INTRODUCTION TO FRACTIONAL CALCULUS

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This course starts from scratch and provides students with the background necessary for the understanding of the fractional calculus.

It is surprisingly, but most scientists and engineers remain unaware of Fractional Calculus; it is not being taught in schools and colleges; and others remain skeptical of this field. There are several reasons for that: several of the definitions proposed for fractional derivatives were inconsistent, meaning they worked in some cases but not in others. The mathematics involved appeared very different from that of integer order. There were almost no practical applications of this field, and it was considered by many as an abstract area containing only mathematical manipulations of little or no use. But recently, the paradigm began to shift from pure mathematical formulation to applications in various fields. During the last decade Fractional Calculus has been applied to almost every field of science, engineering, and mathematics. Some of the areas where Fractional Calculus has made a profound impact include viscoelasticity and rheology, electrical engineering, electrochemistry, biology, biophysics and bioengineering, signal and image processing, mechanics, mechatronics, physics, and control theory.

1. HINTS AND MOTIVATION

1.1. Heat and mass transfer

Let us solve a wave equation

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)u=0, \qquad (1.1)$$

by the factorization method. With this in mind, we factorize a differential operator $\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)$ according to well known algebraic formula $a^2 - b^2 =$ $= (a - b)(a - b)$, as a result equation (1.1) becomes

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)u=\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)u=0. \qquad (1.2)$$

It follows immediately from (1.2) that solution is a sum of two functions f and φ, each of them is solution of the equations

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)f=0, \qquad (1.3a)$$

and

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)\varphi=0. \qquad (1.3b)$$

The solutions of equations (1.3a, b) can be easily found by the method of characteristics; they read

$$f(t, x) = f(x - t),\ \varphi(t, x) = \varphi(x + t). \tag{1.4}$$

A study of heat and mass transfer requires the solution of parabolic equations of the form

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)T = 0\,. \tag{1.5}$$

An attractive idea of solving this equation is by using the factorization method as well. Formally factorizing differential operator yields

$$\left(\frac{\partial^{1/2}}{\partial t^{1/2}} - \frac{\partial}{\partial x}\right)\left(\frac{\partial^{1/2}}{\partial t^{1/2}} + \frac{\partial}{\partial x}\right)T = 0\,. \tag{1.6}$$

Let us consider the equation formed by the right multiplier of the differential operator

$$\left(\frac{\partial^{1/2}}{\partial t^{1/2}} + \frac{\partial}{\partial x}\right)T = 0\,. \tag{1.7}$$

The solution of the latter is the solution of the equation (1.6) as well. It is worth to note that according Fick's and Fourier's laws, the heat and mass fluxes equal to the gradients of the concentration and temperature, respectively. So, in order to determine the fluxes we need to know the spatial distribution of the concentration or temperature. But rewriting equation (1.7) in the following form

$$\frac{\partial^{1/2}}{\partial t^{1/2}}T = -\frac{\partial}{\partial x}T = q\,, \tag{1.8}$$

we arrive at the very important conclusion that *flux in some point $x = x_0$ can be found without knowledge of the spatial temperature distribution as a fractional derivative of 1/2 order of the temperature with respect to time.*

1.2. Motion in one direction

The motion of a system having one degree of freedom is said to take place in one dimension. The most general form of the Lagrangian of such system in fixed external condition is

$$L = 0.5\, m(dx/dt)^2 - U(x). \tag{2.1}$$

The equations of motion corresponding to these Lagrangians can be integrated in a general form. We can start from the first integral of this equation, which gives the law of the energy conservation

$$0.5m(dx/dt)^2 + U(x) = E, \tag{2.2}$$

where E is the total energy of a system.

This a first-order differential equation can be integrated immediately

$$t = \sqrt{0.5m}\int \frac{dx}{\sqrt{E - U(x)}} + \text{constant}. \tag{2.3}$$

Since the kinetic energy is essentially positive, the total energy always exceeds the potential one, i. e. the motion can take place only in those regions of space where $U(x) < E$. The points at which the potential energy equals the total energy,

$$U(x) = E, \tag{2.4}$$

give the limits of the motion. A finite motion in one dimension is oscillatory, the particle moving repeatedly back and forth between two points in the potential well (say between points x_1 and x_2 in Fig.1).

