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Strongly self-absorbing C^*-algebras and Fraïssé limits

Saeed Ghasemi

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STRONGLY SELF-ABSORBING C^*-ALGEBRAS AND FRAÏSSÉ LIMITS

SAEED GHASEMI

Abstract. We show that the Fraïssé limit of a category of unital separable C^*-algebras which is sufficiently closed under tensor products of its objects and morphisms is strongly self-absorbing, given that it has approximate inner half-flip. We use this connection between Fraïssé limits and strongly self-absorbing C^*-algebras to give a self-contained and rather elementary proof for the well known fact that the Jiang-Su algebra is strongly self-absorbing.

1. Introduction

A separable unital C^*-algebra $\mathcal{D}$ is called "strongly self-absorbing" if it is not $*$-isomorphic to $\mathbb{C}$ and there is a $*$-isomorphism $\varphi : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ which is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{D}} \otimes 1_{\mathcal{D}}$. UHF-algebras of infinite type, the Cuntz algebras $\mathcal{O}_2$, $\mathcal{O}_\infty$ and the Jiang-Su algebra $\mathcal{Z}$ are all strongly self-absorbing. These C^*-algebras play a central role in Elliott's classification program of separable nuclear C^*-algebras by K-theoretic data. In fact, the classification program has been almost exclusively focused on the (separable, unital and nuclear) C^*-algebras that tensorially absorb a strongly self-absorbing C^*-algebra $\mathcal{D}$ (called $\mathcal{D}$-stable). Strongly self-absorbing C^*-algebras are systematically studied in [19]. They are automatically simple, nuclear and are either purely infinite or stably finite with at most one tracial state. Among strongly self-absorbing C^*-algebras the Jiang-Su algebra $\mathcal{Z}$ has received special attention in recent years. This is mostly due to the fact that strongly self-absorbing C^*-algebras are all $\mathcal{Z}$-stable [20] and therefore the class of $\mathcal{Z}$-stable C^*-algebras is the largest possible class of $\mathcal{D}$-stable C^*-algebras, for any strongly self-absorbing C^*-algebra $\mathcal{D}$. The remarkable classification of separable, simple, unital, nuclear, and $\mathcal{Z}$-stable C^*-algebras satisfying the UCT by K-theoretic invariants, is the pinnacle of the classification program (cf. [21]).

The Jiang-Su algebra is a simple, separable, unital, nuclear, projectionless and infinite-dimensional C^*-algebra which has a unique tracial state (monotracial) and it is KK-equivalent to the complex numbers. In their original paper [9], Jiang and Su define $\mathcal{Z}$ as the unique inductive limit of a sequence of dimension-drop algebras and unital $*$-homomorphisms which is simple and monotracial. Of course, this definition involves first constructing such an inductive limit and then showing that it is the unique one which is simple and monotracial. Since then many different characterizations of $\mathcal{Z}$ are given (cf. [17], [2] and [8]). It has been shown, already in [9], that $\mathcal{Z}$ is strongly self-absorbing. The original proof of Jiang and Su consists of essentially two main independent parts. One part is to show that $\mathcal{Z}$ has approximate inner half-flip ([9, Proposition 8.3]); recall that a C^*-algebra $\mathcal{D}$ has approximate inner half-flip if the first factor embedding $\mathrm{id} \otimes 1_{\mathcal{D}} : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ is approximately

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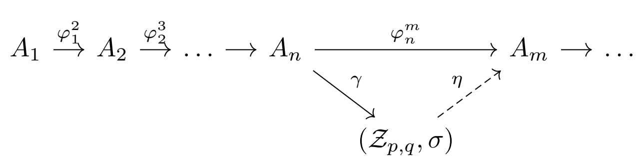
unitarily equivalent to the second factor embedding $1_{\mathcal{D}} \otimes \mathrm{id}_{\mathcal{D}} : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$. This part of the proof follows from basic properties of prime dimension-drop algebras and the unital $*$-embeddings used in the construction of $\mathcal{Z}$. The second part is to show that there is an asymptotically central sequence of inner automorphisms of $\mathcal{Z}$ ([9, Corollary 8.6]). This part heavily relies on the fact that every unital endomorphism of $\mathcal{Z}$ is approximately inner, the proof of which is rather difficult and involves heavy tools from the classification theory, such as KK-theory. Therefore it would be desirable to have a direct proof of the fact that $\mathcal{Z}$ is strongly self-absorbing, which does not depend (at least not so heavily) on the K-theoretic data and the classification tools. Towards this goal, very recently, a new and easier proof has been discovered by A. Schemaitat [18], which uses a characterization of $\mathcal{Z}$ as a stationary inductive limit of a generalized prime dimension-drop algebras and a trace-collapsing endomorphism, given in [17]. In this paper we give a different proof of the fact that $\mathcal{Z}$ is strongly self-absorbing, which follows a more general approach, and does not use any classification results nor any characterization of $\mathcal{Z}$. In fact, we show that the original sequences that are constructed in [9, Proposition 2.5] by Jiang and Su to define $\mathcal{Z}$, can be chosen so that the fact that they all have a same inductive limit (which is $\mathcal{Z}$) follows from a standard approximate intertwining arguments. In other words, the sequences can be chosen such that they are all "Fraïssé sequences" of the same category of prime dimension-drop algebras and hence they have a unique limit. Then, an application of the main result of this paper (Theorem 1.1) shows that the second part of the original proof can be replaced with some elementary arguments concerning the maps between prime dimension-drop algebras and their tensor products. Let us also point out that our proof of the fact that $\mathcal{Z}$ is strongly self-absorbing does not require any Fraïssé theory beyond back and forth arguments (standard approximate intertwining arguments), since in this particular case Fraïssé sequence are built by hand.

It has been shown in [3] and [15] that the category $\mathfrak{Z}$ whose objects are all prime dimension-drop algebras with fixed traces $(\mathcal{Z}_{p,q}, \sigma)$ and the morphisms are unital trace-preserving $*$-embeddings, is a "Fraïssé category" and its "Fraïssé limit" is the Jiang-Su algebra $(\mathcal{Z}, \nu)$ with its unique trace. This is a short and fancy way of saying that there are sequences of objects $A_n = (\mathcal{Z}_{p_n,q_n}, \tau_n)$ and morphisms $\varphi_n^{n+1} : A_n \to A_{n+1}$ in $\mathfrak{Z}$ such that $\mathcal{Z}$ is the inductive limit of the sequence (A_n, φ_n^{n+1}) which satisfies the following properties:

- $(*)$ for every $(\mathcal{Z}_{p,q}, \sigma)$ there exists a unital trace-preserving $*$-embedding $\psi : (\mathcal{Z}_{p,q}, \sigma) \to A_n$, for some n,
- $(**)$ for every n and unital trace-preserving $*$-embedding $\gamma : A_n \to (\mathcal{Z}_{p,q}, \sigma)$ and for every $\epsilon > 0$ and a finite subset F of A_n, there is a unital trace-preserving $*$-embedding $\eta : (\mathcal{Z}_{p,q}, \sigma) \to A_m$ for some $m > n$ such that $\|\eta \circ \gamma(a) - \varphi_n^m(a)\| < \epsilon$ for every a in $\mathcal{F}$, where $\varphi_n^m = \varphi_{m-1}^m \circ \cdots \circ \varphi_n^{n+1}$.



The existence of such a sequence (A_n, φ_n^m) follows from the a fundamental theorem in Fraïssé theory which, roughly speaking, states that any category which has the "joint embedding property", the "near amalgamation property" and satisfies a "separability assumption" (see Section 2 for the details) contains a sequence exhibiting the general behavior of (A_n, φ_n^m), i.e. satisfying the conditions $(*)$ and $(**)$ in the respective category. In a category $\mathfrak{K}$, whenever such sequences exist, a standard approximate intertwining argument guarantees that they all have isomorphic limits. Such sequences are usually called "Fraïssé sequences of $\mathfrak{K}$", revealing the origin of their mainstream study, and what usually makes them interesting is the universality and the (almost) homogeneity of their unique limit, the so called "Fraïssé limit of $\mathfrak{K}$".

In the category $\mathfrak{Z}$, the Fraïssé limit is automatically the Jiang-Su algebra, along with its unique trace. This is because it is not difficult to see that the Fraïssé limit of $\mathfrak{Z}$ (i.e. the inductive limit of any sequence satisfying $(*)$ and $(**)$) is a simple and monotracial C^*-algebra (cf. [15]) and the characterization of the Jiang-Su algebra as the unique unital simple and monotracial inductive limit of sequences of prime dimension-drop algebras (Theorem 6.2 of [9]) implies that it has to be the Jiang-Su algebra. However, one does not need Fraïssé theory nor the mentioned characterization of $\mathcal{Z}$ to prove the existence of a sequence satisfying $(*)$ and $(**)$, or the fact that the limit of such a sequence is $\mathcal{Z}$. Instead, consider the originally constructed sequences intended to define $\mathcal{Z}$ in [9, Proposition 2.5] (the proposition states that certain sequences of prime dimension-drop algebras exist whose limits are simple and monotracial. Let us pretend for a moment that we do not know whether all of these sequences have a same limit). We will show that at the recursive stages of constructing such a sequence one can easily make sure that, with appropriately chosen traces, the constructed sequence would satisfy $(*)$ and $(**)$ as well as the conditions of [9, Proposition 2.5] (i.e. the limit is still simple and monotracial). These sequences are described in Proposition 4.2. As we mentioned earlier, a standard approximate intertwining argument implies that sequences satisfying $(*)$ and $(**)$ have $*$-isomorphic limits. Continuing our pretend amnesia, denote the unique limit of any sequence as in Proposition 4.2 by $\mathcal{Z}$ and call it "the Jiang-Su algebra" (this notation and terminology coincides with the literature, since such sequences satisfy the conditions of the original construction).

Connections between strongly self-absorbing C^*-algebras and Fraïssé limits have been suspected in [3] (see Problem 7.1). In pursuit of such connections, in Section 3 we show that if $\mathfrak{C}$ is a category of unital separable C^*-algebras and unital $*$-homomorphisms, which happens to have a Fraïssé limit $\mathcal{D}$ with approximate inner half-flip, then $\mathcal{D}$ is strongly self-absorbing, if $\mathfrak{C}$ "dominates" a category $\mathfrak{K}$ which contains $\mathfrak{C}$ (both the objects and morphisms) and all the objects of the form $\mathcal{A}\otimes\mathcal{A}$, for $\mathcal{A}$ object of $\mathfrak{C}$, and a "modest but sufficient" set of new morphisms; we call such $\mathfrak{K}$ a $\otimes$-expansion of $\mathfrak{C}$ (Definition 3.4).

