R}^{d+1}$,
$$\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} = \begin{cases} \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} \ \text{if}\ \varrho > 0, \\ 0 \ \text{if}\ \varrho = 0, \mathbf{m} = 0, \\ \infty \ \text{otherwise.} \end{cases}$$

2.1 Turbulent energy and maximal dissipation

In order to define the maximal solutions, we first introduce the *turbulent energy* $\mathcal{E}$,

- $$\mathcal{E} \in L^\infty(0,\infty);$$
- $$\begin{aligned}\int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - P'(\overline{\varrho})(\varrho - \overline{\varrho}) - P(\overline{\varrho}) \right] \ {\rm d}x + D \int_{\overline{\Omega}} {\rm d}\ {\rm tr}[\mathfrak{R}] \leq \mathcal{E} \\ \leq \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) - P'(\overline{\varrho})(\varrho_0 - \overline{\varrho}) - P(\overline{\varrho}) \right] \ {\rm d}x \ \text{a.e. in}\ (0,\infty);\end{aligned}$$

- $$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \leq - \int_\Omega [F(\mathbb{D}_x \mathbf{u}) + F^*(\mathbb{S})] \ \mathrm{d}x \text{ in } \mathcal{D}'(0,\infty). \tag{2.9}$$

In general, the turbulent energy $\mathcal{E}$ is not uniquely determined by $[\varrho, \mathbf{u}]$ and Reynolds defect $\mathfrak{R}$, however, at least one turbulent energy exists. Indeed, in view of the energy inequality (2.8), we may take

$$\mathcal{E}(\tau) = \int_\Omega E\left(\varrho_0, \mathbf{m}_0 \Big| \overline{\varrho}\right) \ \mathrm{d}x - \int_0^\tau \int_\Omega \Big(F(\mathbb{D}_x \mathbf{u}) + F^*(\mathbb{S}) \Big) \ \mathrm{d}x \, \mathrm{d}t,$$

where the right–hand side is non–increasing. Moreover, given $[\varrho, \mathbf{u}]$ we can modify the Reynolds defect $\mathfrak{R}$,

$$\mathfrak{R} \approx \mathfrak{R} + \chi(t)\mathbb{I}, \ \chi \in L^\infty(0,\infty), \ \chi \geq 0,$$

without changing the momentum balance (2.7) in such a way that

$$\int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - P'(\overline{\varrho})(\varrho - \overline{\varrho}) - P(\overline{\varrho}) \right] (\tau, \cdot) \ \mathrm{d}x + D \int_{\overline{\Omega}} \mathrm{d} \ \mathrm{tr}[\mathfrak{R}](\tau) = \mathcal{E}(\tau) \text{ for a.a. } \tau \in (0,\infty)$$

$$\mathcal{E}(0+) = \int_\Omega E\left(\varrho_0, \mathbf{m}_0 \Big| \overline{\varrho}\right) \ \mathrm{d}x. \tag{2.10}$$

In the rest of the paper, we restrict ourselves to the dissipative solutions for which the turbulent energy $\mathcal{E}$ given by (2.10) satisfies (2.9). For definitness, we identify $\mathcal{E}$ with its càdlàg version,

$$\mathcal{E}(\tau) = \mathcal{E}(\tau+).$$

Motivated by Dafermos [5], [6], we introduce the concept of maximal solution. Given two dissipative solutions $[\varrho_1, \mathbf{u}_1]$, $[\varrho_2, \mathbf{u}_2]$ emanating from the same initial data $[\varrho_0, \mathbf{m}_0]$, with the associated turbulent energy $\mathcal{E}_1$, $\mathcal{E}_2$, we say that

$$[\varrho^1, \mathbf{u}^1] \prec [\varrho^2, \mathbf{u}^2] \ \Leftrightarrow \ \mathcal{E}^1 \leq \mathcal{E}^2 \text{ in } [0,\infty). \tag{2.11}$$

To define a maximal solution we first introduce the set

$$\mathcal{U}[\varrho_0, \mathbf{m}_0] = \Big\{ [\varrho, \mathbf{u}, \mathcal{E}] \ \Big| \ [\varrho, \mathbf{u}] \text{ –a dissipative solutions with the initial data } [\varrho_0, \mathbf{m}_0]$$
$$\text{and the associated turbulent energy } \mathcal{E} \Big\}$$

Definition 2.2 (Maximal solution)**.** We say that a dissipative solution $[\varrho, \mathbf{u}]$ emanating from the initial data $[\varrho_0, \mathbf{m}_0]$ with the associated turbulent energy $\mathcal{E}$ is *maximal* if it is minimal with respect to the relation "$\prec$" among all dissipative solutions in $\mathcal{U}[\varrho_0, \mathbf{m}_0]$. More specifically, if $[\tilde{\varrho}, \tilde{\mathbf{u}}]$ is another dissipative solution starting from $[\varrho_0, \mathbf{m}_0]$ with the associated turbulent energy $\widetilde{\mathcal{E}}$ satisfying

$$\widetilde{\mathcal{E}} \leq \mathcal{E} \text{ then } \widetilde{\mathcal{E}} = \mathcal{E}.$$

Remark 2.3. Seeing that

$$\int_\Omega P'(\overline{\varrho})(\varrho - \overline{\varrho}) - P(\overline{\varrho}) \ \mathrm{d}x = - \int_\Omega P(\overline{\varrho}) \ \mathrm{d}x \text{ - a constant}$$

we may consider the turbulent energy in a more concise form

$$\mathcal{E}(\tau) = \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] (\tau, \cdot) \ \mathrm{d}x + D \int_{\overline{\Omega}} \mathrm{d} \ \mathrm{tr}[\mathfrak{R}](\tau)$$

independent of the total mass $M = \overline{\varrho}|\Omega|$.

2.2 Existence of maximal solutions

The existence of a maximal solution can be proved following the line of arguments used in [3]. To begin, it is easy to observe that a minimizer of the functional

$$I[\varrho, \mathbf{u}, \mathfrak{R}] = \int_0^\infty \exp(-t) \mathcal{E}(t) \ \mathrm{d}t$$

over the set of all dissipative solutions in $\mathcal{U}[\varrho_0, \mathbf{m}_0]$ is a maximal solution in the sense of Definition 2.2. Here, the turbulent energy $\mathcal{E}$ is given in terms of $[\varrho, \mathbf{u}, \mathfrak{R}]$ through (2.10).