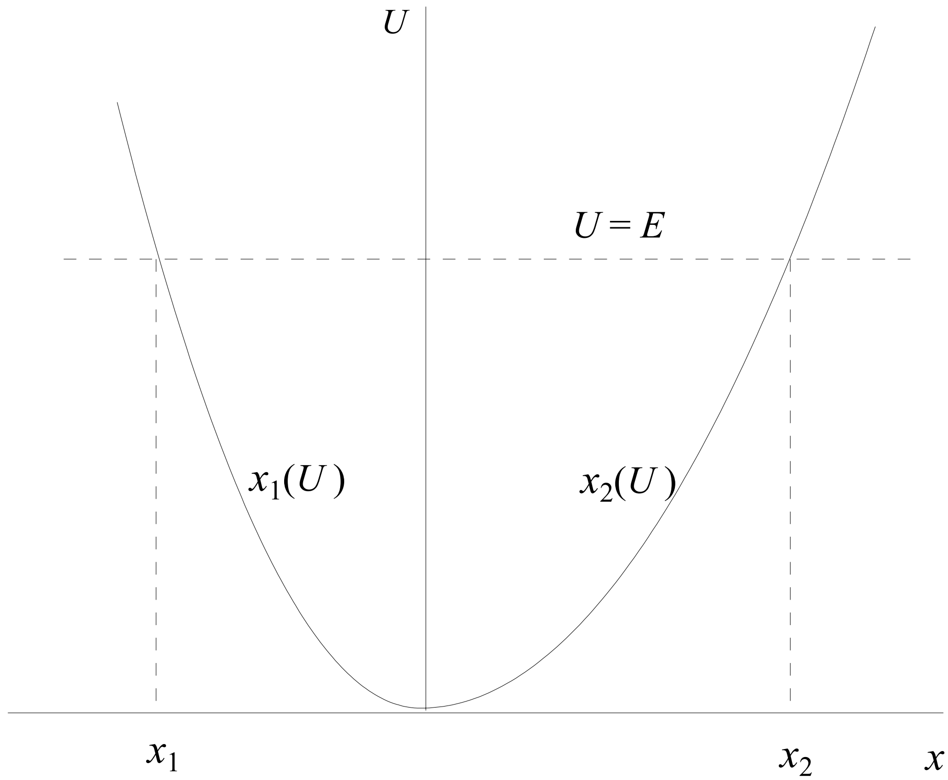

Fig. 1. The motion of particle in potential well.

The period T of the oscillation, i.e. the time during which the particles passes from x_1 and x_2 and back, is twice the time from x_1 to x_2

$$T(E) = \sqrt{2m} \int_{x_1(E)}^{x_2(E)} \frac{dx}{\sqrt{E - U(x)}}, \tag{2.5}$$

where $x_1(E)$ and $x_2(E)$ are roots of equation (2.4) for the given value of E. This formula gives the period of the motion as a function of the total energy of the particle if the potential energy is known. Usually, for example from an experiment, the period of oscillation is known. Therefore, let us consider to what extent the form of the potential energy $U(x)$ of a field in which a particle is oscillating can be deduced from a knowledge of the period of oscillation T as a function of the total energy E. Mathematically, this involves the solution of the integral equation (2.5), in which $U(x)$ is regarded as unknown and $T(E)$ as known.

We shall assume that the required function $U(x)$ has only one minimum in the region of space considered. For convenience, we take the origin at the position of minimum potential energy, and take this minimum energy to be zero (see Fig. 1). In the integral (2.5) we regard the x as a function of U. The function $x(U)$ is two-valued. Accordingly, the integral (2.5) must be divided into two parts before replacing dx by $(dx/dU)dU$: one from $x = x_1$ to $x = 0$ and the other from $x = 0$ to $x = x_2$. We shall denote the function $x(U)$ in these two ranges by $x = x_1(U)$ and $x = x_2(U)$, respectively.

The limits of integration with respect to U are 0 and E, so that we have

$$T(E) = (2m)^{1/2}\int_0^E \left(\frac{dx_2}{dU} - \frac{dx_1}{dU}\right)\frac{dU}{\sqrt{E-U}} \tag{2.6}$$

Dividing both sides of this equation by $(\alpha - E)^{1/2}$, where α is a parameter, and integrating with respect to E from 0 to α result in

$$\int_0^\alpha \frac{T(E)dE}{\sqrt{\alpha - E}} = \sqrt{2m}\int_0^\alpha\int_0^E \left(\frac{dx_2}{dU} - \frac{dx_1}{dU}\right)\frac{dUdE}{\sqrt{(\alpha - E)(E-U)}}$$

or, changing the order of integration,

$$\int_0^\alpha \frac{T(E)dE}{\sqrt{\alpha - E}} = \sqrt{2m}\int_0^\alpha \left(\frac{dx_2}{dU} - \frac{dx_1}{dU}\right) dU \int_U^\alpha \frac{dE}{\sqrt{(\alpha - E)(E - U)}}.$$

The integral over E is reduced to the Beta function B(1/2, 1/2) (for definition of the Beta function see (3.8)) by substitution $u = (E - U)/(\alpha - U)$; its value is π. The integral over U is thus trivial, and we have

$$\int_0^\alpha \frac{T(E)dE}{\sqrt{\alpha - E}} = \pi(2m)^{1/2}[x_2(\alpha) - x_1(\alpha)],$$

since $x_2(0) = x_1(0) = 0$. Writing U in place of α yields the final result:

$$x_2(U) - x_1(U) = \pi^{-1}(2m)^{-1/2}\int_0^U \frac{T(E)dE}{\sqrt{U - E}}. \tag{2.7}$$

Thus the known function $T(E)$ can be used to determine the difference $x_2(U) - x_1(U)$, whereas the functions $x_1(U)$ and $x_2(U)$ themselves remain indeterminate. The indeterminacy of the solution is removed if we impose the condition that the curve $U = U(x)$ must be symmetrical about the U - axis, i. e. that $x_2(U) = - x_1(U) = x(U)$. In this case, the formula (2.7) gives for $x(U)$ the unique expression

$$x(U) = 0.5\pi^{-1}(2m)^{-1/2}\int_0^U \frac{T(E)dE}{\sqrt{U - E}}. \tag{2.8}$$

As you will see later, formula (2.8) can be rewritten in the following form

$$x(U) = (8\pi m)^{1/2} I^{1/2} T(E),$$

where $I^{1/2}f(x)$ is the fractional integral of order 1/2.