Theorem 1.1. *Suppose $\mathfrak{C}$ is a category of unital separable C^*-algebras and unital $*$-homomorphisms and $\mathfrak{C}$ dominates a $\otimes$-expansion of itself. If $\mathfrak{C}$ has a Fraïssé limit $\mathcal{D}$ which has approximate inner half-flip, then $\mathcal{D}$ is strongly self-absorbing.*

The notion of a category dominating a larger category (Definition 2.3) is introduced and used in the context of Fraïssé categories by W. Kubiś (cf. [13]). When a category $\mathfrak{C}$ dominates a larger category $\mathfrak{K}$, then any Fraïssé sequence of $\mathfrak{C}$ (whenever it exists) is also a Fraïssé sequence of $\mathfrak{K}$. This is very helpful, since often it is easier to show that the smaller category $\mathfrak{C}$ has a Fraïssé sequence and it dominates $\mathfrak{K}$ than showing directly that $\mathfrak{K}$ has a Fraïssé sequence. In the proof of Theorem 1.1 we use a weaker version of the notion of Fraïssé sequences, called the "weak Fraïssé sequences" (Definition 2.3). Weak Fraïssé sequences are studied in [14] and they also have isomorphic limits, called the weak Fraïssé limit. An outline of the proof of Theorem 1.1 is as follows. Suppose a category $\mathfrak{C}$ of unital separable C^*-algebras and unital $*$-homomorphisms dominates a category $\mathfrak{K}$ which is a $\otimes$-expansion of $\mathfrak{C}$, and $\mathcal{D}$ is the limit of a Fraïssé sequence $(\mathcal{D}_n, \varphi_n^m)$ of $\mathfrak{C}$. Then $(\mathcal{D}_n, \varphi_n^m)$ is also a Fraïssé sequence of $\mathfrak{K}$, since $\mathfrak{K}$ is dominated by $\mathfrak{C}$. The fact that $\mathfrak{K}$ is a $\otimes$-expansion of $\mathfrak{C}$ guarantees that the sequence $(\mathcal{D}_n \otimes \mathcal{D}_n, \varphi_n^m \otimes \varphi_n^m)$ is a $\mathfrak{K}$-sequence, whose limit is clearly $\mathcal{D} \otimes \mathcal{D}$. Then we will show that $(\mathcal{D}_n \otimes \mathcal{D}_n, \varphi_n^m \otimes \varphi_n^m)$ is a weak Fraïssé sequence of $\mathfrak{K}$. Since $(\mathcal{D}_n, \varphi_n^m)$ is also a weak Fraïssé sequence of $\mathfrak{K}$, by the uniqueness of the weak Fraïssé limit, $\mathcal{D}$ is self-absorbing. Then we use an easy version of the homogeneity property of the weak Fraïssé limit to show that $\mathcal{D}$ is strongly self-absorbing (the reason that we resort to "weak" Fraïssé sequences is because the author does not know whether $(\mathcal{D}_n \otimes \mathcal{D}_n, \varphi_n^m \otimes \varphi_n^m)$ is a Fraïssé sequence of $\mathfrak{K}$).

The last two sections are devoted to give a proof of the fact that $\mathcal{Z}$ is strongly self-absorbing, using Theorem 1.1. In Section 4 we directly show that $\mathcal{Z}$ is the Fraïssé limit of the category $\mathfrak{Z}$. Finally, in the last section we define a $\otimes$-expansion $\mathfrak{T}$ of $\mathfrak{Z}$ which is dominated by $\mathfrak{Z}$. Then Theorem 1.1 implies that $\mathcal{Z}$ is strongly self-absorbing, since $\mathcal{Z}$ has approximate inner half-flip.

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2. PRELIMINARIES

Suppose $\mathfrak{K}$ is a category. We refer to the objects and morphisms of $\mathfrak{K}$ by $\mathfrak{K}$-objects and $\mathfrak{K}$-morphisms, respectively, and sometimes write $\mathcal{A} \in \mathfrak{K}$ if $\mathcal{A}$ is a $\mathfrak{K}$-object. A $\mathfrak{K}$-sequence $(\mathcal{A}_n, \varphi_n^{n+1})$ is a sequence of $\mathfrak{K}$-objects $\mathcal{A}_n$ and $\mathfrak{K}$-morphisms $\varphi_n^{n+1} : \mathcal{A}_n \to \mathcal{A}_{n+1}$, for every n. We often denote such a sequence by $(\mathcal{A}_n, \varphi_n^m)$, where $\varphi_n^m : \mathcal{A}_n \to \mathcal{A}_m$ is defined by $\varphi_n^m = \varphi_{m-1}^m \circ \cdots \circ \varphi_n^{n+1}$ for every $m \geq n$ (let $\varphi_n^n = \mathrm{id}_{\mathcal{A}_n}$). By *limit* we mean the inductive limit (also called *colimit*). In a category $\mathfrak{K}$ of C^*-algebras or more generally Banach spaces, the limit of a $\mathfrak{K}$-sequence always exists in a possibly larger category and it is isometrically isomorphic to the completion of the union of the corresponding chain of spaces. If $\mathcal{A}$ is the limit of the $\mathfrak{K}$-sequence $(\mathcal{A}_n, \varphi_n^m)$, let $\varphi_n^\infty : \mathcal{A}_n \to \mathcal{A}$ denote the induced inductive limit morphism (which may not be $\mathfrak{K}$-morphisms).

Notations. Suppose $\mathcal{A}$ and $\mathcal{B}$ are normed structures. For $i = 0, 1$ and morphisms $\varphi_i : \mathcal{A} \to \mathcal{B}$ and $F \subseteq \mathcal{A}$ we sometimes write $\varphi_0 \approx_{\epsilon, F} \varphi_1$ if and only if of $\|\varphi_0(a) - \varphi_1(a)\| < \epsilon$, for every $a \in F$. We denote the image of the set F under a morphism φ by $\varphi[F]$. We also write $F \Subset \mathcal{A}$ if F is a finite subset of $\mathcal{A}$.

2.1. **Fraïssé sequences and the uniqueness.** For the rest of this section $\mathfrak{C}$ and $\mathfrak{K}$ are always categories of separable C^*-algebras and $*$-homomorphims (although the following notions can be defined for any metric category, cf. [12]). Let us start with the main definition.

Definition 2.1. A $\mathfrak{K}$-sequence $(\mathcal{D}_n, \varphi_n^m)$ is called a *Fraïssé sequence* of $\mathfrak{K}$ if it satisfies,

($\mathscr{U}$) for every $\mathcal{A} \in \mathfrak{K}$ there exists a $*$-homomorphism $\varphi : \mathcal{A} \to \mathcal{D}_n$ in $\mathfrak{K}$ (a $\mathfrak{K}$-morphism), for some $n \in \mathbb{N}$,

($\mathscr{A}$) for every $\epsilon > 0$, $n \in \mathbb{N}$, $F \Subset \mathcal{D}_n$ and for every $\mathfrak{K}$-morphism $\gamma : \mathcal{D}_n \to \mathcal{B}$ there are $m \geq n$ and a $\mathfrak{K}$-morphism $\eta : \mathcal{B} \to \mathcal{D}_m$ such that $\varphi_n^m \approx_{\epsilon,F} \eta \circ \gamma$.

A standard approximate intertwining argument (more generally known as "approximate back-and-forth", in model theory) shows that two Fraïssé sequences must have $*$-isomorphic limits.

Theorem 2.2. *(uniqueness) A Fraïssé limit of $\mathfrak{K}$ (if exists) is unique up to $*$-isomorphism. Moreover, if $(\mathcal{D}_n, \varphi_n^m)$ and $(\mathcal{E}_n, \psi_n^m)$ are both Fraïssé sequences of $\mathfrak{K}$ with limits $\mathcal{D}$ and $\mathcal{E}$, respectively, and $\theta : \mathcal{D}_k \to \mathcal{E}_\ell$ is a $\mathfrak{K}$-morphism, then for every $\epsilon > 0$ and $F \Subset \mathcal{D}_k$ there is a $*$-isomorphism $\Phi : \mathcal{D} \to \mathcal{E}$ such that the diagram*

$$\begin{array}{ccc} \mathcal{D}_k & \xrightarrow{\varphi_n^\infty} & \mathcal{D} \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \Phi} \\ \mathcal{E}_\ell & \xrightarrow{\psi_n^\infty} & \mathcal{E} \end{array}$$

ϵ-commutes on F, i.e. $\psi_n^\infty \circ \theta \approx_{\epsilon,F} \Phi \circ \varphi_n^\infty$.

Proof. Suppose $(\mathcal{D}_n, \varphi_n^m)$ and $(\mathcal{E}_n, \psi_n^m)$ are both Fraïssé sequences of $\mathfrak{K}$ with respective limits $\mathcal{D}$ and $\mathcal{E}$. Recursively construct sequences $(m_i)_{i\in\mathbb{N}}$, $(n_i)_{i\in\mathbb{N}}$ of natural numbers, finite sets $F_i' \Subset \mathcal{D}_{n_i}$, $G_i' \Subset \mathcal{E}_{m_i}$ and $\mathfrak{K}$-morphisms $\gamma_i : \mathcal{D}_{n_i} \to \mathcal{E}_{m_i}$ and $\eta_i : \mathcal{E}_{m_i} \to \mathcal{D}_{n_{i+1}}$ such that for every i,

- $\mathcal{D} = \overline{\bigcup_i \varphi_{n_i}^\infty[F_i']}$ and $\mathcal{E} = \overline{\bigcup_i \psi_{m_i}^\infty[G_i']}$,
- $\varphi_{n_i}^{n_{i+1}}[F_i'] \subseteq F_{i+1}'$ and $\psi_{m_i}^{m_{i+1}}[G_i'] \subseteq G_{i+1}'$,
- $\gamma_i[F_i'] \subseteq G_i'$ and $\eta_i[G_i'] \subseteq F_{i+1}'$,
- $\eta_i \circ \gamma_i \approx_{\frac{1}{2^{i-1}},F_i'} \varphi_{n_i}^{n_{i+1}}$ and $\gamma_{i+1} \circ \eta_i \approx_{\frac{1}{2^i},G_i'} \psi_{m_i}^{m_{i+1}}$.

This guarantees the existence of a $*$-isomorphism $\Phi : \mathcal{D} \to \mathcal{E}$ (cf. [16, Proposition 2.3.2]).

$$\begin{array}{ccccccccc} \mathcal{D}_{n_1} & \xrightarrow{\varphi_{n_1}^{n_2}} & \mathcal{D}_{n_2} & \xrightarrow{\varphi_{n_2}^{n_3}} & \mathcal{D}_{n_3} & \longrightarrow & \dots & & \mathcal{D} \\ & {\scriptstyle \gamma_1}\searrow \quad \nearrow{\scriptstyle \eta_1} & & {\scriptstyle \gamma_2}\searrow \quad \nearrow{\scriptstyle \eta_2} & & & & & \downarrow{\scriptstyle \Phi} \\ & \mathcal{E}_{m_1} & \xrightarrow{\psi_{m_1}^{m_2}} & \mathcal{E}_{m_2} & \xrightarrow{\psi_{m_2}^{m_3}} & \dots & & & \mathcal{E} \end{array}$$

To start, let $\epsilon_n = 2^{-n}$, fix $F_n \Subset \mathcal{D}_n$ and $G_n \Subset \mathcal{E}_n$ such that

$$\mathcal{D} = \overline{\bigcup_n \varphi_n^\infty[F_n]} \qquad \text{and} \qquad \mathcal{E} = \overline{\bigcup_n \psi_n^\infty[G_n]}.$$

Let $n_1 = 1$ and $F_1' = F_1$. Using the condition ($\mathscr{U}$) for the Fraïssé sequence $(\mathcal{E}_n, \psi_n^m)$, there are m_1 and $\gamma_1 : \mathcal{D}_{n_1} \to \mathcal{E}_{m_1}$ in $\mathfrak{K}$. By the condition ($\mathscr{A}$) for the Fraïssé sequence $(\mathcal{D}_n, \varphi_n^m)$, find $\eta_1 : \mathcal{E}_{m_1} \to \mathcal{D}_{n_2}$ in $\mathfrak{K}$, for some $n_2 > n_1$, such that

$$\eta_1 \circ \gamma_1 \approx_{\epsilon_1, F_1'} \varphi_{n_1}^{n_2}.$$

Let $G_1' = \gamma_1[F_1'] \cup G_{m_1}$. Similarly, using the condition ($\mathscr{A}$) for $(\mathcal{E}_n, \psi_n^m)$, find $m_2 > m_1$ and $\gamma_2 : \mathcal{D}_{n_2} \to \mathcal{E}_{m_2}$ such that

$$\gamma_2 \circ \eta_1 \approx_{\epsilon_2, G_1'} \psi_{m_1}^{m_2}.$$

Let $F_2' = \eta_1[G_1'] \cup F_{n_2} \cup \varphi_{n_1}^{n_2}[F_1']$. Again, find $\eta_2 : \mathcal{E}_{m_2} \to \mathcal{D}_{n_3}$, for some $n_3 > n_2$ such that

$$\eta_2 \circ \gamma_2 \approx_{\epsilon_3, F_2'} \varphi_{n_2}^{n_3}.$$

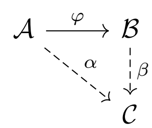
Let $G'_2 = \gamma_2[F'_2] \cup G_{m_2} \cup \psi_{m_1}^{m_2}[G'_1]$. Continuing this process gives us the required approximate intertwining. For the second statement, in the proof above let $n_1 = k$, $m_1 = \ell$, $F_1 = F$, $\gamma_1 = \theta$ and pick ϵ_i so that $\sum_{i=1}^{\infty} \epsilon_i < \epsilon$. □

For this reason, given a category with Fraïssé sequences we may refer to their unique limit as "the" Fraïssé limit. The following notion is introduced in [13] for abstract categories.