Let $\{[\varrho_n, \mathbf{u}_n]\}_{n=1}^\infty$, with the associated $\{\mathcal{E}_n\}_{n=1}^\infty$, be a minimizing sequence of I on $\mathcal{U}[\varrho_0, \mathbf{m}_0]$. In view of the uniform bounds resulting from the energy inequality (2.8) and Helly's theorem, we may extract a suitable subsequence (not relabeled) such that

$$\begin{aligned}
\varrho_n &\to \varrho \text{ in } C_{\mathrm{weak,loc}}([0,\infty); L^\gamma(\Omega)),\\
\mathbf{u}_n &\to \mathbf{u} \text{ weakly in } L^q([0,\infty); W_0^{1,q}(\Omega; \mathbb{R}^d)),\\
\varrho_n \mathbf{u}_n \equiv \mathbf{m}_n &\to \mathbf{m} \text{ in } C_{\mathrm{weak,loc}}([0,\infty); L^{\frac{2\gamma}{\gamma+1}}(\Omega)),\\
\mathcal{E}_n &\to \mathcal{E} \text{ pointwise in } [0,\infty),\\
\mathbb{S}_n &\to \mathbb{S} \text{ weakly in } L^1_{\mathrm{loc}}([0,\infty) \times \Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}),\\
\mathfrak{R}_n &\to \mathfrak{R}^\infty \text{ weakly-(*) in } L^\infty(0,\infty; \mathcal{M}(\overline{\Omega}; \mathbb{R}^{d\times d}_{\mathrm{sym}})),\\
\mathfrak{R}_n^{\mathrm{conv}} \equiv \left(1_{\varrho_n > 0} \frac{\mathbf{m}_m \otimes \mathbf{m}_n}{\varrho_n} - 1_{\varrho > 0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) &\to \mathfrak{R}^{\mathrm{conv}} \text{ weakly-(*) in } L^\infty(0,\infty; \mathcal{M}(\overline{\Omega}; \mathbb{R}^{d\times d}_{\mathrm{sym}})),\\
\mathfrak{R}_n^p \equiv (p(\varrho_n) - p(\varrho)) &\to \mathfrak{R}^p \text{ weakly-(*) in } L^\infty(0,\infty; \mathcal{M}(\overline{\Omega})),\\
\mathfrak{R}_n^{\mathrm{kin}} \equiv \left(\frac{1}{2} \frac{|\mathbf{m}_n|^2}{\varrho_n} - \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} \right) &\to \mathfrak{R}^{\mathrm{kin}} \text{ weakly-(*) in } L^\infty(0,\infty; \mathcal{M}(\overline{\Omega})),\\
\mathfrak{R}_n^P \equiv (P(\varrho_n) - P(\varrho)) &\to \mathfrak{R}^P \text{ weakly-(*) in } L^\infty(0,\infty; \mathcal{M}(\overline{\Omega})).
\end{aligned}$$

Repeating the arguments used in [1, Section 3.4], we successively deduce that

$$\mathbf{m} = \varrho \mathbf{u};$$

$$\mathcal{E} = \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - P'(\overline{\varrho})(\varrho - \overline{\varrho}) - P(\overline{\varrho}) \right] \ \mathrm{d}x + \int_{\overline{\Omega}} \mathrm{d}\mathfrak{R}^{\mathrm{kin}} + \int_{\overline{\Omega}} \mathrm{d}\mathfrak{R}^{\mathrm{P}} + D \int_{\overline{\Omega}} \mathrm{d} \ \mathrm{tr}[\mathfrak{R}^\infty];$$

$$\left[\int_\Omega \varrho\varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \Big[\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \Big] \ \mathrm{d}x \, \mathrm{d}t, \ \varrho(0, \cdot) = \varrho_0,$$

for any $\tau \geq 0$, and any test function $\varphi \in C^1_{\mathrm{loc}}([0,\infty) \times \overline{\Omega})$;

$$\left[\int_\Omega \varrho \mathbf{u} \cdot \boldsymbol{\varphi} \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \Big[\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} - \mathbb{S} : \nabla_x \boldsymbol{\varphi} \Big] \ \mathrm{d}x$$
$$+ \int_0^\tau \int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\Big[\mathfrak{R}^{\mathrm{conv}} + \mathfrak{R}^p \mathrm{Id} \Big](t) \, \mathrm{d}t + \int_0^\tau \int_\Omega \nabla_x \boldsymbol{\varphi} : \mathrm{d} \ \mathfrak{R}^\infty(t) \ \mathrm{d}t, \ \varrho \mathbf{u}(0, \cdot) = \mathbf{m}_0,$$

for any $\tau \geq 0$ and any test function $\boldsymbol{\varphi} \in C^1_c([0,\infty) \times \Omega; \mathbb{R}^d)$;

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E} \leq - \int_\Omega \Big(F(\mathbb{D}_x \mathbf{u}) + F^*(\mathbb{S}) \Big) \ \mathrm{d}x \, \mathrm{d}t \text{ in } \mathcal{D}'(0,\infty),$$
$$\mathcal{E}(0+) = \int_\Omega E\left(\varrho_0, \mathbf{m}_0 \Big| \overline{\varrho} \right) \ \mathrm{d}x.$$

Next, the convexity hypothesis (2.1) implies that

$$\mathfrak{R}^P \geq \frac{a}{d} \mathrm{tr}[\mathfrak{R}^p \mathrm{Id}], \text{ while, obviously, } \mathfrak{R}^{\mathrm{kin}} \geq \frac{1}{2} \mathrm{tr}[\mathfrak{R}^{\mathrm{conv}}]$$