2. THE FRACTIONAL INTEGRAL OF ORDER α

I would like to recall you Cauchy formula for repeated integration, that reduces the calculation of the n - fold primitive of a function $f(t)$ to a single integral of the convolution type:

$$I^n f(t) \equiv \int_0^t \int_0^t \ldots \int_0^t f(t)(dt)^n = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau \,, \qquad (3.1)$$

where $t > 0$, n is a positive integer.

Now we are ready to make a very important step leading us to the fractional integral. To this end, let us remember the properties of the Gamma function $\Gamma(z)$ and its relation with the factorial of a non-negative integer n, denoted by $n!$:

$$(n-1)! = \Gamma(n),\ \Gamma(n+1) = n\Gamma(n),\ \Gamma(z+1) = z\Gamma(z),\ \operatorname{Re} z > 0, \qquad (3.2)$$

$$\Gamma(1) = \Gamma(2) = 1,\ \Gamma(1/2) = \pi^{1/2},\ \Gamma(3/2) = \pi^{1/2}/2, \qquad (3.2a)$$

$$\Gamma(-1/2) = -\,2\pi^{1/2},\ \Gamma(-3/2) = (4/3)\ \pi^{1/2}, \qquad (3.2b)$$

The gamma function is commonly defined by a definite integral due to Leonhard Euler

$$\Gamma(z) = \int_0^\infty e^{-u} u^{z-1} du \,. \qquad (3.3)$$

Extending equation (1.1) from positive integer values of the index n to any positive real values α yields the ***Fractional Integral of orde***r[(S1)] $\alpha > 0$:

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \,. \qquad (3.4)$$

It is easy to see from equation (3.1) that applying n-fold differential operator $D^n \equiv d^n/dt^{\,n}$ to I^n results in the identity operator $I^0 = E$; this means

$$D^n I^n f(t) = I^0 f(t) = Ef(t) = f(t). \tag{3.5}$$

From equations (3.4) and (3.5) an interesting conjecture can be deduced: since

$$I^0 f(t) \;=\; \frac{1}{\Gamma(0)}\int_0^\tau (t-\tau)^{-1} f(\tau)d\tau = f(t), \tag{3.6}$$

the function

$$F_0(t) = t^{-1}/\Gamma(0) = \delta(t), \tag{3.6a}$$

is the Dirac δ - function.

I would like to recall that the Dirac δ - function can be loosely thought of as a function on the real line which is zero everywhere except at the origin, where it is infinite

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}, \tag{3.6b}$$

and which is also constrained to satisfy the identity

$$\int_{-\infty}^{+\infty} \delta(x)dx = 1. \tag{3.6c}$$

It is worthy of note that the Dirac δ - function can be defined more strictly as a linear functional

$$\int_{-\infty}^{+\infty} f(x)\delta(x-a)dx = f(a), \tag{3.6d}$$

where $f(x)$ is so-called *test function* (conventional and well-behaved function).

Moreover, it can be shown that functions

$$F_{-n}(t) = t^{-n-1}/\Gamma(-n) = \delta^{(n)}(t),\ n = 0, 1, ... \tag{3.6b}$$

are the generalized derivatives of order n of the Dirac delta function.

It can be shown that the I operator is both commutative and additive; that is

$$I^{\alpha}I^{\beta}f(t) = I^{\beta}I^{\alpha}f(t) = I^{\alpha+\beta}f(t) = \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-\tau)^{\alpha+\beta-1} f(\tau)d\tau\,. \tag{3.7}$$

This property is called the *semigroup* property of fractional integral operators.

Let us derive the following important and common result of the action of I^{α} operator on the power function t^{β}. To accomplish this, the Eulerian integral, *i. e.* the Beta function can be used:

$$\mathrm{B}(q, p) = \int_0^1 (1-u)^{p-1}u^{q-1}du = \Gamma(p)\,\Gamma(q)/\,\Gamma(p+q),\ \mathrm{Re}\ \mathrm{p},\ q > 0. \tag{3.8}$$

So, we have to take the integral

$$I^{\alpha}\,t^{\beta} = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^{\beta}d\tau\,. \tag{3.9}$$

Introducing a new variable $u = \tau/t$, integral reduces to the Beta function of the following form

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^{\beta}d\tau = \frac{t^{\alpha+\beta}}{\Gamma(\alpha)}\int_0^1 (1-u)^{\alpha-1}u^{(\beta+1)-1}du\,. \tag{3.10}$$

From Eqs. (3.8) and (3.10) it immediately follows the final result:

$$I^{\alpha}\,t^{\beta} = t^{\alpha+\beta}\,\Gamma(\beta+1)/\,\Gamma(\alpha+\beta+1). \tag{3.11}$$