Definition 2.3. A category $\mathfrak{C}$ *dominates* $\mathfrak{K}$ if $\mathfrak{C}$ is a subcategory of $\mathfrak{K}$ and for any $\epsilon > 0$ satisfies the following conditions:

($\mathscr{C}$) for every $\mathcal{A} \in \mathfrak{K}$ there are $\mathcal{C} \in \mathfrak{C}$ and $\varphi : \mathcal{A} \to \mathcal{C}$ in $\mathfrak{K}$, i.e. $\mathfrak{C}$ is cofinal in $\mathfrak{K}$,

($\mathscr{D}$) for every $\varphi : \mathcal{A} \to \mathcal{B}$ in $\mathfrak{K}$ with $\mathcal{A} \in \mathfrak{C}$ and for every $F \Subset \mathcal{A}$, there exist $\beta : \mathcal{B} \to \mathcal{C}$ in $\mathfrak{K}$ with $\mathcal{C} \in \mathfrak{C}$ and a $\mathfrak{C}$-morphism $\alpha : \mathcal{A} \to \mathcal{C}$ such that $\alpha \approx_{\epsilon,F} \beta \circ \varphi$.



2.2. **The existence of Fraïssé sequences.** We will define the notion of Fraïssé categories (only) for categories of separable C^*-algebras and $*$-homomorphisms. These categories are guaranteed to contain Fraïssé sequences.

Definition 2.4. we say,

- $\mathfrak{C}$ has *the joint embedding property* (also sometimes called "directed", which is more appropriate for our setting) if for $\mathcal{A}, \mathcal{B} \in \mathfrak{C}$ there are $\mathcal{C} \in \mathfrak{C}$ and $\mathfrak{C}$-morphisms $\varphi : \mathcal{A} \to \mathcal{C}$ and $\psi : \mathcal{B} \to \mathcal{C}$.
- $\mathfrak{C}$ has *the near amalgamation property* if for every $\epsilon > 0$, $\mathcal{A} \in \mathfrak{C}$, $F \Subset \mathcal{A}$ and $\mathfrak{C}$-morphisms $\varphi : \mathcal{A} \to \mathcal{B}$, $\psi : \mathcal{A} \to \mathcal{C}$ there are $\mathcal{D} \in \mathfrak{C}$ and $\mathfrak{C}$-morphisms $\varphi' : \mathcal{B} \to \mathcal{D}$ and $\psi' : \mathcal{C} \to \mathcal{D}$ such that $\|\varphi' \circ \varphi(a), \psi' \circ \psi(a)\| < \epsilon$ for every $a \in F$
- $\mathfrak{C}$ is *separable* if $\mathfrak{C}$ is dominated by a countable subcategory (a subcategory with countably many objects and morphisms).

A category is called a *Fraïssé category* if it has the joint embedding property, the near amalgamation property and it is separable.

It is well known that Fraïssé categories have Fraïssé sequences and their Fraïssé limits are unique, universal and almost homogeneous in the respective categories. To see a proof of the next theorem refer to [13]. Although in the statements and definitions of [13] there are no finite sets F and ϵ is always 0, since the objects of our categories are separable, a routine adjustment of each proof immediately implies the corresponding statement. In fact, the proofs of universality and almost homogeneity follow an approximate intertwining argument.

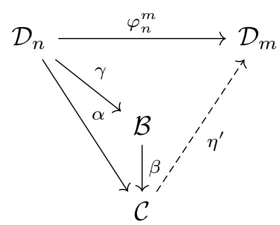
Theorem 2.5. *Assume $\mathfrak{C}$ is a Fraïssé category. Then $\mathfrak{C}$ has a Fraïssé sequence. If $(\mathcal{D}_n, \varphi_n^m)$ is a Fraïssé sequence of $\mathfrak{C}$ and $\mathcal{D} = \varinjlim(\mathcal{D}_n, \varphi_n^m)$ is the unique Fraïssé limit, then*

- *$\mathcal{D}$ is universal: if $\mathcal{B}$ is the limit of a $\mathfrak{C}$-sequence, then there is a $*$-homomorphism $\varphi : \mathcal{B} \to \mathcal{D}$, and*
- *$\mathcal{D}$ is almost homogeneity: for every $\epsilon > 0$, $\mathfrak{C}$-objects $\mathcal{A}, \mathcal{B} \subseteq \mathcal{D}$, every $F \Subset \mathcal{A}$ and $*$-isomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ in $\mathfrak{C}$, there is a $*$-automorphism $\theta : \mathcal{D} \to \mathcal{D}$ such that $\theta \approx_{\epsilon,F} \varphi$.*

If every $\mathfrak{C}$-morphism is a $*$-embedding (injective $*$-homomorphism), then $\mathcal{D}$ is universal in the sense that the limit of any $\mathfrak{C}$-sequence can be $*$-embedded into $\mathcal{D}$.

Proposition 2.6. *Suppose a category $\mathfrak{C}$ dominates $\mathfrak{K}$ and $(\mathcal{D}_n, \varphi_n^m)$ is a Fraïssé sequence of $\mathfrak{C}$, then $(\mathcal{D}_n, \varphi_n^m)$ is also a Fraïssé sequence of $\mathfrak{K}$.*

Proof. The fact that $(\mathcal{D}_n, \varphi_n^m)$ satisfies the condition $(\mathscr{U})$ in $\mathfrak{K}$ follows from $(\mathscr{C})$ and the fact that $(\mathcal{D}_n, \varphi_n^m)$ satisfies $(\mathscr{U})$ in $\mathfrak{C}$. For the condition $(\mathscr{A})$, suppose $\epsilon > 0$, $F \Subset \mathcal{D}_n$ and a $\mathfrak{K}$-morphism $\gamma : \mathcal{D}_n \to \mathcal{B}$ are given. By $(\mathscr{D})$ find $\mathcal{C} \in \mathfrak{C}$, a $\mathfrak{K}$-morphism $\beta : \mathcal{B} \to \mathcal{C}$ and a $\mathfrak{C}$-morphism $\alpha : \mathcal{D}_n \to \mathcal{C}$ such that $\beta \circ \gamma \approx_{\epsilon/2, F} \alpha$. Since $(\mathcal{D}_n, \varphi_n^m)$ is a Fraïssé sequence of $\mathfrak{C}$, there is a $*$-homomorphism $\eta' : \mathcal{C} \to \mathcal{D}_m$ in $\mathfrak{C}$, for some $m > n$, such that $\varphi_n^m \approx_{\epsilon/2, F} \eta' \circ \alpha$.



Then we have $\varphi_n^m \approx_{\epsilon, F} \eta \circ \gamma$, where η is the $\mathfrak{K}$-morphism defined by $\eta = \eta' \circ \beta$. Therefore $(\mathcal{D}_n, \varphi_n^m)$ a Fraïssé sequence of $\mathfrak{K}$. □

Remark 2.7. The Fraïssé theory was introduced by R. Fraïssé [5] to study the correspondence between countable first-order homogeneous structures and properties of the classes of their finitely generated substructures. It was extended to metric structures by Ben Yaacov [1] in continuous model theory, and by Kubiś [12] in the framework of (metric-enriched) categories. The category theoretical approach was recently used in [6] to study the AF-algebras that are Fraïssé limits of categories of finite-dimensional C^*-algebras.

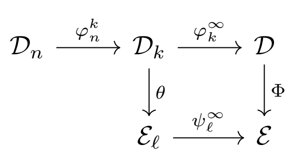
2.3. **Weak Fraïssé sequences and the uniqueness.** A $\mathfrak{K}$-sequence $(\mathcal{D}_n, \varphi_n^m)$ is called a *weak Fraïssé sequence* of $\mathfrak{K}$ if it satisfies,

- $(\mathscr{U})$ for every $\mathcal{A} \in \mathfrak{K}$ there exists a $\mathfrak{K}$-morphism $\varphi : \mathcal{A} \to \mathcal{D}_n$, for some n,
- $(\mathscr{W}\mathscr{A})$ for every $\epsilon > 0$, $n \in \mathbb{N}$, $F \Subset \mathcal{D}_n$ there is a natural number $m \geq n$ such that for every $\gamma : \mathcal{D}_m \to \mathcal{B}$ in $\mathfrak{K}$, there are $k > m$ and $\eta : \mathcal{B} \to \mathcal{D}_k$ in $\mathfrak{K}$ such that $\varphi_n^k \approx_{\epsilon, F} \eta \circ \gamma \circ \varphi_n^m$.

The limit of a weak Fraïssé sequence of $\mathfrak{K}$ is called *weak Fraïssé limit* of $\mathfrak{K}$. The following is proved in [14, Lemma 4.9] for abstract categories.

Theorem 2.8. *(uniqueness) A weak Fraïssé limit of $\mathfrak{K}$, whenever it exists, is unique up to $*$-isomorphism.*

Suppose $(\mathcal{D}_i, \varphi_i^j)$ and $(\mathcal{E}_i, \psi_i^j)$ are both weak Fraïssé sequences of $\mathfrak{K}$ with respective limits $\mathcal{D}$ and $\mathcal{E}$. For every $\epsilon > 0$, $n \in \mathbb{N}$ and $F \Subset \mathcal{D}_n$ there is $k \geq n$ such that for every $\theta : \mathcal{D}_k \to \mathcal{E}_\ell$ in $\mathfrak{K}$ there is a $*$-isomorphism $\Phi : \mathcal{D} \to \mathcal{E}$ such that the diagram



ϵ-commutes on F, i.e. $\psi_\ell^\infty \circ \theta \circ \varphi_n^k \approx_{\epsilon, F} \Phi \circ \varphi_n^\infty$.

Proof. Suppose $(\mathcal{D}_i, \varphi_i^j)$ and $(\mathcal{E}_i, \psi_i^j)$ are weak Fraïssé sequences of $\mathfrak{K}$, with respective limits $\mathcal{D}$ and $\mathcal{E}$. First, to show the uniqueness, we recursively find sequences $(n_i)_{i\in\mathbb{N}}$, $(k_i)_{i\in\mathbb{N}}$, $(m_i)_{i\in\mathbb{N}}$, $(\ell_i)_{i\in\mathbb{N}}$ of natural numbers, finite sets $F_i' \Subset \mathcal{D}_{k_i}$, $G_i' \Subset \mathcal{E}_{\ell_i}$ and $\mathfrak{K}$-morphisms $\gamma_i : \mathcal{D}_{k_i} \to \mathcal{E}_{m_i}$ and $\eta_i : \mathcal{E}_{\ell_i} \to \mathcal{D}_{n_{i+1}}$ such that for every i we have (see Diagram (2.1)),

- $n_1 = 1$, $n_i \leq k_i < n_{i+1}$ and $m_i \leq \ell_i < m_{i+1}$,
- $\mathcal{D} = \overline{\bigcup_i \varphi_{k_i}^{\infty}[F_i']}$ and $\mathcal{E} = \overline{\bigcup_i \psi_{\ell_i}^{\infty}[G_i']}$,
- $\varphi_{k_i}^{k_{i+1}}[F_i'] \subseteq F_{i+1}'$ and $\psi_{\ell_i}^{\ell_{i+1}}[G_i'] \subseteq G_{i+1}'$,
- $\psi_{m_i}^{\ell_i} \circ \gamma_i[F_i'] \subseteq G_i'$ and $\varphi_{n_{i+1}}^{k_{i+1}} \circ \eta_i[G_i'] \subseteq F_{i+1}'$,
- $\eta_i \circ \psi_{m_i}^{\ell_i} \circ \gamma_i \approx_{\frac{1}{2^i}, F_i'} \varphi_{k_i}^{n_{i+1}}$,
- $\gamma_{i+1} \circ \varphi_{n_{i+1}}^{k_{i+1}} \circ \eta_i \approx_{\frac{1}{2^i}, G_i'} \psi_{\ell_i}^{m_{i+1}}$.