Consequently, introducing a new Reynolds stress

$$\mathfrak{R} = \mathfrak{R}^{\mathrm{conv}} + \mathfrak{R}^p \mathbb{I} + \mathfrak{R}^\infty \in L^\infty(0,\infty; \mathcal{M}^+(\overline{\Omega}; \mathbb{R}^{d\times d}_{\mathrm{sym}}))$$

we may infer that

$$\mathcal{E}(\tau) \geq \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - P'(\overline{\varrho})(\varrho - \overline{\varrho}) - P(\overline{\varrho}) \right] \ \mathrm{d}x + D \int_{\overline{\Omega}} \mathrm{d} \ \mathrm{tr}[\mathfrak{R}]$$

for a.e. $\tau \in (0,\infty)$. Thus modifying $\mathfrak{R}$

$$\mathfrak{R} \approx \mathfrak{R} + \chi \mathbb{I}, \ \chi \in L^\infty(0,\infty), \ \chi \geq 0,$$

we achieve

$$\mathcal{E}(\tau) = \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - P'(\overline{\varrho})(\varrho - \overline{\varrho}) - P(\overline{\varrho}) \right] \ \mathrm{d}x + D \int_{\overline{\Omega}} \mathrm{d} \ \mathrm{tr}[\mathfrak{R}]$$

for a.e. $\tau \in (0,\infty)$. In other words, $[\varrho, \mathbf{u}]$ is a dissipative solution, with the associated turbulent energy $\mathcal{E}$ minimizing the functional I; whence maximal.

We have shown the following result.

Proposition 2.4 (Existence of maximal solutions)**.** *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded Lipschitz domain. Suppose that F and p comply with the hypotheses* (2.1)*,* (2.3)*,* (2.4)*. Let the initial data $[\varrho_0, \mathbf{m}_0]$ be given,*

$$\varrho_0 \geq 0, \ \int_\Omega E\left(\varrho_0, \mathbf{m}_0 \ \middle| \overline{\varrho}\right) \ \mathrm{d}x < \infty.$$

Then the problem (1.1)–(1.5) *admits a maximal dissipative solution $[\varrho, \mathbf{u}]$ in $(0, \infty) \times \Omega$ in the sense specified in Definition 2.2, meaning minimal with respect to the relation $\prec$ in $\mathcal{U}[\varrho_0, \mathbf{m}_0]$.*

3 Long–time behavior of maximal solutions

We are ready to state our main result concerning the long time behavior of maximal solutions.

Theorem 3.1 (Long time behavior)**.** *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded Lipschitz domain. Suppose that F and p comply with the hypotheses* (2.1)*,* (2.3)*,* (2.4)*. Let $[\varrho, \mathbf{u}]$ be a maximal dissipative solution the problem* (1.1)–(1.5) *in $(0, \infty) \times \Omega$ in $\mathcal{U}[\varrho_0, \mathbf{m}_0]$, with the associate Reynolds defect $\mathfrak{R}$ and the turbulent energy $\mathcal{E}$,*

$$\mathcal{E}(\tau) \to \mathcal{E}_\infty \ \text{as} \ \tau \to \infty.$$

Then

$$\int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - P'(\overline{\varrho})(\varrho - \overline{\varrho}) - P(\overline{\varrho})\right](\tau, \cdot) \ \mathrm{d}x \to \mathcal{E}_\infty \ \text{as} \ \tau \to \infty,$$

in particular,

$$\operatorname{ess}\lim_{\tau \to \infty} \|\mathfrak{R}(\tau)\|_{\mathcal{M}(\overline{\Omega}; \mathbb{R}^{d \times d}_{\rm sym})} = 0.$$

Proof. As the energy $E\left(\varrho, \mathbf{m} \ \middle| \overline{\varrho}\right)$ is convex and the functions ϱ, $\mathbf{m}$ weakly continuous in the time variable, we have

$$\mathcal{E}_\infty \geq \limsup_{\tau \to \infty} \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - P'(\overline{\varrho})(\varrho - \overline{\varrho}) - P(\overline{\varrho})\right](\tau, \cdot) \ \mathrm{d}x.$$

Consequently, it is enough to show

$$\int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - P'(\overline{\varrho})(\varrho - \overline{\varrho}) - P(\overline{\varrho})\right](T, \cdot) \ \mathrm{d}x \equiv \int_\Omega E\left(\varrho, \mathbf{m} \middle| \overline{\varrho}\right)(T, \cdot) \ \mathrm{d}x \geq \mathcal{E}_\infty \tag{3.1}$$

for any $T \geq 0$.

Arguing by contradiction we suppose there exists $T \geq 0$ such that

$$\mathcal{E}_\infty > \int_\Omega E\left(\varrho, \mathbf{m} \middle| \overline{\varrho}\right)(T, \cdot) \ \mathrm{d}x.$$

In accordance with Proposition 2.4, the problem (1.1)–(1.5) admits a dissipative solution $[\varrho_T, \mathbf{u}_T]$ in $(T,\infty)\times\Omega$, starting from the initial data $[\varrho(T,\cdot), \mathbf{m}(T,\cdot)]$, and such that the associated turbulent energy $\mathcal{E}_T$ satisfies

$$\mathcal{E}_T(\tau) \leq \int_\Omega E\left(\varrho, \mathbf{m}\middle|\overline{\varrho}\right)(T,\cdot)\ \mathrm{d}x < \mathcal{E}_\infty \leq \mathcal{E}(\tau) \text{ for all } \tau \geq T. \tag{3.2}$$

Finally, we define a new dissipative solution solution $[\tilde{\varrho}, \tilde{\mathbf{u}}]$,

$$[\tilde{\varrho}, \tilde{\mathbf{u}}](\tau,\cdot) = \left\{ \begin{array}{l} [\varrho, \mathbf{u}](\tau,\cdot) \text{ if } 0 \leq \tau < T, \\ \\ [\varrho_T, \mathbf{u}_T](\tau,\cdot) \text{ if } \tau \geq T. \end{array} \right.$$

with the associated turbulent energy

$$\widetilde{\mathcal{E}}(\tau) = \left\{ \begin{array}{l} \mathcal{E}(\tau) \text{ if } 0 \leq \tau < T, \\ \\ \mathcal{E}_T(\tau) \text{ if } \tau \geq T. \end{array} \right.$$

In view of (3.2), however, $[\tilde{\varrho}, \tilde{\mathbf{u}}] \prec [\varrho, \mathbf{u}]$ and $[\varrho, \mathbf{u}]$ is not maximal in contrast with our hypothesis.