We can also come at this result via the Laplace transform $L[f(t)] = F(s) =$ $= \int_0^\infty f(t) e^{-st} dt$. Taking the Laplace transform of $I^n f(t)$ yields $L[I^n f(t)] = F(s)/s^n$, where n is the integer. We assert

$$I^\alpha f(t) = L^{-1}[F(s)/s^\alpha], \tag{3.12}$$

where $L^{-1}[\ ]$ denotes the inverse Laplace transform, α is the positive real value. The Laplace transform of t^β is $L[t^\beta] = \Gamma(\beta + 1)/s^{\beta + 1}$. Accounting for equation (3.12) we arrive at the same result given by equation (3.11):

$$I^\alpha t^\beta = L^{-1}[\Gamma(\beta + 1)/s^{\alpha + \beta + 1}] = \frac{\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 1)} L^{-1}[\Gamma(\alpha + \beta + 1)/s^{\alpha + \beta + 1}] =$$

$$= t^{\alpha + \beta}\Gamma(\beta + 1)/\ \Gamma(\alpha + \beta + 1). \tag{3.13}$$

It is convenient to introduce the following causal function $F_\alpha(t)$ which is vanishing at $t < 0$:

$$F_\alpha(t) = t^{\alpha - 1}/\Gamma(\alpha). \tag{3.14}$$

Let us now recall the convolution integral with two function $f(t)$ and $g(t)$, which reads

$$f(t) * g(t) = \int_0^t f(t - \tau) g(\tau) d\tau . \tag{3.15}$$

Then we note from equations (3.4) and (3.14) that the fractional integral of order α can be considered as the *convolution between* $F_\alpha(t)$ and $f(t)$, *i.e.*

$$I^\alpha f(t) = F_\alpha(t) * f(t). \tag{3.16}$$

Let us prove an important *composition rule* based on the Beta function

$$F_\alpha(t) * F_\beta(t) = F_{\alpha+\beta}(t), \tag{3.17}$$

which can be used to prove the semigroup property (3.7) of fractional integral operators. We have

$$F_\alpha(t) * F_\beta(t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \frac{\tau^{\beta-1}}{\Gamma(\beta)} d\tau =$$

$$= \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (1-\eta)^{\alpha-1} \eta^{\beta-1} d\eta = \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}, \tag{3.18}$$

where $\eta = \tau/t$.

2.1. Geometric interpretation of fractional integration

Let us rewrite fractional integral (3.4) in the following form:

$$I^\alpha f(t) = \int_0^t f(\tau) dg(\tau; t), \tag{3.19}$$

$$g(\tau; t) = [t^\alpha - (t-\tau)^\alpha]/\Gamma(\alpha+1). \tag{3.20}$$

Considering the integral (3.20) for a fixed t, then it becomes simply a Riemann-Stieltjes integral.

Now let us take the axes τ, g, and f. In the plane (τ, g) we plot the function $g(\tau; t)$ for $0 \le \tau \le t$. Along the obtained curve we “build a fence” of the varying height $f(\tau)$, so the top edge of the "fence" is a three-dimensional line $(\tau, g(\tau; t), f(\tau))$, $0 \le \tau \le t$. This "fence" can be projected onto two surfaces (see Fig. 2):

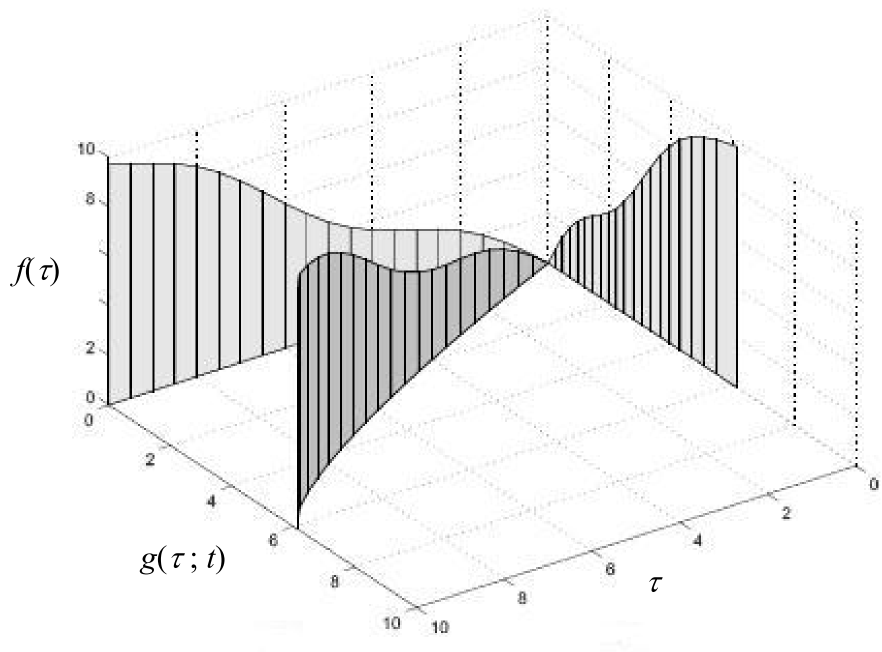

Fig. 2. The “fence” and its shadows: $I^1f(t)$ and $I^\alpha f(t)$.

a) the area of the projection of this "fence" onto the plane (τ, f) corresponds to the value of the integral $I^1f(t) = \int_0^t f(\tau)d\tau$;

b) the area of the projection of the same "fence" onto the plane (g, f) corresponds to the value of the integral (3.19), or, what is the same, to the value of the fractional integral (3.4).