$$
\begin{array}{ccccccccccccc}
\mathcal{D}_1 & \xrightarrow{\varphi_1^{k_1}} & \mathcal{D}_{k_1} & \xrightarrow{\varphi_{k_1}^{n_2}} & \mathcal{D}_{n_2} & \xrightarrow{\varphi_{n_2}^{k_2}} & \mathcal{D}_{k_2} & \xrightarrow{\varphi_{k_2}^{n_3}} & \mathcal{D}_{n_3} & \xrightarrow{\varphi_{n_3}^{k_3}} & \mathcal{D}_{k_3} & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle \gamma_1} & & {\scriptstyle \eta_1}\uparrow & & \downarrow{\scriptstyle \gamma_2} & & {\scriptstyle \eta_2}\uparrow & & \downarrow{\scriptstyle \gamma_3} & & \\
\mathcal{E}_1 & \xrightarrow{\psi_1^{m_1}} & \mathcal{E}_{m_1} & \xrightarrow{\psi_{m_1}^{\ell_1}} & \mathcal{E}_{\ell_1} & \xrightarrow{\psi_{\ell_1}^{m_2}} & \mathcal{E}_{m_2} & \xrightarrow{\psi_{m_2}^{\ell_2}} & \mathcal{E}_{\ell_2} & \xrightarrow{\psi_{\ell_2}^{m_3}} & \mathcal{E}_{m_3} & \longrightarrow & \dots
\end{array}
\tag{2.1}
$$

Then the $\mathfrak{K}$-morphisms $\alpha_i : \mathcal{D}_{k_i} \to \mathcal{E}_{\ell_i}$ and $\beta_i : \mathcal{E}_{\ell_i} \to \mathcal{D}_{k_{i+1}}$ given by $\alpha_i = \psi_{m_i}^{\ell_i} \circ \gamma_i$ and $\beta_i = \varphi_{n_{i+1}}^{k_{i+1}} \circ \eta_i$ produce an approximate intertwining between the two sequences, which guarantees the existence of a $*$-isomorphism $\Phi : \mathcal{D} \to \mathcal{E}$ (cf. [16, Proposition 2.3.2]).

To start, let $\epsilon_n = 2^{-n}$ and fix sequences $F_n \Subset \mathcal{D}_n$ and $G_n \Subset \mathcal{E}_n$ such that

$$\varphi_n^{n+1}[F_n] \subseteq F_{n+1}, \qquad \psi_n^{n+1}[G_n] \subseteq G_{n+1},$$

and

$$\mathcal{D} = \overline{\bigcup_n \varphi_n^{\infty}[F_n]}, \qquad \mathcal{E} = \overline{\bigcup_n \psi_n^{\infty}[G_n]}.$$

Let $n_1 = 1$. Since $(\mathcal{D}_i, \varphi_i^j)$ is a weak Fraïssé sequence of $\mathfrak{K}$, we can find $k_1 \geq 1$ such that

(1) the condition $(\mathscr{W}\mathscr{A})$ holds for ϵ_1, n_1 and F_1 at k_1: that is, for every $\gamma : \mathcal{D}_{k_1} \to \mathcal{B}$ in $\mathfrak{K}$, there are $k > k_1$ and $\eta : \mathcal{B} \to \mathcal{D}_k$ in $\mathfrak{K}$ such that $\eta \circ \gamma \circ \varphi_1^{k_1} \approx_{\epsilon_1, F_1} \varphi_1^k$.

Let $F_1' = \varphi_1^{k_1}[F_1]$. Using the condition $(\mathscr{U})$ for the weak Fraïssé sequence $(\mathcal{E}_i, \psi_i^j)$, there is a natural number $m_1 \geq 1$ and a $\mathfrak{K}$-morphism $\gamma_1 : \mathcal{D}_{k_1} \to \mathcal{E}_{m_1}$. Again, since $(\mathcal{E}_i, \psi_i^j)$ is a weak Fraïssé sequence, find $\ell_1 \geq m_1$ such that for the sequence $(\mathcal{E}_i, \psi_i^j)$

(2) the condition $(\mathscr{W}\mathscr{A})$ holds for ϵ_1, m_1 and $G_{m_1} \cup \gamma_1[F_1']$ at ℓ_1.

Let $G_1' = \psi_{m_1}^{\ell_1}[G_{m_1} \cup \gamma_1[F_1']]$. By (1) there are $n_2 > k_1$ and a $\mathfrak{K}$-morphism $\eta_1 : \mathcal{E}_{\ell_1} \to \mathcal{D}_{n_2}$ such that (notice the choice of F_1')

$$\eta_1 \circ \psi_{m_1}^{\ell_1} \circ \gamma_1 \approx_{\epsilon_1, F_1'} \varphi_{k_1}^{n_2}.$$

Find $k_2 \geq n_2$ such that for the sequence $(\mathcal{D}_i, \varphi_i^j)$

(3) the condition $(\mathscr{W}\mathscr{A})$ holds for ϵ_2, n_2 and $F_{n_2} \cup \eta_1[G_1']$ at k_2.

Let $F_2' = \varphi_{n_2}^{k_2}[F_{n_2} \cup \eta_1[G_1']]$. By (2) there are $m_2 > \ell_1$ and a $\mathfrak{K}$-morphism $\gamma_2 : \mathcal{D}_{k_2} \to \mathcal{E}_{m_2}$ such that

$$\gamma_2 \circ \varphi_{n_2}^{k_2} \circ \eta_1 \approx_{\epsilon_1, G_1'} \psi_{\ell_1}^{m_2}.$$

Find $\ell_2 \leq m_2$ such that for the sequence $(\mathcal{E}_i, \psi_i^j)$

(4) the condition $(\mathscr{W}\mathscr{A})$ holds for ϵ_2, m_2 and $G_{m_2} \cup \psi_{\ell_1}^{m_2}[G_1'] \cup \gamma_2[F_2']$ at k_2.

Let $G_2' = \psi_{m_2}^{\ell_2}[G_{m_2} \cup \gamma_2[F_2']] \cup \psi_{\ell_1}^{\ell_2}[G_1']$. By (3) there are $n_3 > k_2$ and a $\mathfrak{K}$-morphism $\eta_2 : \mathcal{E}_{\ell_2} \to \mathcal{D}_{n_3}$ such that

$$\eta_2 \circ \psi_{m_2}^{\ell_2} \circ \gamma_2 \approx_{\epsilon_2, F_2'} \varphi_{k_2}^{n_3}.$$

Continuing this process produces the required approximate intertwining according to Diagram (2.1).

For the second statement, in the proof above start with $n_1 = n$, $F_1 = F$ and let $k = k_1$, $\gamma_1 = \theta$ and pick ϵ_i so that $\sum_{i=1}^{\infty} \epsilon_i < \epsilon$. □

2.4. **The existence of weak Fraïssé sequences.** Clearly in a category $\mathfrak{K}$ a Fraïssé sequence is also weak Fraïssé and therefore Fraïssé categories contain weak Fraïssé sequences. However, a weakening of the notions of the near amalgamation property and separability of a category is sufficient (and necessary) to guarantee the existence of weak Fraïssé sequences. These categories are called "weak Fraïssé categories" and they are studied in [14]. As with Fraïssé sequences, if a category $\mathfrak{C}$ dominates a larger category $\mathfrak{K}$, then a weak Fraïssé sequence of $\mathfrak{C}$ (if it exists) is also a weak Fraïssé sequence of $\mathfrak{K}$. This remains true even if $\mathfrak{C}$ "weakly dominates" $\mathfrak{K}$ (Proposition 2.11).

Definition 2.9. A category $\mathfrak{C}$ *weakly dominates* $\mathfrak{K}$ if $\mathfrak{C}$ is a subcategory of $\mathfrak{K}$ and for any given $\epsilon > 0$, we have

($\mathscr{C}$) for every $\mathcal{A} \in \mathfrak{K}$ there are $\mathcal{C} \in \mathfrak{C}$ and $\varphi : \mathcal{A} \to \mathcal{C}$ in $\mathfrak{K}$, i.e. $\mathfrak{C}$ is cofinal in $\mathfrak{K}$,

($\mathscr{W}\mathscr{D}$) for every $\mathcal{A} \in \mathfrak{C}$ and for every $F \Subset \mathcal{A}$, there exist $\zeta : \mathcal{A} \to \mathcal{A}'$ in $\mathfrak{C}$ such that for every $\varphi : \mathcal{A}' \to \mathcal{B}$ in $\mathfrak{K}$, there are $\beta : \mathcal{B} \to \mathcal{C}$ in $\mathfrak{K}$ with $\mathcal{C} \in \mathfrak{C}$ and $\alpha : \mathcal{A} \to \mathcal{C}$ in $\mathfrak{C}$ such that $\alpha \approx_{\epsilon, F} \beta \circ \varphi$.

The following notion of "weak amalgamation property" has been first identified by Ivanov [7] and later independently by Kechris and Rosendal [10] in model theory. A category $\mathfrak{C}$ is called weakly Fraïssé if

- it has the joint embedding property,
- $\mathfrak{C}$ has the *weak near amalgamation property*: for every $\epsilon > 0$, $\mathcal{A} \in \mathfrak{K}$ and $F \Subset \mathcal{A}$ there is $\zeta : \mathcal{A} \to \mathcal{A}'$ such that for any $\mathfrak{K}$-morphisms $\varphi : \mathcal{A}' \to \mathcal{B}$, $\psi : \mathcal{A}' \to \mathcal{C}$ there are $\mathcal{D} \in \mathfrak{K}$ and $\mathfrak{K}$-morphisms $\varphi' : \mathcal{B} \to \mathcal{D}$ and $\psi' : \mathcal{C} \to \mathcal{D}$ such that $\|\varphi' \circ \varphi \circ \zeta(a), \psi' \circ \psi \circ \zeta(a)\| < \epsilon$, for every $a \in F$.
- $\mathfrak{C}$ is *weakly separable*: it is weakly dominated by a countable subcategory.

The proof of the following theorem is a straightforward metric adaptation of Theorem 4.6 of [14] (by adding 2^{-n} and finite subsets to the proof).

Theorem 2.10. *A category has a weak Fraïssé sequence if and only if it is weakly Fraïssé.*

The abstract category theoretic version of the next proposition is Lemma 4.3 of [14] and the proof can again be adjusted to work for the metric case; similar to the proof of Proposition 2.6.

Proposition 2.11. *Suppose a category $\mathfrak{C}$ dominates $\mathfrak{K}$ and $(\mathcal{D}_n, \varphi_n^m)$ is a weak Fraïssé sequence of $\mathfrak{C}$, then $(\mathcal{D}_n, \varphi_n^m)$ is also a weak Fraïssé sequence of $\mathfrak{K}$.*

3. Strongly self-absorbing C^*-algebras and Fraïssé limits

For C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ we let $\mathcal{A} \otimes \mathcal{B}$ denote their "minimal" or "spacial" tensor product. A C^*-algebra is called "self-absorbing" if it is $*$-isomorphic to its (minimal) tensor product with itself. Recall that $*$-homomorphisms $\varphi_i : \mathcal{A} \to \mathcal{B}$ $(i = 1, 2)$ between separable C^*-algebras $\mathcal{A}, \mathcal{B}$ are *approximately unitarily equivalent* if there is a sequence $(u_n)_{n\in\mathbb{N}}$ of unitaries in the multiplier algebra of $\mathcal{B}$ such that

$$\lim_{n\to\infty} \|u_n^* \varphi_1(a) u_n - \varphi_2(a)\| = 0,$$

for every $a \in \mathcal{A}$.

Definition 3.1. Let $\mathcal{D}$ be a separable unital C^*-algebra.