□

4 Concergence to equilibria

In order to establish convergence of dissipative solutions to equilibria, extra hypotheses must be imposed on the structural properties of F and p, specifically, on the the exponents γ and q appearing in (2.2) and (2.5), respectively.

4.1 Renormalization

In addition to (2.6), we need its renormalized version

$$\left[\int_\Omega B(\varrho)\varphi\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[B(\varrho)\partial_t\varphi + B(\varrho)\mathbf{u}\cdot\nabla_x\varphi + \Big(B(\varrho) - B'(\varrho)\varrho\Big)\mathrm{div}_x\mathbf{u}\right]\ \mathrm{d}x\,\mathrm{d}t \tag{4.1}$$

to be satisfied for any $0 \leq \tau \leq T$, any test function $\varphi \in C^1_c([0,\infty)\times\overline{\Omega})$, and any $B \in C^1(\mathbb{R})$, $B' \in C_c(\mathbb{R})$.

In accordance with the DiPerna–Lions theory [8], the renormalized equation (4.1) follows from (2.6) as soon as

$$\frac{1}{\gamma} + \frac{1}{q} \leq 1. \tag{4.2}$$

4.2 Bounds on kinetic energy

In order to prove convergence to equilibria, better bounds on the kinetic energy are necessary. More specifically, we need

$$\sup_{T\geq 0}\int_T^{T+1}\int_\Omega \varrho^\alpha |\mathbf{u}|^{2\alpha}\ \mathrm{d}x\,\mathrm{d}t < \infty \text{ for some } \alpha > 1. \tag{4.3}$$

To obtain (4.3), we write

$$\|\varrho|\mathbf{u}|^2\|_{L^\alpha(\Omega)} \leq \|\varrho\mathbf{u}\|_{L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^d)}\|\mathbf{u}\|_{L^p(\Omega;\mathbb{R}^d)},\ \frac{1}{\alpha} = \frac{\gamma+1}{2\gamma} + \frac{1}{p}.$$

On the other hand, by Sobolev's embedding theorem,

$$\begin{aligned}
\|\mathbf{u}\|_{L^\infty(\Omega;\mathbb{R}^d)} &\lesssim \|\mathbf{u}\|_{W^{1,q}(\Omega;\mathbb{R}^d)} \text{ if } q > d,\\
\|\mathbf{u}\|_{L^p(\Omega;\mathbb{R}^d)} &\lesssim \|\mathbf{u}\|_{W^{1,q}(\Omega;\mathbb{R}^d)} \text{ for any } 1 \leq p < \infty \text{ if } q = d,\\
\|\mathbf{u}\|_{L^p(\Omega;\mathbb{R}^d)} &\lesssim \|\mathbf{u}\|_{W^{1,q}(\Omega;\mathbb{R}^d)} \text{ for } 1 \leq p \leq \frac{dq}{d-q} \text{ if } q < d.
\end{aligned}$$

Consequently, the desired bound (4.3) follows from the energy inequality (2.8) as soon as

$$\frac{\gamma+1}{2\gamma} + \frac{d-q}{dq} < 1. \tag{4.4}$$

Moreover, a short inspection of the existence proof in [1] reveals that the traceless part of the Reynolds stress $\mathfrak{R}$,

$$\mathfrak{R} - \frac{1}{d}\mathrm{tr}[\mathfrak{R}]\ \mathbb{I} = \overline{\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} - \frac{1}{d}\frac{|\mathbf{m}|^2}{\varrho}\mathbb{I}} - \left(\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} - \frac{1}{d}\frac{|\mathbf{m}|^2}{\varrho}\mathbb{I}\right),$$

admits a bound similar to (4.3), namely,

$$\sup_{T\geq 0}\int_T^{T+1}\int_\Omega \left|\mathfrak{R} - \frac{1}{d}\mathrm{tr}[\mathfrak{R}]\mathbb{I}\right|^\alpha\ \mathrm{d}x\,\mathrm{d}t < \infty \text{ for some } \alpha > 1. \tag{4.5}$$

In other words, under the hypothesis (4.4), there exists a dissipative solution with the associated Reynolds stress $\mathfrak{R}$ satisfying (4.5).

Remark 4.1. As a matter of fact, the traceless part

$$\mathfrak{R} - \frac{1}{d}\mathrm{tr}[\mathfrak{R}]\ \mathbb{I} = \overline{\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} - \frac{1}{d}\frac{|\mathbf{m}|^2}{\varrho}\mathbb{I}} - \left(\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} - \frac{1}{d}\frac{|\mathbf{m}|^2}{\varrho}\mathbb{I}\right)$$

actually *vanishes* for the dissipative solutions constructed by the method of [1]. Indeed the sequence of approximate solutions $\{[\varrho_n, \mathbf{u}_n]\}_{n=1}^\infty$ constructed in [1, Section 3.4] satisfies

$$\varrho_n \mathbf{u}_n \to \varrho \mathbf{u} \text{ in } C_{\text{weak}}([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^d)),\ \mathbf{u}_n \to \mathbf{u} \text{ weakly in } L^q([0,T]; W_0^{1,q}(\Omega;\mathbb{R}^d))$$

for arbitrary $T > 0$. Moreover, the hypothesis (4.4) implies that

$$L^{\frac{2\gamma}{\gamma+1}}(\Omega; \text{weak}) \hookrightarrow\hookrightarrow W^{-1,q}(\Omega);$$

whence

$$\overline{\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} - \frac{1}{d}\frac{|\mathbf{m}|^2}{\varrho}\mathbb{I}} - \left(\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} - \frac{1}{d}\frac{|\mathbf{m}|^2}{\varrho}\mathbb{I}\right) = 0$$

4.3 Bounds on the viscous stress

In addition to the lower bound (2.4), we suppose that

$$F(\mathbb{D}) \lesssim 1 + |\mathbb{D}|^r \text{ for some } r < \infty,\ \partial F(0) = \{0\}, \tag{4.6}$$

which implies

$$F^*(\mathbb{S}) > 0 \text{ for all } \mathbb{S} \neq 0,\ F^*(\mathbb{S}) \gtrsim |\mathbb{S}|^\alpha \text{ for some } \alpha > 1, \text{ and all } |\mathbb{S}| \geq 1. \tag{4.7}$$

4.4 Convergence

We are ready to state our main result concerning convergence to equilibria of dissipative solutions.