Obviously, if $\alpha = 1$, then $g(\tau;\, t) = \tau$, and both "shadows" are equal. This shows that the classical definite integration is a particular case of the fractional one.

3. THE FRACTIONAL DERIVATIVE OF ORDER α

Let us consider the Abel integral equation of the first kind

$$\frac{1}{\Gamma(\alpha)}\int_0^t u(\tau)(t-\tau)^{\alpha-1}d\tau = f(t),\ 0 < \alpha < 1, \tag{4.1}$$

where $f(t)$ is a given function. It can be easily recognized that this equation can be expressed in terms of a fractional integral, *i.e.*

$$I^{\alpha} u(\tau) = f(t)\,, \tag{4.2}$$

and consequently solved in terms of a fractional derivative, according to

$$u(t) = D^{\alpha} f(\tau). \tag{4.3}$$

To this end we need to extend the property (3.5) from the positive integers to real values; this means

$$D^{\alpha} I^{\alpha} = E. \tag{3.5a}$$

Let us now solve (4.1) using the Laplace transform. Noting from equations (3.7) and (3.10) that $I^{\alpha} u(t) = F_{\alpha}(t) * u(t) \div U(s)/s^{\alpha}$, we then obtain

$$U(s)/ s^{\alpha} = F(s) \Rightarrow U(s) = s^{\alpha} F(s),\ F(s) = L[f(s)]. \tag{4.4}$$

Now we can choose two different ways to get the inverse Laplace transform from (4.4), according to the standard rules. Writing (4.4) as

$$U(s) = s[F(s)/s^{1-\alpha}] \tag{4.4a}$$

we obtain

$$u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha}} d\tau\,. \tag{4.5a}$$

On the other hand, writing (4.4) as

$$U(s) = [sF(s) - f(0)]/s^{1-\alpha} + f(0)/ s^{1-\alpha}, \tag{4.4b}$$

we obtain

$$u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau + f(0)\frac{t^{-\alpha}}{\Gamma(1-\alpha)}. \quad (4.5b)$$

Pitfall

Equation (4.5b) cannot be obtained from equation (4.5) directly by differentiation of the integral involving parameter t according to Leibniz's rule

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,\tau)d\tau = \int_{a(t)}^{b(t)} \frac{\partial f(t,\tau)}{\partial t} d\tau + f[t,b(t)]\frac{db(t)}{dt} - f[(t,a(t)]\frac{da(t)}{dt}. \quad (4.6)$$

To derive equation (4.5b) from equation (4.5a), first it needs to perform the integration of (4.5a) by parts:

$$u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^\alpha} d\tau =$$

$$= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt}\left[\frac{1}{\alpha-1} f(\tau)(t-\tau)^{1-\alpha}\Big|_0^t + \frac{1}{\alpha-1}\int_0^t \frac{f'(\tau)d\tau}{(t-\tau)^{\alpha-1}} \right] = \quad (4.7)$$

$$= \frac{f(0)t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\left[\frac{1}{\alpha-1}\int_0^t \frac{f'(\tau)d\tau}{(t-\tau)^{\alpha-1}} \right].$$

And only now differentiation of the integral with respect to t according to formula (4.6) can be fulfilled yielding equation (4.5b).

As a result we are arriving at the explicit formula defining the *Fractional Derivative of order* α(S1):

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^\alpha} d\tau, \quad (4.8)$$

or,

$$D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^{\alpha}} d\tau + f(0) \frac{t^{-\alpha}}{\Gamma(1-\alpha)}. \tag{4.9}$$

Using equation (4.8) let us find the fractional derivative of order α from the power function t^{γ}:

$$D^{\alpha} t^{\gamma} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} \tau^{\gamma} d\tau =$$

$$= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} t^{\gamma+1-\alpha} \int_0^1 (1-\eta)^{-\alpha} \eta^{\gamma} d\eta =$$

$$= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} t^{\gamma+1-\alpha} \int_0^1 (1-\eta)^{(1-\alpha)-1} \eta^{(1+\gamma)-1} d\eta =$$

$$= \frac{\gamma+1-\alpha}{\Gamma(1-\alpha)} t^{\gamma-\alpha} \frac{\Gamma(1-\alpha)\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha+1)}. \tag{4.10}$$

Using the following property of the Gamma function, *i. e.* $\Gamma(z + 1) = z\Gamma(z)$, finally we arrive at

$$D^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha}, \ \gamma > -1, \ \alpha > 0, \ t > 0. \tag{4.11}$$

Of course, the properties (4.11) are a natural generalization of those known when the order is a positive integer. To show that, let us assume that $f(t)$ is a monomial of the form $f(t) = t^k$ with k integer. The first derivative is as usual $Df(t) \equiv df(t)/dt = kt^{k-1}$. Repeating this n time gives more general results

$$D^n f(t) = [k!/(k-n)!] t^{k-n}. \tag{4.12}$$

Replacing the integers n, k with the real α, γ and the factorials with the Gamma function, leads us to equation (4.11).