- $\mathcal{D}$ has *approximate inner half-flip* if the maps $\mathrm{id}_{\mathcal{D}} \otimes 1_{\mathcal{D}}$ and $1_{\mathcal{D}} \otimes \mathrm{id}_{\mathcal{D}}$ from $\mathcal{D}$ to $\mathcal{D} \otimes \mathcal{D}$ are approximately unitarily equivalent.
- $\mathcal{D}$ is *strongly self-absorbing* if $\mathcal{D} \ncong \mathbb{C}$ and there is a $*$-isomorphism $\varphi : \mathcal{D} \to \mathcal{D}\otimes\mathcal{D}$ which is approximately unitary equivalent to $\mathrm{id}_{\mathcal{D}} \otimes 1_{\mathcal{D}}$.

Remark 3.2. Any strongly self-absorbing C^*-algebra $\mathcal{D}$ has *approximate inner flip* [19], which is a stronger notion than approximate inner half-flip and states that the flip $*$-automorphism $\sigma_{\mathcal{D}} : \mathcal{D}\otimes\mathcal{D} \to \mathcal{D}\otimes\mathcal{D}$, which sends $a\otimes b$ to $b\otimes a$, is approximately unitarily equivalent to the identity map on $\mathcal{D}\otimes\mathcal{D}$. The notion of approximate inner flip was studied for C^*-algebras by Effros and Rosenberg [4], inspired by a profound result of Connes about hyperfinite II_1 factor. Any C^*-algebra with approximate inner (half-)flip is automatically simple and nuclear (cf. [4] and [11]).

We need the following elementary lemma.

Lemma 3.3. *Suppose $(\mathcal{D}_i, \varphi_i^j)$ is a sequence of unital C^*-algebras and unital $*$-embeddings and $\mathcal{D} = \varinjlim(\mathcal{D}_i, \varphi_i^j)$. Then $\mathcal{D}$ has approximate inner half-flip if and only if for every $\epsilon > 0$, $n \in \mathbb{N}$, $F \Subset \mathcal{D}_n$ there is a unitary $u \in \mathcal{D}_m \otimes \mathcal{D}_m$ for some $m \geq n$ such that*

$$\|\varphi_n^m(a) \otimes 1_{\mathcal{D}_m} - u^*(1_{\mathcal{D}_m} \otimes \varphi_n^m(a))u\| < \epsilon$$

for every $a \in F$.

Proof. The inverse direction is trivial. For the forward direction, suppose $\epsilon > 0$, $n \in \mathbb{N}$ and $F \Subset \mathcal{D}_n$ are given. We can assume $\epsilon < 1$ and $\mathcal{F}$ is contained in the unit ball of $\mathcal{D}_n$. Find a unitary $v \in \mathcal{D} \otimes \mathcal{D}$ such that

$$\|\varphi_n^\infty(a) \otimes 1_{\mathcal{D}} - v^*(1_{\mathcal{D}} \otimes \varphi_n^\infty(a))v\| < \epsilon/5$$

for every $a \in F$. Note that $\mathcal{D}\otimes\mathcal{D}$ is the limit of the sequence $(\mathcal{D}_i\otimes\mathcal{D}_i, \varphi_i^j\otimes\varphi_i^j)$. For some $m \geq n$ there is an element b in $\mathcal{D}_m\otimes\mathcal{D}_m$ such that $\|v - \varphi_m^\infty\otimes\varphi_m^\infty(b)\| < \epsilon/10$. Then we have $\|bb^* - 1_{\mathcal{D}_m\otimes\mathcal{D}_m}\| < \epsilon/5$ and $\|b^*b - 1_{\mathcal{D}_m\otimes\mathcal{D}_m}\| < \epsilon/5$. It is easy to check that there is a unitary $u \in \mathcal{D}_m \otimes \mathcal{D}_m$ such that $\|u - b\| < \epsilon/5$. For every $a \in F$ we have

$$\begin{aligned}
&\|\varphi_n^m(a) \otimes 1_{\mathcal{D}_m} - u^*(1_{\mathcal{D}_m} \otimes \varphi_n^m(a))u\| \\
&\qquad \leq \|\varphi_n^m(a) \otimes 1_{\mathcal{D}_m} - b^*(1_{\mathcal{D}_m} \otimes \varphi_n^m(a))b\| + 2\epsilon/5 \\
&\qquad = \|\varphi_n^\infty(a) \otimes 1_{\mathcal{D}} - \varphi_m^\infty \otimes \varphi_m^\infty(b)^*(1_{\mathcal{D}} \otimes \varphi_n^\infty(a))\varphi_m^\infty \otimes \varphi_m^\infty(b)\| + 2\epsilon/5 \\
&\qquad \leq \|\varphi_n^\infty(a) \otimes 1_{\mathcal{D}} - v^*(1_{\mathcal{D}} \otimes \varphi_n^\infty(a))v\| + 4\epsilon/5 < \epsilon.
\end{aligned}$$

∎

In this section $\mathfrak{C}$ and $\mathfrak{K}$ are always categories of unital separable C^*-algebras and unital $*$-homomorphisms.

Definition 3.4. We say that $\mathfrak{K}$ is a $\otimes$-*expansion* of $\mathfrak{C}$ if

- $\mathfrak{C}$ is a subcategory of $\mathfrak{K}$,
- $\mathcal{A} \otimes \mathcal{A} \in \mathfrak{K}$ for every $\mathcal{A} \in \mathfrak{C}$,
- if $\varphi : \mathcal{A} \to \mathcal{B}$ is in $\mathfrak{C}$ then $\varphi \otimes \varphi : \mathcal{A} \otimes \mathcal{A} \to \mathcal{B} \otimes \mathcal{B}$ belongs to $\mathfrak{K}$,
- for every $\mathcal{A} \in \mathfrak{C}$ the maps $\mathrm{id}_{\mathcal{A}} \otimes 1_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and $1_{\mathcal{A}} \otimes \mathrm{id}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ belong to $\mathfrak{K}$,
- for every $\mathcal{B} \in \mathfrak{K}$ and a unitary $u \in \mathcal{B}$, the inner $*$-automorphism $\mathrm{Ad}_u : \mathcal{B} \to \mathcal{B}$ belongs to $\mathfrak{K}$.

Now we are ready to prove the main result of this section. The key tool in the proof is the uniqueness of the weak Fraïssé limit, whenever it exists in a category (Theorem 2.8).

Theorem 3.5. *Suppose $\mathfrak{C}$ is a category of unital separable C^*-algebras and unital $*$-homomorphisms and $\mathfrak{C}$ (weakly) dominates a $\otimes$-expansion of itself. If $\mathfrak{C}$ has a (weak) Fraïssé limit $\mathcal{D}$ with approximate inner half-flip, then $\mathcal{D}$ is strongly self-absorbing.*

Proof. Suppose $\mathcal{D} = \varinjlim_i (\mathcal{D}_i, \varphi_i^j)$, where $(\mathcal{D}_i, \varphi_i^j)$ is a (weak) Fraïssé sequence of $\mathfrak{C}$ and assume $\mathfrak{K}$ is a $\otimes$-expansion of $\mathfrak{C}$ which is (weakly) dominated by $\mathfrak{C}$. By Proposition 2.6 (Proposition 2.11) the C^*-algebra $(\mathcal{D}_i, \varphi_i^j)$ is a (weak) Fraïssé sequence of the category $\mathfrak{K}$. The sequence $(\mathcal{D}_i \otimes \mathcal{D}_i, \varphi_i^j \otimes \varphi_i^j)$ is a $\mathfrak{K}$-sequence and its limit is $\mathcal{D} \otimes \mathcal{D}$. We will show that $(\mathcal{D}_i \otimes \mathcal{D}_i, \varphi_i^j \otimes \varphi_i^j)$ is also a weak Fraïssé sequence of $\mathfrak{K}$. Hence, the uniqueness of weak Fraïssé limits (Theorem 2.8) implies that $\mathcal{D}$ is self-absorbing. Then we use the second statement of Theorem 2.8 in order to show that $\mathcal{D}$ is strongly self-absorbing.

To show that $(\mathcal{D}_i \otimes \mathcal{D}_i, \varphi_i^j \otimes \varphi_i^j)$ is a weak Fraïssé sequence of $\mathfrak{K}$, note that the condition $(\mathscr{U})$ of 2.3 is satisfied; for any $\mathcal{B} \in \mathfrak{K}$, by the condition $(\mathscr{C})$, there is $\mathcal{A} \in \mathfrak{C}$ and $\varphi : \mathcal{B} \to \mathcal{A}$ in $\mathfrak{K}$. Since $\mathcal{D}$ satisfies $(\mathscr{U})$ in the category $\mathfrak{C}$, for some m there is a $\mathfrak{K}$-morphism $\psi : \mathcal{A} \to \mathcal{D}_m$. Then the map $(\mathrm{id}_{\mathcal{D}_m} \otimes 1_{\mathcal{D}_m}) \circ \psi \circ \varphi$ is a $\mathfrak{K}$-morphism from $\mathcal{B}$ to $\mathcal{D}_m \otimes \mathcal{D}_m$.

To see that $(\mathcal{D}_i \otimes \mathcal{D}_i, \varphi_i^j \otimes \varphi_i^j)$ satisfies $(\mathscr{W}\mathscr{A})$, suppose $\epsilon > 0$, $n \in \mathbb{N}$ and $F \Subset \mathcal{D}_n \otimes \mathcal{D}_n$ are given. Let G be a finite subset of $\mathcal{D}_n$ such that $F \subseteq_{\epsilon/2} G \otimes G$, where

$$G \otimes G := \{\sum_{i=1}^{k} a_i \otimes b_i : a_i, b_i \in G\}.$$

Since $\mathcal{D}$ has approximate inner half-flip, by Lemma 3.3 we can find $m \geq n$ and a unitary $u \in \mathcal{D}_m \otimes \mathcal{D}_m$ such that

$$\|u(\varphi_n^m(a) \otimes 1)u^* - 1 \otimes \varphi_n^m(a)\| < \epsilon/2 \tag{3.1}$$

for every $a \in G$. By increasing m, if necessary, we can assume that for $(\mathcal{D}_i, \varphi_i^j)$ the condition $(\mathscr{W}\mathscr{A})$ holds in the category $\mathfrak{K}$ for $\epsilon/2$, n and G at m (recall that $(\mathcal{D}_i, \varphi_i^j)$ is also the (weak) Fraïssé sequence of $\mathfrak{K}$), that is

(1) for every $\alpha : \mathcal{D}_m \to \mathcal{B}$ in $\mathfrak{K}$ there exist $k > m$ and $\beta : \mathcal{B} \to \mathcal{D}_k$ in $\mathfrak{K}$ such that $\|\beta \circ \alpha \circ \varphi_n^m(a) - \varphi_n^k(a)\| < \epsilon/2$ for every $a \in G$.

We claim that in the category $\mathfrak{K}$ the sequence $(\mathcal{D}_i \otimes \mathcal{D}_i, \varphi_i^j \otimes \varphi_i^j)$ satisfies the condition $(\mathscr{W}\mathscr{A})$ for ϵ, n and F at m. Without loss of generality and after possibly dividing ϵ by a constant, it is enough to show that

(∗) for any $\mathfrak{K}$-morphism $\gamma : \mathcal{D}_m \otimes \mathcal{D}_m \to \mathcal{B}$ there are $k > m$ and $\eta : \mathcal{B} \to \mathcal{D}_k \otimes \mathcal{D}_k$ in $\mathfrak{K}$ such that $\|\eta \circ \gamma \circ (\varphi_n^m \otimes \varphi_n^m)(x) - \varphi_n^k \otimes \varphi_n^k(x)\| < \epsilon/2$ where x is of the form $a \otimes 1$ or $1 \otimes a$ for $a \in G$.