Theorem 4.2 (Convergence to equilibria). *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded Lipschitz domain. Suppose that F and p comply with the hypotheses* (2.1), (2.3), (2.4), *and* (4.6). *In addition, suppose that the exponents γ and q appearing in* (2.2) *and* (2.5), *respectively, satisfy*

$$\frac{\gamma+1}{2\gamma} + \frac{d-q}{dq} < 1. \tag{4.8}$$

Let $[\varrho, \mathbf{u}]$ be a dissipative solution to the problem (1.1)–(1.5) *satisfying the renormalized equation of continuity* (4.1), *with the associated Reynolds stress $\mathfrak{R}$ such that*

$$\sup_{T\geq 0} \int_T^{T+1} \int_\Omega \left| \mathfrak{R} - \frac{1}{d}\text{tr}[\mathfrak{R}]\mathbb{I} \right|^\alpha \ \mathrm{d}x\,\mathrm{d}t < \infty \text{ for some } \alpha > 1. \tag{4.9}$$

Finally, suppose that

$$\operatorname{ess}\lim_{\tau\to\infty} \|\mathfrak{R}(\tau)\|_{\mathcal{M}(\overline{\Omega};\mathbb{R}^{d\times d}_{\text{sym}})} = 0. \tag{4.10}$$

Then

$$\varrho\mathbf{u}(\tau,\cdot) \to 0 \text{ in } L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^d),\ \varrho(\tau,\cdot) \to \overline{\varrho} \text{ in } L^\gamma(\Omega),\ \overline{\varrho} = \frac{1}{|\Omega|}\int_\Omega \varrho_0 \ \mathrm{d}x,$$

as $\tau \to \infty$.

Remark 4.3. As pointed out in Section 4.2, the problem (1.1)–(1.5) admits a dissipative solution satisfying (4.9) as soon as the hypothesis (4.8) holds.

Remark 4.4. As stated in Theorem 3.1, the hypothesis (4.10) holds for any *maximal* dissipative solution in the sense of Definition 2.2.

The rest of this section is devoted to the proof of Theorem 4.2, carried over in several steps.

4.4.1 Uniform pressure estimates

In order to derive the estimates implying equi–integrability of the pressure $p(\varrho)$, we introduce the so–called Bogovskii operator $\mathcal{B}$ that may be seen as a suitable branch of the inverse divergence div_x^{-1}, see Bogovskii [2]. For reader's convenience, we recall the basic properties of the operator $\mathcal{B}$ proved in Bogovskii [2], Galdi [13], and Geissert, Heck, and Hieber [14] (see also [9, Theorem 11.17]).

- $$\mathcal{B} : \left\{ f \in L^p(\Omega) \ \Big| \ \int_\Omega f \ \mathrm{d}x = 0 \right\} \to W_0^{1,p}(\Omega; \mathbb{R}^d)$$
  is a bounded linear operator for any $1 < p < \infty$, specifically,
  $$\| \mathcal{B}[f] \|_{W_0^{1,p}(\Omega;\mathbb{R}^d)} \stackrel{<}{\sim} \| f \|_{L^p(\Omega)},\ 1 < p < \infty. \tag{4.11}$$
- $$\mathrm{div}_x \mathcal{B}[f] = f;$$
- if $f \in L^p(\Omega)$, $\int_\Omega f \ \mathrm{d}x = 0$, and, in addition,
  $$f = \mathrm{div}_x \mathbf{g},\ \mathbf{g} \in L^r(\Omega; \mathbb{R}^d),\ \mathrm{div}_x \mathbf{g} \in L^p(\Omega),\ \mathbf{g} \cdot \mathbf{n}|_{\partial\Omega} = 0,$$
  then
  $$\| \mathcal{B}[f] \|_{L^r(\Omega;\mathbb{R}^d)} \stackrel{<}{\sim} \| \mathbf{g} \|_{L^r(\Omega;\mathbb{R}^d)},\ 1 < r < \infty; \tag{4.12}$$
- if $f \in W^{k,p}(\Omega)$, k=1,2,..., $1 < p < \infty$, $\int_\Omega f \ \mathrm{d}x = 0$, then $\mathcal{B}[f] \in W^{k+1,p}(\Omega; \mathbb{R}^d)$.

It follows from (4.1) that the renormalized equation of continuity holds in $(0, \infty) \times R^d$ provided ϱ, $\mathbf{u}$ are extended to be zero outside Ω:

$$\left[\int_{\mathbb{R}^d} B(\varrho) \varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\mathbb{R}^d} \Big[B(\varrho) \partial_t \varphi + B(\varrho) \mathbf{u} \cdot \nabla_x \varphi + \Big(B(\varrho) - B'(\varrho)\varrho \Big) \mathrm{div}_x \mathbf{u} \Big] \ \mathrm{d}x \, \mathrm{d}t$$

for any $0 \leq \tau \leq T$, any test function $\varphi \in C_c^1([0, \infty) \times \mathbb{R}^d)$, and any $B \in C^1(\mathbb{R})$, $B' \in C_c(\mathbb{R})$. Consequently, we may apply the standard regularization procedure via convolution with a family of regularizing kernels $\{\theta_\varepsilon\}_{\varepsilon > 0}$ in the $x-$variable to obtain

$$\partial_t [B(\varrho)]_\varepsilon + \mathrm{div}_x \left([B(\varrho)]_\varepsilon \mathbf{u} \right) + \left[\left(B'(\varrho)\varrho - B(\varrho) \right) \mathrm{div}_x \mathbf{u} \right]_\varepsilon = E_\varepsilon$$

with the error term,

$$E_\varepsilon = \mathrm{div}_x \left([B(\varrho)]_\varepsilon \mathbf{u} \right) - [\mathrm{div}_x (B(\varrho)\mathbf{u})]_\varepsilon \,,$$

where we have denoted $[v]_\varepsilon = \theta_\varepsilon * v$. In view of (2.5) and a version of Korn–Poincaré inequality,

$$\mathbf{u} \in L^q(0,T; W_0^{1,q}(\Omega; \mathbb{R}^d)).$$

As B is bounded, we may use the DiPerna–Lions theory [8], notably Friedrich's commutator lemma (see also [9, Lemma 11.12]), to conclude

$$E_\varepsilon \to 0 \text{ in } L^r_{\mathrm{loc}}(0,T; L^r(\Omega)) \text{ as } \varepsilon \to 0 \text{ for any } 1 \le r < q. \tag{4.13}$$