Figure 3 illustrates the half derivative (maroon curve) of the function $y = t$ (blue curve) together with the first derivative (red curve).

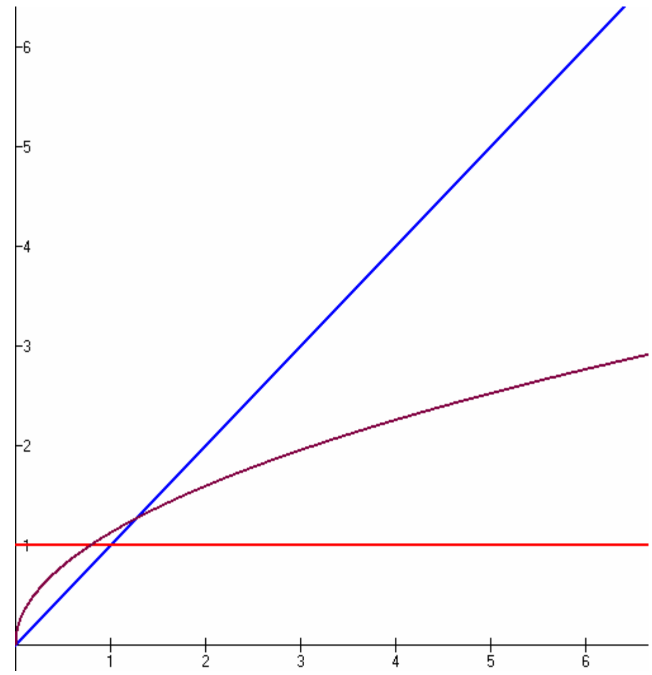

Fig.3. Half derivative (maroon curve) of the function $y = t$ (blue curve) together with the first derivative (red curve).

Note the remarkable fact that the fractional derivative $D^{\alpha}f(t)$ is not zero for the constant function $f(t) \equiv 1$ if $\alpha \notin \mathbf{N}$, where $\mathbf{N}$ is the set of positive integers. In fact, equation (4.11) with $\gamma = 0$ teaches us that

$$D^{\alpha}1 = t^{-\alpha}/\Gamma(1-\alpha),\ \alpha \geq 0,\ t > 0. \tag{4.13}$$

$D^{\alpha}1$ is equal to zero only for $\alpha \in \mathbf{N}$ due to the poles of the Gamma function in the points $0, -1, -2, \ldots$. By looking at equation (4.11), we observe that

$$D^{\alpha}t^{\alpha-1} = 0,\ \alpha > 0,\ t > 0. \tag{4.14}$$

The half-derivative of t is

$$D^{1/2}t = [\Gamma(2)/\,\Gamma(3/2)]t^{1/2} = 2(t/\pi)^{1/2}. \tag{4.15}$$

Repeating this process gives

$$D^{1/2}2(t/\pi)^{1/2} = 2\pi^{-1/2}D^{1/2}t^{1/2} = 2\pi^{-1/2}\,[\Gamma(3/2)/\,\Gamma(1)]t^{0} = 1, \tag{4.16}$$

which is indeed the expected result of

$$(D^{1/2}D^{1/2})t = 1. \tag{4.17}$$

It is important to keep in mind that in contrast to the semigroup property of the operators of fractional integration ($I^{\alpha}I^{\,\beta} = I^{\,\beta}I^{\alpha} = I^{\alpha+\beta}$), the operators of fractional differentiation D^{α} do not satisfy either the semigroup property, or the (weaker) commutative property. To show how the semigroup property does not necessarily hold for the standard fractional derivative, two simple examples are provided for which

$$D^{\alpha}D^{\,\beta}f(t) = D^{\,\beta}D^{\alpha}f(t) \neq D^{\alpha+\beta}f(t), \tag{4.18}$$

$$D^{\alpha}D^{\,\beta}g(t) \neq D^{\,\beta}D^{\alpha}g(t) = D^{\alpha+\beta}g(t). \tag{4.19}$$

In the first example let us take $f(t) = t^{-1/2}$ and $\alpha = \beta = 1/2$. Then, using equation (4.11), we get $D^{1/2}f(t) \equiv 0$, $D^{1/2}D^{1/2}f(t) \equiv 0$, but $D^{1/2+1/2}f(t) = D^{1}f(t) = -t^{-3/2}/2$. In the second one let us take $g(t) = t^{1/2}$ and $\alpha = 1/2$, $\beta = 3/2$. Then, again using equation (4.11) and property of the Gamma function $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$ allowing to find the value of $\Gamma(-1/2) = -2\pi^{1/2}$, we obtain $D^{1/2}g(t) = \pi^{1/2}/2$, $D^{3/2}g(t) \equiv 0$, but $D^{1/2}D^{3/2}g(t) \equiv 0$, $D^{3/2}D^{1/2}g(t) = -t^{-3/2}/4$ and $D^{1/2+3/2}g(t) = D^{2}g(t) = -t^{3/2}/4$.