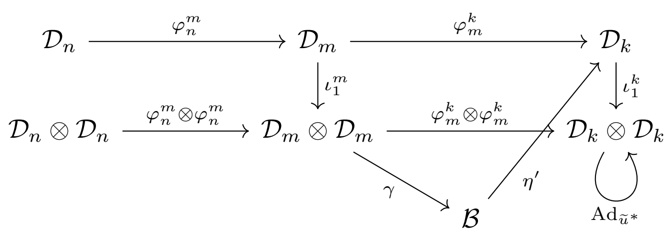
Let $\iota_i^m : \mathcal{D}_m \to \mathcal{D}_m \otimes \mathcal{D}_m$ $(i = 1, 2)$ be the unital $*$-embeddings defined by

$$\iota_1^m(a) = a \otimes 1 \qquad \text{and} \qquad \iota_2^m(b) = 1 \otimes b.$$

The map $\gamma \circ \iota_1^m : \mathcal{D}_m \to \mathcal{B}$ is a $\mathfrak{K}$-morphism, hence by (1) there are $k > m$ and a unital $*$-homomorphism $\eta' : \mathcal{B} \to \mathcal{D}_k$ in $\mathfrak{K}$ such that

$$\|\eta' \circ \gamma \circ \iota_1^m \circ \varphi_n^m(a) - \varphi_n^k(a)\| < \epsilon/2 \tag{3.2}$$

for every $a \in G$. Let $\lambda : \mathcal{D}_m \otimes \mathcal{D}_m \to \mathcal{D}_k$ be the $\mathfrak{K}$-morphism $\lambda = \eta' \circ \gamma$.



Let $\theta : \mathcal{D}_m \otimes \mathcal{D}_m \to \mathcal{D}_k \otimes \mathcal{D}_k$ be the -homomorphism defined by $\theta = (\lambda \circ \iota_2^m) \otimes \varphi_m^k$ (θ is not necessarily a $\mathfrak{K}$-morphism). Put $\widetilde{u} = \theta(u)$ and note that $\widetilde{u}$ commutes with the image of the map $\iota_1^k \circ \lambda \circ \iota_1^m$. By linearity, it is enough to check this for $u = a \otimes b$: for every $c \in \mathcal{D}_m$ we have

$$\begin{aligned}\widetilde{u}[\iota_1^k \circ \lambda \circ \iota_1^m(c)] &= [\lambda(1 \otimes a) \otimes \varphi_m^k(b)][\lambda(c \otimes 1) \otimes 1] \\ &= \lambda(c \otimes a) \otimes \varphi_m^k(b) = [\lambda(c \otimes 1) \otimes 1][\lambda(1 \otimes a) \otimes \varphi_m^k(b)] \\ &= [\iota_1^k \circ \lambda \circ \iota_1^m(c)]\widetilde{u}.\end{aligned}$$

Applying θ to (3.1) gives us

$$\|\widetilde{u}(\lambda(1 \otimes \varphi_n^m(a)) \otimes 1)\widetilde{u}^* - 1 \otimes \varphi_n^k(a)\| < \epsilon/2, \tag{3.3}$$

for every $a \in G$.

Take $x = a \otimes 1$ for $a \in G$. The inequality 3.2 and the fact that $\widetilde{u}$ commutes with $\lambda(\varphi_n^m(a) \otimes 1) \otimes 1$ imply that

$$\begin{aligned}\| \mathrm{Ad}_{\widetilde{u}^*} \circ \iota_1^k \circ \lambda \circ (\varphi_n^m \otimes \varphi_n^m)(x) - (\varphi_n^k \otimes \varphi_n^k)(x)\| &= \|\lambda(\varphi_n^m(a) \otimes 1) \otimes 1 - \varphi_n^k(a) \otimes 1\| \\ &= \|\lambda(\varphi_n^m(a) \otimes 1) - \varphi_n^k(a)\| < \epsilon/2.\end{aligned}$$

Assume $x = 1 \otimes a$ for $a \in G$. Then by the inequality 3.3 we have

$$\begin{aligned}&\| \mathrm{Ad}_{\widetilde{u}^*} \circ \iota_1^k \circ \lambda \circ (\varphi_n^m \otimes \varphi_n^m)(x) - (\varphi_n^k \otimes \varphi_n^k)(x)\| \\ &\qquad = \|\widetilde{u}(\lambda(1 \otimes \varphi_n^m(a)) \otimes 1)\widetilde{u}^* - 1 \otimes \varphi_n^k(a)\| < \epsilon/2.\end{aligned}$$

The $\mathfrak{K}$-morphism $\eta = \mathrm{Ad}_{\widetilde{u}^*} \circ \iota_1^m \circ \eta'$ satisfies (∗) as required and therefore by the uniqueness of the weak Fraïssé limit $\mathcal{D} \otimes \mathcal{D}$ is $*$-isomorphic to $\mathcal{D}$.

To see that $\mathcal{D}$ is strongly self-absorbing, for each natural number i consider the natural induced $*$-embeddings $\varphi_i^\infty : \mathcal{D}_i \to \mathcal{D}$ and the $*$-embeddings $\theta_i : \mathcal{D}_i \to \mathcal{D} \otimes \mathcal{D}$ defined by $\theta_i = (\varphi_i^\infty \otimes \varphi_i^\infty) \circ \iota_i^2$. Fix finite subsets G_i of $\mathcal{D}_i$ such that $\varphi_i^{i+1}[G_i] \subseteq G_{i+1}$

and $\bigcup_i^\infty \varphi_i^\infty[G_i]$ is dense in $\mathcal{D}$. Since $(\mathcal{D}_i, \varphi_i^j)$ is a weak Fraïssé sequence of $\mathfrak{K}$, by the second statement of Theorem 2.8, for every n there is $m \geq n$ and a $*$-isomorphism $\Phi_n : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ such that $\|\theta_m \circ \varphi_n^m(a) - \Phi_n \circ \varphi_n^\infty(a)\| < 2^{-n}$ for every $a \in G_n$.

$$\begin{array}{ccccc}
\mathcal{D}_n & \xrightarrow{\varphi_n^m} & \mathcal{D}_m & \xrightarrow{\varphi_m^\infty} & \mathcal{D} \\
 & & \downarrow{\scriptstyle \iota_m^2} & & {\scriptstyle \Phi_n}\downarrow \\
 & & \mathcal{D}_m \otimes \mathcal{D}_m & \xrightarrow{\varphi_n^\infty \otimes \varphi_n^\infty} & \mathcal{D} \otimes \mathcal{D}
\end{array}$$

Clearly for every i and $j > i$, the diagram

$$\begin{array}{ccc}
\mathcal{D}_i & \xrightarrow{\varphi_i^\infty} & \mathcal{D}_j \\
\downarrow{\scriptstyle \iota_i^2} & & \downarrow{\scriptstyle \iota_j^2} \\
\mathcal{D}_i \otimes \mathcal{D}_i & \xrightarrow{\varphi_i^j \otimes \varphi_i^j} & \mathcal{D}_j \otimes \mathcal{D}_j
\end{array}$$

commutes. Hence for every n and $k, \ell > n$ we have $\Phi_k|_{\varphi_n^\infty[G_n]} \approx_{1/2^n} \Phi_\ell|_{\varphi_n^\infty[G_n]}$. Therefore $\Phi = \lim_n \Phi_n$ is a well-defined $*$-isomorphism from $\mathcal{D}$ onto $\mathcal{D} \otimes \mathcal{D}$.

We claim that Φ is approximately unitarily equivalent to $\mathrm{id}_\mathcal{D} \otimes 1_\mathcal{D}$. Suppose $\epsilon > 0$ and $F \Subset \mathcal{D}$ are given. Without loss of generality let us assume that F is a subset of $\varphi_n^\infty[G_n]$, for a large enough n. Find $m \geq n$ such that

$$\begin{aligned}
\Phi(\varphi_n^\infty(a)) \approx_{\epsilon/3} \Phi_m(\varphi_n^\infty(a)) &\approx_{\epsilon/3} \varphi_m^\infty \otimes \varphi_m^\infty(1 \otimes \varphi_n^m(a)) \\
&= 1 \otimes \varphi_n^\infty(a)
\end{aligned} \tag{3.4}$$

for every $a \in G_n$. By increasing m, if necessary, find a unitary u in $\mathcal{D}_m \otimes \mathcal{D}_m$ such that

$$\|\varphi_n^m(a) \otimes 1 - u^*(1 \otimes \varphi_n^m(a))u\| < \epsilon/3.$$

Let $\widetilde{u} = \varphi_m^\infty \otimes \varphi_m^\infty(u)$. Apply $\varphi_m^\infty \otimes \varphi_m^\infty$ to the above inequality to get

$$\|\varphi_n^\infty(a) \otimes 1 - \widetilde{u}^*(1 \otimes \varphi_n^\infty(a))\widetilde{u}\| < \epsilon/3 \tag{3.5}$$

for every $a \in G_n$. Hence, from (3.4) and (3.5) we have

$$\|\varphi_n^\infty(a) \otimes 1 - \widetilde{u}^* \Phi(\varphi_n^\infty(a))\widetilde{u}\| < \epsilon$$

for every $a \in G_n$. Therefore Φ is approximately unitarily equivalent to $\mathrm{id}_\mathcal{D} \otimes 1_\mathcal{D}$. $\square$

4. The Jiang-Su algebra as a Fraïssé limit

Let us start by recalling some definitions and fundamental facts about dimension-drop algebras and the Jiang-Su algebra from [9]. For every positive integer n, by M_n we denote the C^*-algebra of all complex $n \times n$-matrices, with the unit 1_n. For positive integers p, q the *dimension-drop algebra* $\mathcal{Z}_{p,q}$ is defined by

$$\mathcal{Z}_{p,q} = \{f \in C([0,1], M_{pq}) : f(0) \in M_p \otimes 1_q \text{ and } f(1) \in 1_p \otimes M_q\}.$$

A dimension-drop algebra is called *prime* if $(p, q) = 1$. A unital $*$-embedding $\varphi : \mathcal{Z}_{p,q} \to \mathcal{Z}_{p',q'}$ between dimension-drop algebras is called *diagonalizable* (in [15]) if there are continuous maps $\{\xi_i : [0,1] \to [0,1] : i = 1, \dots, k\}$ and a unitary $u \in C([0,1], M_{p'q'})$ such that

$$\varphi(f) = \mathrm{Ad}_u \circ \begin{bmatrix} f \circ \xi_1 & & 0 \\ & \ddots & \\ 0 & & f \circ \xi_k \end{bmatrix}$$

for every $f \in \mathcal{Z}_{p,q}$. Let $\Lambda^{\varphi} = \{\xi_1, \xi_2, \dots, \xi_k\}$ and define $\Delta^{\varphi} : \mathcal{Z}_{p,q} \to C([0,1], M_{p'q'})$ by

$$\Delta^{\varphi}(f) = \begin{bmatrix} f \circ \xi_1 & & 0 \\ & \ddots & \\ 0 & & f \circ \xi_k \end{bmatrix}$$

for every $f \in \mathcal{Z}_{p,q}$.

Proposition 4.1. [9] *There is a sequence $(\mathcal{A}_n, \varphi_n^m)$ of prime dimension-drop algebras and diagonalizable $*$-embeddings such that for every $\xi \in \Lambda^{\varphi_n^m}$ the diameter of the image of ξ is not greater than $1/2^{m-n}$. The limit of any such sequence is unital, simple and has a unique tracial state.*

In [9] Jiang and Su use tools from the classification theory to show that there is up to $*$-isomorphisms a unique simple and monotracial (i.e. has a unique tracial state) C^*-algebra which is the limit of a sequence of prime dimension-drop algebras and unital $*$-embeddings. They denote the limit of a (any) sequence as in Proposition 4.1 by $\mathcal{Z}$, which is nowadays called the Jiang-Su algebra. A closer look at the proof of Proposition 4.1 in [9] shows that one can choose the sequences so that they satisfy some extra properties.