Next, we use

$$\boldsymbol{\varphi} = \psi(t) \mathcal{B} \left[[B(\varrho)]_\varepsilon - \frac{1}{|\Omega|} \int_\Omega [B(\varrho)]_\varepsilon \ \mathrm{d}x \right], \ \psi \in C^1_c(0,\infty), \ \psi \ge 0,$$

as a test function in the momentum equation (2.7). After a straightforward manipulation, we obtain

$$\begin{aligned}
&\int_0^\infty \psi \int_\Omega p(\varrho)[B(\varrho)]_\varepsilon \ \mathrm{d}x \, \mathrm{d}t - \frac{1}{|\Omega|} \int_0^\infty \psi \left(\int_\Omega p(\varrho) \ \mathrm{d}x \right) \left(\int_\Omega [B(\varrho)]_\varepsilon \ \mathrm{d}x \right) \mathrm{d}t \\
&\qquad + \frac{1}{d} \int_0^\infty \psi \int_\Omega [B(\varrho)]_\varepsilon \ \mathrm{d}\mathrm{tr}[\mathfrak{R}] \, \mathrm{d}t - \frac{1}{d|\Omega|} \int_0^\infty \psi \left(\int_\Omega \mathrm{d}\mathrm{tr}[\mathfrak{R}] \right) \left(\int_\Omega [B(\varrho)]_\varepsilon \ \mathrm{d}x \right) \mathrm{d}t \\
&= - \int_0^\infty \psi \int_\Omega \left[\varrho \mathbf{u} \otimes \mathbf{u} - \mathbb{S} + \left(\mathfrak{R}(t) - \frac{1}{d} \mathrm{tr}[\mathfrak{R}] \mathbb{I} \right) \right] : \nabla_x \mathcal{B} \left[[B(\varrho)]_\varepsilon - \frac{1}{|\Omega|} \int_\Omega [B(\varrho)]_\varepsilon \ \mathrm{d}x \right] \ \mathrm{d}x \, \mathrm{d}t \\
&\qquad - \int_0^\infty \partial_t \psi \int_\Omega \varrho \mathbf{u} \cdot \mathcal{B} \left[[B(\varrho)]_\varepsilon - \frac{1}{|\Omega|} \int_\Omega [B(\varrho)]_\varepsilon \ \mathrm{d}x \right] \ \mathrm{d}x \, \mathrm{d}t \\
&\qquad + \int_0^\infty \psi \int_\Omega \varrho \mathbf{u} \cdot \mathcal{B} [\mathrm{div}_x ([B(\varrho)]_\varepsilon \mathbf{u})] \ \mathrm{d}x \, \mathrm{d}t \\
&\qquad + \int_0^\infty \psi \int_\Omega \varrho \mathbf{u} \cdot \mathcal{B} \left[(B'(\varrho)\varrho - B(\varrho)) \, \mathrm{div}_x \mathbf{u} - \frac{1}{|\Omega|} \int_\Omega (B'(\varrho)\varrho - B(\varrho)) \, \mathrm{div}_x \mathbf{u} \ \mathrm{d}x \right]_\varepsilon \ \mathrm{d}x \, \mathrm{d}t \\
&\qquad - \int_0^\infty \psi \int_\Omega \varrho \mathbf{u} \cdot \mathcal{B} \left[E_\varepsilon - \frac{1}{|\Omega|} \int_\Omega E_\varepsilon \ \mathrm{d}x \right] \ \mathrm{d}x \, \mathrm{d}t.
\end{aligned}$$

Now observe that, in view of the hypothesis (4.8), the error estimate (4.13), and the regularization property of the operator $\mathcal{B}$ stated in (4.11), we may let $\varepsilon \to 0$ obtaining

$$\begin{aligned}
\int_0^\infty \psi \int_\Omega p(\varrho)[B(\varrho)] \ \mathrm{d}x\,\mathrm{d}t \leq \frac{1}{|\Omega|} \int_0^\infty \psi \left(\int_\Omega p(\varrho) \ \mathrm{d}x \right) \left(\int_\Omega [B(\varrho)] \ \mathrm{d}x \right) \mathrm{d}t \\
+ \frac{1}{d|\Omega|} \int_0^\infty \psi \left(\int_\Omega \mathrm{d}\mathrm{tr}[\mathfrak{R}] \right) \left(\int_\Omega [B(\varrho)] \ \mathrm{d}x \right) \mathrm{d}t \\
- \int_0^\infty \psi \int_\Omega \left[\varrho \mathbf{u} \otimes \mathbf{u} - \mathbb{S} + \left(\mathfrak{R}(t) - \frac{1}{d} \mathrm{tr}[\mathfrak{R}] \mathbb{I} \right) \right] : \nabla_x \mathcal{B} \left[[B(\varrho)] - \frac{1}{|\Omega|} \int_\Omega [B(\varrho)] \ \mathrm{d}x \right] \ \mathrm{d}x\,\mathrm{d}t \\
- \int_0^\infty \partial_t \psi \int_\Omega \varrho \mathbf{u} \cdot \mathcal{B} \left[[B(\varrho)] - \frac{1}{|\Omega|} \int_\Omega [B(\varrho)] \ \mathrm{d}x \right] \ \mathrm{d}x\,\mathrm{d}t \\
+ \int_0^\infty \psi \int_\Omega \varrho \mathbf{u} \cdot \mathcal{B}[\mathrm{div}_x([B(\varrho)]\mathbf{u})] \ \mathrm{d}x\,\mathrm{d}t \\
+ \int_0^\infty \psi \int_\Omega \varrho \mathbf{u} \cdot \mathcal{B} \left[(B'(\varrho)\varrho - B(\varrho)) \,\mathrm{div}_x \mathbf{u} - \frac{1}{|\Omega|} \int_\Omega (B'(\varrho)\varrho - B(\varrho)) \,\mathrm{div}_x \mathbf{u} \ \mathrm{d}x \right] \ \mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{4.14}$$

Finally, using the hypothesis (4.9), together with the bounds (4.3), (4.7), and the regularizing properties of $\mathcal{B}$ stated in (4.11), (4.12), we may infer that

- validity of (4.14) can be extended to $B(\varrho) = \varrho^\beta$ for some $\beta > 0$;
- the integrals on the right–hand side of (4.14) are uniformly bounded with respect to the time shifts of ψ.