4. APPLICATIONS

4.1. Eigenfunction

The eigenfunctions of the fractional derivatives D^{α} are defined as the solutions of the fractional differential equation

$$D^{\alpha}f(t) = \lambda f(t), \tag{5.1}$$

where λ is the eigenvalue.

The solution of equation (5.1) can be easily found by means of the Laplace transform

$$L[f(t)] \equiv F(s) = \int_0^{\infty} f(t)\exp(-st)dt\,. \tag{5.2}$$

The Laplace transform of $D^{\alpha}f(t)$ is simply $s^{\alpha}F(s)$. Applying (5.2) to equation (5.1), the solution in the imaginary space reads

$$F(s) = 1/(s^{\alpha} - \lambda). \tag{5.3}$$

Using equation (A11d), the inverse Laplace transform gives the solution of equation (5.1):

$$f(t) = t^{\alpha-1}E_{\alpha,\,\alpha}(\lambda t^{\alpha}), \tag{5.4}$$

where $E_{\alpha,\,\beta}(t)$ is the generalized Mittag-Leffler function [see (A11b)] for the definition).

4.2. Summation of series

The series

$$S = t^{1/2}/\Gamma(3/2) + t^{3/2}/\Gamma(5/2) + t^{5/2}/\Gamma(7/2) + \ldots \tag{5.5}$$

can be obtained from the following expansion

$$e^t = 1/\Gamma(1) + t/\Gamma(2) + t^2/\Gamma(3) + \ldots \qquad (5.6)$$

by applying the operator $I^{1/2}$ to the latter. Using result from Table 3, the sum S reads

$$S = I^{1/2}e^t = e^t \mathrm{erf}\, t^{1/2}, \qquad (5.7)$$

It should be noted, that applied the fractional integration for the series summation is advisable when the coefficients of expansion contain Gamma-function of a fractional argument. This often occurs in the theory of heat- and mass-transfer.

4.3. Oscillation of a particle in symmetrical potential well

As was shown early, the coordinate x of a particle depends on potential energy U by the following manner

$$x(U) = (8\pi m)^{1/2} I^{1/2} T(E), \qquad (5.8)$$

where $T(E)$ is the period of oscillation as a function of the particle total energy E.

Taking the fractional derivative $D^{1/2}$ from equation (5.8) yields

$$T(U) = (8\pi m)^{-1/2} D^{1/2}\, x(E). \qquad (5.9)$$

4.4. Rheological laws

Rheology is the study of the deformation and flow of matter under the influence of an applied stress, which might be, for example, a shear stress or extensional stress. The experimental characterization of a material's rheological behavior is known as *rheometry*, although the term *rheology* is frequently used synonymously with rheometry, particularly by experimentalists. Theoretical aspects of rheology are the relation of the flow/deformation behavior of material and its internal structure (e.g. the orientation and elongation of polymer molecules), and the flow/deformation behavior of materials that cannot be described by classical fluid mechanics or elasticity. This is also often called *non-Newtonian fluid mechanics* in the case of fluids. Highly elastic strain of polymers can be described by the following rheological law

$$y(t) = \text{const}\, I^{\nu} F(t),\ 0 < \nu < 1, \tag{5.10}$$

where y is the strain, $F(t)$ is the stress.

4.5. Heat and mass flux determination

Let us consider the heat of the semi-infinite area with the initial temperature be equal to zero. The mathematical statement of the problem is as follows:

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) T = 0,\ 0 < x < \infty, \tag{5.11}$$

$$T(0, t) = T_s(t), \tag{5.12}$$

$$T(\infty, t) = 0, \tag{5.13}$$

$$T(x, 0) = 0. \tag{5.14}$$

Factorizing equation (5.11) yields

$$\left(\frac{\partial^{1/2}}{\partial t^{1/2}} - \frac{\partial}{\partial x}\right)\left(\frac{\partial^{1/2}}{\partial t^{1/2}} + \frac{\partial}{\partial x}\right)T = 0. \tag{5.15}$$

Let us consider the equation formed by the right multiplier of the differential operator

$$\left(\frac{\partial^{1/2}}{\partial t^{1/2}} + \frac{\partial}{\partial x}\right)T = 0. \tag{5.16}$$

It should be noted here that the solution of the equation formed by the left multiplier

$$\left(\frac{\partial^{1/2}}{\partial t^{1/2}} - \frac{\partial}{\partial x}\right)T = 0, \tag{5.17}$$

does not satisfy the condition given by equation (5.13). To show this, applying the Laplace transform to (5.17) results in the ordinary differential equation of the first order:

$$d\Im/dx = s^{1/2}\Im, \tag{5.18}$$

where $\Im(x, s) = L[T(x, t)]$ is the Laplace transform of T.

By the separation of variables, the solution of equation (5.18) can be easily found:

$$\Im = \text{const} \times \exp\,(s^{1/2}x). \tag{5.19}$$

As x turns to the infinity, $\Im$ turns to the infinity as well.