Proposition 4.2. *There is a sequence $(\mathcal{Z}_{p_n,q_n}, \varphi_n^m)$ of prime dimension-drop algebras and diagonalizable $*$-embeddings such that*

- (†) *for every $m \geq n$, if $\Lambda^{\varphi_n^m} = \{\xi_1 \dots, \xi_k\}$ then we have $\xi_1(x) \leq \xi_2(x) \cdots \leq \xi_k(x)$ for all $x \in [0,1]$ and the diameter of the image of each ξ_i is not greater than $1/2^{m-n}$.*
- (‡) *$p_m q_m$ is a multiple of m, for every natural number m.*

Proof. It is been already pointed out in Remark 2.6 of [9] that $\{\xi_i\}$ can be arranged in the increasing order. In the $(m+1)$-th stage of the recursive construction of the proof in [9, Proposition 2.5], when the sequence $(\mathcal{Z}_{p_i,q_i}, \varphi_i)_{i \leq m}$ is already chosen, we can choose k_0 and k_1 such that

$$k_0 > 2q_m, \quad k_1 > 2p_m, \quad (k_0 p_m, k_1 q_m) = 1$$

and moreover make sure that $k_0 k_1$ is a multiple of $m+1$ (distribute the factors of the prime factorization of $m+1$ appropriately among k_0 and k_1). Then $p_{m+1} = k_0 p_m$ and $q_{m+1} = k_1 q_m$ satisfy the property that $p_{m+1} q_{m+1}$ is a multiple of $m+1$. □

In Theorem 4.6 we will show that sequences as in Proposition 4.2 have $*$-isomorphic limits.

The Category $\mathfrak{Z}$. We define the Category $\mathfrak{Z}$ as in [3] and [15]. Let $\mathfrak{Z}$ denote the category whose objects are all prime dimension-drop algebras $(\mathcal{Z}_{p,q}, \tau)$ with fixed faithful traces. The set of $\mathfrak{Z}$-morphisms from $(\mathcal{Z}_{p,q}, \tau)$ to $(\mathcal{Z}_{p',q'}, \tau')$ is the set of all unital trace-preserving $*$-embeddings.

Remark 4.3. The main result of [15] states that $\mathfrak{Z}$ is a Fraïssé category and its Fraïssé limit is simple and monotracial. Note that the fixed traces are only for the purpose of the near amalgamation property. Namely, the unital $*$-embeddings with the same prime dimension-drop algebra in their domains can be nearly amalgamated if for some traces on their respective codomains they induce the same trace on the domain.

It turns out that in order to show that $\mathcal{Z}$ is the Fraïssé limit of $\mathfrak{Z}$, we do not need to use the fact that $\mathfrak{Z}$ is a Fraïssé category, instead in Theorem 4.6 we show directly (using the tools that are already developed in [15]) that any sequence as in Proposition 4.2 can be turned into a Fraïssé sequence of $\mathfrak{Z}$, by fixing appropriate traces. The proof of Theorem 4.6 is similar to the one of [15, Proposition 4.10], of the fact that $\mathfrak{Z}$ has the near amalgamation property. As a result, to carry out the proof of Theorem 4.6, we need the same tools that are developed in [15] (listed below in Proposition 4.4), all of which have straightforward proofs.

First recall that, essentially by Riesz representation theorem, traces (by "trace" we mean a tracial state) on a dimension-drop algebra $\mathcal{Z}_{p,q}$ correspond to probability (Radon) measures on $[0,1]$. In fact, the trace space $T(\mathcal{Z}_{p,q})$ is affinely homeomorphic to the space of all probability measures on $[0,1]$. Given a probability measure τ on $[0,1]$, we use the same letter τ to denote the "corresponding trace" on $\mathcal{Z}_{p,q}$ which defined by

$$\tau(A) = \int_0^1 \mathrm{Tr}(f(x)) d\tau(x)$$

for every $f \in \mathcal{Z}_{p,q}$, where Tr is the unique trace on M_{pq}. A measure τ on $[0,1]$ is faithful if and only if the corresponding trace is faithful. A measure is called *diffuse* if any measurable set of non-zero measure can be partitioned into two measurable set of non-zero measures. For the maps $\xi, \zeta : [0,1] \to [0,1]$ we write $\xi \leq \zeta$ if and only if $\xi(x) \leq \zeta(x)$ for any $x \in [0,1]$.

Proposition 4.4. *(1)* [15, Proposition 4.3] *For every $(\mathcal{Z}_{p,q}, \tau)$ in $\mathfrak{Z}$ there is a $\mathfrak{Z}$-morphism $\psi : (\mathcal{Z}_{p,q}, \tau) \to (\mathcal{Z}_{p,q}, \sigma)$, for any trace σ on $\mathcal{Z}_{p,q}$ which corresponds to a diffuse measure on* $[0,1]$.

(2) [15, Proposition 4.4] *Suppose p and q are coprime natural numbers. There exists $N \in \mathbb{N}$ such that if p', q' are coprimes natural numbers larger than N such that pq divides $p'q'$, there exists a $\mathfrak{Z}$-morphism $\varphi : (\mathcal{Z}_{p,q}, \tau) \to (\mathcal{Z}_{p',q'}, \tau')$, for any faithful diffuse measures τ, τ' on* $[0,1]$.

(3) [15, Proposition 4.7] *Suppose $\varphi : (\mathcal{Z}_{p,q}, \tau) \to (\mathcal{Z}_{p',q'}, \tau')$ is a $\mathfrak{Z}$-morphism and pq divides $p'q'$. For every $\epsilon > 0$ and $F \Subset \mathcal{Z}_{p,q}$ there is a diagonalizable $\mathfrak{Z}$-morphism $\psi : (\mathcal{Z}_{p,q}, \tau) \to (\mathcal{Z}_{p',q'}, \tau')$ such that $\|\varphi(f) - \psi(f)\| < \epsilon$ for every $f \in F$ and $\Lambda^\psi = \{\xi_1, \dots, \xi_k\}$ satisfy $\xi_1 \leq \dots \leq \xi_k$.*

(4) [15, Proposition 4.8] *Suppose τ, τ' correspond to diffuse faithful measures on $[0,1]$. For every $\epsilon > 0$ and $(\mathcal{Z}_{p,q}, \tau) \in \mathfrak{Z}$ there is a diagonalizable $\mathfrak{Z}$-morphism $\psi : (\mathcal{Z}_{p,q}, \tau) \to (\mathcal{Z}_{p',q'}, \tau')$ such that the diameter of the image of each $\xi \in \Lambda^\psi$ is less than ϵ.*

(5) [15, Lemma 4.2] *Suppose $\mathcal{Z}_{p,q}$ and $\mathcal{Z}_{p',q'}$ are prime dimension-drop algebras such that pq divides $p'q'$. There is a unitary $w \in C([0,1], M_{p'q'})$ such that for any $\psi : \mathcal{Z}_{p,q} \to C([0,1], M_{p'q'})$ of the form*

$$\psi(f) = \mathrm{diag}(f \circ \zeta_1, \dots, f \circ \zeta_k),$$

where $\zeta_1 \leq \dots \leq \zeta_k$ are continuous maps from $[0,1]$ into $[0,1]$, the image of $\mathrm{Ad}_w \circ \psi$ is included in $\mathcal{Z}_{p',q'}$.

(6) [15, Lemma 4.9] *Suppose $\varphi, \psi : \mathcal{Z}_{p,q} \to \mathcal{Z}_{p',q'}$ are diagonalziable $*$-embeddings such that $\Delta^\varphi = \Delta^\psi$. For any $\epsilon > 0$ and $F \Subset \mathcal{Z}_{p,q}$ there exists a unitary w in $C([0,1], M_{p',q'})$ such that the inner $*$-automorphism Ad_w preserves $\mathcal{Z}_{p',q'}$ and $\|\mathrm{Ad}_w \circ \psi(f) - \varphi(f)\| < \epsilon$ for every $f \in F$.*

Remark 4.5. The composition $\varphi' \circ \varphi$ of two diagonalizable morphism is again diagonalizable. In fact, if $\Lambda^{\varphi} = \{\xi_1, \dots \xi_k\}$ and $\Lambda^{\varphi'} = \{\xi'_1, \dots \xi'_{k'}\}$, then we can arrange so that $\Lambda^{\varphi' \circ \varphi} = \{\xi_1 \circ \xi'_1, \dots, \xi_1 \circ \xi'_{k'}, \dots, \xi_k \circ \xi'_1, \dots, \xi_k \circ \xi'_{k'}\}$. Hence if the diameter of the image of each map in Λ^{φ} is less than ϵ, then the diameter of the image of each map in $\Lambda^{\varphi' \circ \varphi}$ is also less than ϵ.

Theorem 4.6. *Sequences as in Proposition 4.2 have $*$-isomorphic limits.*

Proof. Suppose $(\mathcal{A}_n, \varphi_n^m)$ is a sequence as in Proposition 4.2 and $\mathcal{A}_n = \mathcal{Z}_{p_n,q_n}$. Fix a sequence $\{\nu_n : \nu_n \in T(\mathcal{A}_n)\}_{n \in \mathbb{N}}$ of traces which correspond to diffuse measures on $[0,1]$. It is easy to check that $\tau_n = \lim_{m\to\infty} \nu_m \circ \varphi_n^m$ defines a trace on $\mathcal{A}_n$ which also corresponds to a diffuse measure on $[0,1]$. Each $\varphi_n^m : (\mathcal{A}_n, \tau_n) \to (\mathcal{A}_m, \tau_m)$ is a trace-preserving $*$-embedding, since $\tau_n = \tau_m \circ \varphi_n^m$, for all $m \geq n$. The limit of the sequence $(\mathcal{A}_n, \varphi_n^m)$ is a monotracial C^*-algebra and $\tau = \lim_n \tau_n$ is its unique trace.

Let $A_n = (\mathcal{A}_n, \tau_n)$. Then (A_n, φ_n^m) is a $\mathfrak{Z}$-sequence. We claim that (A_n, φ_n^m) is a Fraïssé sequence of $\mathfrak{Z}$. The condition $(\mathscr{U})$ of Definition 2.1 is clear from (2) of Proposition 4.4, since by (‡) for any pair p, q of coprime natural numbers, $p_n q_n$ divides pq, if $n > pq$. To see $(\mathscr{A})$, suppose $\epsilon > 0$, n, $F \Subset \mathcal{A}_n$ and a $\mathfrak{Z}$-morphism $\gamma : A_n \to (\mathcal{Z}_{p,q}, \sigma)$ are given. Take $\delta > 0$ such that

$$\|f(x) - f(y)\| < \epsilon \quad \text{if} \quad |x - y| < \delta$$

for every $f \in F$ and $x, y \in [0,1]$. Find $m > n$ such that $2^{n-m} < \delta$, i.e. the diameter of the image of each $\xi \in \Lambda^{\varphi_n^m}$ is less than δ. Suppose

$$\varphi_n^m(f) = \mathrm{Ad}_u \circ \mathrm{diag}(f \circ \xi_1, \dots, f \circ \xi_\ell).$$

By (1), (3) and (4) of Proposition 4.4, without loss of generality, we can assume that σ corresponds to a diffuse measure, γ is a diagonalizable $\mathfrak{Z}$-morphism such that if $\Lambda^{\gamma} = \{\zeta_1, \dots, \zeta_k\}$ then it satisfies $\zeta_1 \leq \dots \leq \zeta_k$, and the diameter of the image each ζ_i is less than $\delta/3$.

By increasing m, if necessary, we can make sure that $p_m q_m$ is a multiple of pq and find a diagonalizable $\mathfrak{Z}$-morphism $\gamma' : (\mathcal{Z}_{p,q}, \sigma) \to A_m$ such that $\Lambda^{\gamma'} = \{\zeta'_1, \dots, \zeta'_{k'}\}$ satisfies $\zeta'_1 \leq \dots \leq \zeta'_{k'}$ and $kk' = \ell$. Therefore

$$|\Lambda^{\gamma' \circ \gamma}| = |\Lambda^{\gamma}||\Lambda^{\gamma'}| = kk' = \ell = |\Lambda^{\varphi_n^m}|.$$

Then, if $\Lambda^{\gamma' \circ \gamma} = \{\zeta''_1, \dots, \zeta''_\ell\}$ we have $\zeta''_1 \leq \dots \leq \zeta_\ell$ and the diameter of the image each ζ''_i is less than $\delta/3$.