We conclude that

$$\sup_{T \geq 0} \int_T^{T+1} \int_\Omega p(\varrho)\varrho^\beta \ \mathrm{d}x\,\mathrm{d}t < \infty \text{ for some } \beta > 0, \tag{4.15}$$

and, by virtue of the hypothesis (2.1),

$$\sup_{T \geq 0} \int_T^{T+1} \int_\Omega P(\varrho)\varrho^\beta \ \mathrm{d}x\,\mathrm{d}t < \infty \text{ for some } \beta > 0. \tag{4.16}$$

4.4.2 Convergence of density and momentum averages

We are ready to show convergence to the equilibrium state. We introduce the time shifts

$$\varrho_n(t,x) = \varrho(t+n,x), \ \mathbf{u}_n(t,x) = \mathbf{u}(t+n,x), \ \mathbf{m}_n(t,x) = \mathbf{m}(t+n,x) \text{ etc.}$$

In view of the bound

$$\mathbf{u} \in L^q(0,\infty; W_0^{1,q}(\Omega;\mathbb{R}^d)) < \infty,$$

we have

$$\mathbf{u}_n \to 0 \text{ in } L^q(0,1;W_0^{1,q}(\Omega;\mathbb{R}^d)).$$

Moreover, as

$$\varrho \in L^\infty(0,\infty;L^\gamma(\Omega;\mathbb{R}^d)),$$

and the kinetic energy is controlled by (4.3), we deduce that

$$\int_0^1 \int_\Omega \frac{1}{2}\frac{|\mathbf{m}_n|^2}{\varrho_n} \ \mathrm{d}x \to 0 \text{ as } n \to \infty. \tag{4.17}$$

The next step is to show strong a.e. convergence of $\{\varrho_n\}_{n=1}^\infty$. We start observing that

$$\varrho_n \to \varrho_\infty \text{ weakly-(*) in } L^\infty(0,1;L^\gamma(\Omega)),\ \ \varrho_\infty \geq 0.$$

Moreover, letting $n \to \infty$ in the equation of continuity (2.6) we deduce

$$\varrho_\infty = \varrho_\infty(x) \text{ is independent of } t. \tag{4.18}$$

Next, we perform the limit $n \to \infty$ in the momentum equation (2.7). Here, the crucial fact is that $\mathfrak{R}$ vanishes as stated in (4.10). Consequently, in accordance with (4.7),

$$\mathbb{S} \in L^\alpha(0,\infty;L^\alpha(\Omega;\mathbb{R}^{d\times d}_{\rm sym})),$$

and we obtain

$$\nabla_x \overline{p(\varrho)} = 0 \text{ in } \mathcal{D}'((0,1)\times\Omega), \tag{4.19}$$

where

$$p(\varrho_n) \to \overline{p(\varrho)} \text{ weakly in } L^1((0,T)\times\Omega).$$

Here, similarly to (4.18), the convergence holds up to a subsequence which we do not relabel.

In order to show strong convergence of $\{\varrho_n\}_{n=1}^\infty$, we consider (4.14), with

$$B(\varrho) = \varrho^\alpha,\ \text{with } 0 < \alpha < \beta,$$

where β is the exponent in (4.15). Letting $n \to \infty$ in (4.14) we obtain

$$\int_0^1 \int_\Omega \overline{p(\varrho)\varrho^\alpha} \ \mathrm{d}x\,\mathrm{d}t \leq \frac{1}{|\Omega|}\int_\Omega \overline{p(\varrho)} \ \mathrm{d}x \int_\Omega \overline{\varrho^\alpha} \ \mathrm{d}x, \tag{4.20}$$

where, similarly to (4.19), the bar denotes the corresponding weak limits.

Now, testing (4.19) on

$$\psi \mathcal{B}\left[\overline{\varrho^\alpha} - \frac{1}{|\Omega|}\int_\Omega \overline{\varrho^\alpha} \ \mathrm{d}x\right]$$

we obtain

$$\int_0^1 \int_\Omega \overline{p(\varrho)}\ \overline{\varrho^\alpha} \ \mathrm{d}x\,\mathrm{d}t = \frac{1}{|\Omega|}\int_\Omega \overline{p(\varrho)} \ \mathrm{d}x \int_\Omega \overline{\varrho^\alpha} \ \mathrm{d}x,$$

which, together with (4.20), gives rise to

$$\int_0^1 \int_\Omega \overline{p(\varrho)\varrho^\alpha} \ \mathrm{d}x \leq \int_0^1 \int_\Omega \overline{p(\varrho)} \ \overline{\varrho^\alpha} \ \mathrm{d}x \, \mathrm{d}t. \tag{4.21}$$

As p is strictly increasing, relation (4.21) implies

$$\varrho_n \to \varrho \text{ in measure in } (0,1) \times \Omega \tag{4.22}$$

by means of the standard monotonicity argument (cf. [9, Theorem 11.26]).