We claim that $\|\xi_i - \zeta''_i\| < \delta$ for every $i \leq \ell$. If not, then for some $j \leq \ell$ and some $t \in [0,1]$ we have $\xi_j(t) \geq \zeta''_j(t) + \delta$. Set $c = \min \xi_{j+1}$ (if $j = d$ let $c = 1$) and $d = \max \zeta''_j$. Note that

- the image of ζ''_i is included in $[0, d]$ for every $1 \leq i \leq j$,
- if $\mathrm{Im}(\xi_i) \cap [0, c) \neq \emptyset$, then $i \leq j$,
- $c > d + \delta/3$.

Since φ_n^m and $\gamma' \circ \gamma$ are trace-preserving

$$j = \sum_{i=1}^{j} \tau_m((\zeta''_j)^{-1}[0,d]) \leq \ell \tau_n([0,d]) < \ell \tau_n([0,c]) \leq \sum_{i=1}^{j} \tau_m((\xi_j)^{-1}[0,1]) = j,$$

which is a contradiction. The claim implies that $\Delta^{\varphi_n^m} \approx_{\epsilon,F} \Delta^{\gamma' \circ \gamma}$.

Suppose $\gamma' \circ \gamma(f) = \mathrm{Ad}_v \circ \mathrm{diag}(f \circ \zeta_1'', \dots, f \circ \zeta_\ell'')$. By (4) of Proposition 4.4, there is a unitary w such that the images of the maps $\varphi = \mathrm{Ad}_{wu^*} \circ \varphi_n^m$ and $\psi = \mathrm{Ad}_{wv^*} \circ (\gamma' \circ \gamma)$ are included in $\mathcal{Z}_{p',q'}$. Clearly

$$\varphi = \mathrm{Ad}_w \circ \Delta^{\varphi_n^m} \quad \text{and} \quad \psi = \mathrm{Ad}_w \circ \Delta^{\gamma' \circ \gamma}.$$

Since $\Delta^\varphi = \Delta^{\varphi_n^m}$ and $\Delta^\psi = \Delta^{\gamma' \circ \gamma}$, by (6) of Proposition 4.4, there are unitaries w_0 and w_1 such that the inner $*$-automorphism Ad_{w_0} and Ad_{w_1} both preserve $\mathcal{Z}_{p_m,q_m}$ and

$$\|\varphi(f) - \mathrm{Ad}_{w_0} \circ \varphi_n^m(f)\| < \epsilon \quad \text{and} \quad \|\psi(f) - \mathrm{Ad}_{w_1} \circ \gamma' \circ \gamma(f)\| < \epsilon,$$

for any $f \in F$. Finally for any $f \in F$ we have

$$\begin{aligned}
\mathrm{Ad}_{w_0^* w_1} \circ \gamma' \circ \gamma(f) &\approx_\epsilon \mathrm{Ad}_{w_0^*} \circ \psi(f) \\
&= \mathrm{Ad}_{w_0^* w} \circ \Delta^{\gamma' \circ \gamma}(f) \\
&\approx_\epsilon \mathrm{Ad}_{w_0^* w} \circ \Delta^{\varphi_n^m}(f) \\
&= \mathrm{Ad}_{w_0^*} \circ \varphi(f) \\
&\approx_\epsilon \varphi_n^m(f).
\end{aligned}$$

Therefore the $\mathfrak{Z}$-morphism $\eta = \mathrm{Ad}_{w_0^* w_1} \circ \gamma'$ satisfies $\eta \circ \gamma \approx_{3\epsilon, F} \varphi_n^m$, as required. This finished the proof. □

In fact, we have shown the following.

Corollary 4.7. *The category $\mathfrak{Z}$ contains Fraïssé sequences and the Fraïssé limit of $\mathfrak{Z}$ is $(\mathcal{Z}, \nu)$, where ν is the unique trace of $\mathcal{Z}$.*

5. $\mathcal{Z}$ is strongly self-absorbing

In this short section first we define a category $\mathfrak{T}$ which is a $\otimes$-expansion of $\mathfrak{Z}$ and it is dominated by $\mathfrak{Z}$. This would complete all the necessary ingredients to use Theorem 3.5 in order to prove that $\mathcal{Z}$ is strongly self-absorbing.

The Category $\mathfrak{T}$. Let $\mathfrak{T}$ denote a category whose objects are

$$\mathfrak{Z} \cup \{(\mathcal{Z}_{p,q} \otimes \mathcal{Z}_{p,q}, \tau \otimes \tau) : (\mathcal{Z}_{p,q}, \tau) \in \mathfrak{Z}\}$$

and $\mathfrak{T}$-morphisms are exactly finite compositions of the maps of the form below.

(i) $\mathfrak{Z}$-morphisms,
(ii) $\mathrm{Ad}_u : (\mathcal{Z}_{p,q} \otimes \mathcal{Z}_{p,q}, \tau \otimes \tau) \to (\mathcal{Z}_{p,q} \otimes \mathcal{Z}_{p,q}, \tau \otimes \tau)$ for every unitary $u \in \mathcal{Z}_{p,q} \otimes \mathcal{Z}_{p,q}$ and for every $(\mathcal{Z}_{p,q}, \tau) \in \mathfrak{Z}$,
(iii) the $*$-embeddings $\mathrm{id} \otimes 1_{pq} : (\mathcal{Z}_{p,q}, \tau) \to (\mathcal{Z}_{p,q} \otimes \mathcal{Z}_{p,q}, \tau \otimes \tau)$ and $1_{pq} \otimes \mathrm{id} : (\mathcal{Z}_{p,q}, \tau) \to (\mathcal{Z}_{p,q} \otimes \mathcal{Z}_{p,q}, \tau \otimes \tau)$, for every $(\mathcal{Z}_{p,q}, \tau) \in \mathfrak{Z}$.
(iv) $\varphi \otimes \varphi : (\mathcal{Z}_{p,q} \otimes \mathcal{Z}_{p,q}, \tau \otimes \tau) \to (\mathcal{Z}_{p',q'} \otimes \mathcal{Z}_{p',q'}, \tau' \otimes \tau')$ for every unital trace-preserving $*$-embeddings (a $\mathfrak{Z}$-morphism) $\varphi : (\mathcal{Z}_{p,q}, \tau) \to (\mathcal{Z}_{p',q'}, \tau')$,
(v) the unital $*$-homomorphism $\rho : (\mathcal{Z}_{p,q} \otimes \mathcal{Z}_{p,q}, \tau \otimes \tau) \to (\mathcal{Z}_{p^2,q^2}, \tau)$ defined by $\rho(g)(t) = g(t,t)$, for every $(\mathcal{Z}_{p,q}, \tau) \in \mathfrak{Z}$.

Note that $\mathcal{Z}_{p,q} \otimes \mathcal{Z}_{p,q}$ is not $*$-isomorphic to any dimension-drop algebra, since its center is $C([0,1]^2)$. The following lemma justifies why $\mathfrak{T}$ is in fact a category, i.e., it is closed under compositions of its morphisms.

Lemma 5.1. *Any $\mathfrak{T}$-morphism $\varphi : (\mathcal{Z}_{p,q}, \tau) \to (\mathcal{Z}_{p',q'}, \tau')$ is a trace-preserving unital $*$-embedding i.e. φ is a $\mathfrak{Z}$-morphism.*

Proof. Any such φ is either already a $\mathfrak{Z}$-morphism or it is of the form

$$(\mathcal{Z}_{p,q},\tau) \xrightarrow{\varphi_1} (\mathcal{Z}_{p_1,q_1},\tau_1) \xrightarrow[\text{or } 1\otimes\mathrm{id}]{\mathrm{id}\otimes 1} (\mathcal{Z}_{p_1,q_1}\otimes \overset{\mathrm{Ad}_u\,\circlearrowright}{\mathcal{Z}_{p_1,q_1}}, \tau_1\otimes\tau_1) \xrightarrow{\varphi_2\otimes\varphi_2} \dots$$

$$\dots \xrightarrow{\varphi_2\otimes\varphi_2} (\mathcal{Z}_{p_2,q_2}\otimes \underset{\circlearrowleft\,\mathrm{Ad}_v}{\mathcal{Z}_{p_2,q_2}}, \tau_2\otimes\tau_2) \xrightarrow{\rho} (\mathcal{Z}_{p_2^2,q_2^2},\tau_2) \xrightarrow{\varphi_3} (\mathcal{Z}_{p',q'},\tau')$$

for some $\varphi_1,\varphi_2,\varphi_3$ in $\mathfrak{Z}$, unitaries u, v and a $*$-homomorphism ρ from (v). All of these maps except ρ are trace-preserving $*$-embeddings. Therefore it is enough to show that $\varphi : (\mathcal{Z}_{p,q},\tau) \to (\mathcal{Z}_{p^2,q^2},\tau)$ is trace-preserving, when $\varphi = \rho\circ(\mathrm{id}\otimes 1_{pq})$ (or $\varphi = \rho\circ(1_{pq}\otimes\mathrm{id})$). First note that φ is a $*$-embedding such that

$$\varphi(f)(x) = f(x)\otimes 1_{pq}$$

for every $x\in[0,1]$ and $f\in\mathcal{Z}_{p,q}$. We have

$$\begin{aligned}\tau(\varphi(f)) &= \int_0^1 \mathrm{Tr}(f(x)\otimes 1_{pq})d\tau(x)\\ &= \int_0^1 \mathrm{Tr}(f(x))d\tau(x) = \tau(f).\end{aligned}$$

□

Similarly, any $\mathfrak{T}$-morphism from objects of the form $(\mathcal{Z}_{p,q},\tau)$ to the objects of the form $(\mathcal{Z}_{p',q'}\otimes\mathcal{Z}_{p',q'},\tau'\otimes\tau')$ is automatically a trace-preserving unital $*$-embedding.

Lemma 5.2. *$\mathfrak{T}$ is a $\otimes$-expansion of $\mathfrak{Z}$.*

Proof. This is clear from Definition 3.4 and the definition of $\mathfrak{T}$. □

Lemma 5.3. *The category $\mathfrak{Z}$ is dominating in $\mathfrak{T}$.*

Proof. The condition $(\mathscr{C})$ of Definition 2.3 is clear, since for any $(\mathcal{Z}_{p,q}\otimes\mathcal{Z}_{p,q},\tau\otimes\tau)\in\mathfrak{T}$ the diagonal map $\rho : (\mathcal{Z}_{p,q}\otimes\mathcal{Z}_{p,q},\tau\otimes\tau)\to(\mathcal{Z}_{p^2,q^2},\tau)$ of the form (v) is a $\mathfrak{T}$-morphism.

For $(\mathscr{D})$, note that by Lemma 5.1, any $\mathfrak{T}$-morphism $\varphi : (\mathcal{Z}_{p,q},\tau)\to(\mathcal{Z}_{p',q'},\tau')$ is in $\mathfrak{Z}$ and if $\psi : (\mathcal{Z}_{p,q},\tau)\to(\mathcal{Z}_{p',q'}\otimes\mathcal{Z}_{p',q'},\tau'\otimes\tau')$ is a $\mathfrak{T}$-morphism, then $\rho\circ\psi : (\mathcal{Z}_{p,q},\tau)\to(\mathcal{Z}_{p'^2,q'^2},\tau')$ is in $\mathfrak{Z}$, where $\rho : (\mathcal{Z}_{p',q'}\otimes\mathcal{Z}_{p',q'},\tau'\otimes\tau')\to(\mathcal{Z}_{p'^2,q'^2},\tau')$ is again the diagonal map of the form (v). □

Corollary 5.4. *The Jiang-Su algebra $\mathcal{Z}$ is strongly self-absorbing.*

Proof. The Jiang-Su algebra with its unique trace $(\mathcal{Z},\nu)$ is the Fraïssé limit of $\mathfrak{Z}$ (Corollary 4.7) and has approximate inner half-flip ([9, Proposition 8.3]). The category $\mathfrak{T}$ is a $\otimes$-expansion of $\mathfrak{Z}$ (Lemma 5.2) and $\mathfrak{Z}$ dominates $\mathfrak{T}$ (Lemma 5.3). Therefore it follows from Theorem 3.5 that $\mathcal{Z}$ is strongly self-absorbing. □

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Institute of Mathematics, Czech Academy of Sciences, Czech Republic
E-mail address: `ghasemi@math.cas.cz`