Finally, we deduce from (4.19) that ϱ_∞ is independent of x; whence ϱ_∞ coincides with the constant equilibrium state

$$\varrho_\infty = \overline{\varrho}.$$

Using (4.17), together with the uniform bound (4.17) and the strong convergence of the density stated in (4.22), we conclude that there is a sequence of times $\tau_n \to \infty$ such that

$$\int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - P'(\overline{\varrho})(\varrho - \overline{\varrho}) - P(\overline{\varrho}) \right] (\tau_n, \cdot) \ \mathrm{d}x \to 0.$$

As shown in Theorem 3.1, the energy functional admits a limit

$$\int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - P'(\overline{\varrho})(\varrho - \overline{\varrho}) - P(\overline{\varrho}) \right] (\tau, \cdot) \ \mathrm{d}x \to \mathcal{E}_\infty \text{ as } \tau \to \infty;$$

whence $\mathcal{E}_\infty = 0$ and the proof of Theorem 4.2 is complete.

5 Unconditional convergence

Theorem 4.2 may seem rather awkward as extra hypotheses are imposed not only on the structural properties of p and F but also on the solution itself. In this section, we try to remedy the problem at the expense of stronger restrictions on the exponents γ and q. As observed in Remark 4.1, the problem (1.1)–(1.5) admits a dissipative solution with a vanishing traceless component of $\mathfrak{R}$ as soon as the exponents γ and q satisfy (4.8). This motivates the following modification of the set $\mathcal{U}[\varrho_0, \mathbf{m}_0]$ that we replace by

$$\widetilde{\mathcal{U}}[\varrho_0, \mathbf{m}_0] = \Big\{ [\varrho, \mathbf{u}, \mathcal{E}] \ \Big| \ [\varrho, \mathbf{u}] \text{ –a dissipative solutions with the initial data } [\varrho_0, \mathbf{m}_0]$$
$$\text{and the associated turbulent energy } \mathcal{E}, \ \mathfrak{R} - \frac{1}{d} \mathrm{tr}[\mathfrak{R}] \mathbb{I} = 0 \Big\}$$

Theorem 5.1 (Unconditional convergence to equilibria)**.** *Let $\Omega \subset \mathbb{R}^d$, $d = 2,3$ be a bounded Lipschitz domain. Suppose that F and p comply with the hypotheses* (2.1), (2.3), (2.4), *and* (4.6). *In addition, suppose that the exponents γ and q appearing in* (2.2) *and* (2.5), *respectively, satisfy*

$$\frac{1}{\gamma} + \frac{1}{q} \leq 1 \ \textit{if} \ q > \frac{d}{2}, \ \ \frac{\gamma + 1}{2\gamma} + \frac{d - q}{dq} < 1 \ \textit{if} \ q \leq \frac{d}{2}. \tag{5.1}$$

Let $[\varrho, \mathbf{u}]$ be a solution to the problem (1.1)–(1.5) maximal in $\widetilde{\mathcal{U}}[\varrho_0, \mathbf{m}_0]$ in the sense of Definition 2.2, meaning minimal in $\widetilde{\mathcal{U}}[\varrho_0, \mathbf{m}_0]$ with respect to the relation $\prec$.

Then

$$\varrho\mathbf{u}(\tau, \cdot) \to 0 \text{ in } L^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^d), \ \varrho(\tau, \cdot) \to \overline{\varrho} \text{ in } L^\gamma(\Omega), \ \overline{\varrho} = \frac{1}{|\Omega|} \int_\Omega \varrho_0 \ \mathrm{d}x,$$

as $\tau \to \infty$.

Remark 5.2. It is easy to check that (5.1) implies

$$\frac{1}{\gamma} + \frac{1}{q} \leq 1 \text{ and } \frac{\gamma+1}{2\gamma} + \frac{d-q}{dq} < 1.$$

Proof. First observe that, in view the hypothesis (5.1) and Remark 4.1, the set $\widetilde{\mathcal{U}}[\varrho_0, \mathbf{m}_0]$ is non–empty for any finite energy data $[\varrho_0, \mathbf{m}_0]$. Now it is enough to show that any $[\varrho, \mathbf{u}]$ maximal in $\widetilde{\mathcal{U}}[\varrho_0, \mathbf{m}_0]$ satisfies the hypotheses of Theorem 4.2.

To begin,

$$\mathfrak{R} - \frac{1}{d}\mathrm{tr}[\mathfrak{R}]\mathbb{I} = 0,$$

in particular the hypothesis (4.9) holds.

Next, repeating the arguments of the proof of Theorem 3.1 we can show that the Reynolds stress $\mathfrak{R}$ associated to $[\varrho, \mathbf{u}]$ vanishes for $\tau \to \infty$ as required in (4.10).

Finally, it follows from (5.1) (cf. Remark 5.2) and the DiPerna–Lions theory [8], that $[\varrho, \mathbf{u}]$ satisfies the renormalized equation of continuity (4.1). Thus the solution $[\varrho, \mathbf{u}]$ complies with all hypotheses of Theorem 4.2, which completes the proof.

□

6 Concluding remarks

The results presented above can be extended in a straightforward manner to the system driven by a potential external force:

$$\partial_t \varrho + \mathrm{div}_x(\varrho\mathbf{u}) = 0,$$
$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x\mathbb{S} + \varrho\nabla_x G, \ G = G(x).$$

Indeed the corresponding energy functional reads

$$\int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - \varrho G\right] \ \mathrm{d}x,$$

which can be rewritten as

$$\int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - P'(\tilde{\varrho})(\varrho - \tilde{\varrho}) - P(\tilde{\varrho})\right] \ \mathrm{d}x$$

modulo an additive constant. Here $\tilde{\varrho}$ is the associated equilibrium state solving

$$\nabla_x \tilde{\varrho} = \tilde{\varrho} \nabla_x G \text{ in } \Omega. \tag{6.1}$$

Apparently, equation (6.1) admits infinitely many solution, however uniqueness can be restored for certain potentials F by prescribing the total mass

$$M = \int_\Omega \tilde{\varrho} \ \mathrm{d}x. \tag{6.2}$$

As shown in [11], [12], the problem (6.1), (6.2) admits a unique non–negative solution $\tilde{\varrho}$ as soon as the level sets

$$[G > k] = \left\{ x \in \Omega \ \middle| \ G(x) > k \right\}$$

are connected for any k. Under these circumstances, Theorems 3.1, 4.2, 5.1 remain valid with obvious modifications in the proof.

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