Coming back to (5.16), let us write this equation at $x = 0$:

$$\frac{\partial^{1/2}}{\partial t^{1/2}} T(0, t) = D^{1/2} T_s(t) = -\frac{\partial}{\partial x} T\bigg|_{x=0} = q_s, \tag{5.20}$$

where q_s is the heat flux at $x = 0$.

Thus, the heat flux has been found without knowing the spatial temperature distribution. Accounting for equations (3.2a) and (4.8), the heat flux at $x = 0$ can be written in the explicit form

$$q_s = \frac{1}{\sqrt{\pi}} \frac{d}{dt} \int_0^t \frac{T_s(\tau) d\tau}{\sqrt{t - \tau}} . \tag{5.21}$$

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APPENDIX I. General properties of semidifferentiation and semiintegration

A binomial coefficient $\binom{q}{j} = (-1)^j \dfrac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}$, where j is a nonnegative integer and q may take any value.

Table 1. Values of $\binom{1/2}{j}, \binom{-1/2}{j}$ and their cumulative sums.

J	$\binom{1/2}{J}$	$\sum_{j=0}^{J}\binom{1/2}{j}$	$\binom{-1/2}{J}$	$\sum_{j=0}^{J}\binom{-1/2}{j}$
0	1	1	1	1
1	1/2	3/2	– 1/2	1/2
2	– 1/8	11/8	3/8	7/8
3	1/16	23/16	– 5/16	9/16
4	– 5/128	179/128	35/128	107/128
5	7/256	365/256	– 63/256	151/256
$\to \infty$	$\dfrac{(-1)^{J+1}}{2\sqrt{\pi}J^{3/2}}$	$\to 2$	$\dfrac{(-1)^{J}}{\sqrt{\pi J}}$	$\to 2^{-1/2}$

Table 2. General properties

f	$D^{1/2}f$	$I^{1/2}f$
$f_1 \pm f_2$	$D^{1/2}f_1 \pm D^{1/2}f_2$	$I^{1/2}f_1 + I^{1/2}f_2$
Cf	$C\,D^{1/2}f$	$C\,I^{1/2}f$
$f(kt)$	$k^{1/2}\,{D_{kt}}^{1/2}f(kt)$	$k^{-1/2}{I_{kt}}^{1/2}f(kt)$
$tf(t)$	$tD^{1/2}f + 0.5I^{1/2}f$	$tI^{1/2}f - 0.5I^{3/2}f$
df/dt	$D^{3/2}f - 0.5t^{-3/2}f(0)/\pi^{1/2}, f(0) \neq \infty$	$f(0)/\,\pi^{1/2} + D^{1/2}f, f(0) \neq \infty$

Table 3. Exponential, related function and special functions(S2)

f	$D^{1/2}f$	$I^{1/2}f$
$\exp(t)$	$(\pi t)^{-1/2} + \exp(t)\mathrm{erf}\,(t^{1/2})$	$\exp(t)\mathrm{erf}\,(t^{1/2})$
$\exp(-t)$	$(\pi t)^{-1/2} - 2\pi^{-1/2}D(t^{1/2})$	$2\pi^{-1/2}D(t^{1/2})$
$\exp(t)\mathrm{erf}\,(t^{1/2})$	$\exp(t)$	$\exp(t) - 1$
$\mathrm{Daw}(t^{1/2})$	$0.5\pi^{1/2}\exp(-t)$	$0.5\pi^{1/2}[1 - \exp(-t)]$
$\exp(t)\mathrm{erfc}\,(t^{1/2})$	$(\pi t)^{-1/2} - \exp(t)\mathrm{erf}\,(t^{1/2})$	$1 - \exp(t)\mathrm{erf}\,(t^{1/2})$
$\exp(t)\mathrm{erfc}\,(-\,t^{1/2})$	$(\pi t)^{-1/2} + \exp(t)\mathrm{erf}\,(-\,t^{1/2})$	$\exp(t)\mathrm{erf}\,(-\,t^{1/2}) - 1$

APPENDIX II. Special functions closely related to the fractional differentiation and integration.

$K(x)$ and $E(x)$ denote the complete elliptic integrals of the first

(A1) $$K(x) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - x\sin^2\theta}}$$

and second

(A2) $$E(x) = \int_0^{\pi/2} \sqrt{1 - x\sin^2\theta}\, d\theta$$

kinds.

The incomplete gamma function

(A3) $$\gamma^*(\nu, z) = e^{-z} \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\nu + n + 1)}.$$

If Re $z > 0$, then $\gamma^*(\nu, z)$ has the integral representation

(A4) $$\gamma^*(\nu, z) = \frac{1}{\Gamma(\nu) z^{\nu}} \int_0^z t^{\nu-1} e^{-t} dt.$$

Useful properties of the incomplete gamma function are its recursion

(A4a) $$\gamma^*(\nu - 1, z) = z\gamma^*(\nu, z) + \exp(-z)/\Gamma(\nu),$$

and its value

(A4b) $$\gamma^*(1/2, z) = \mathrm{erf}(z^{0.5})/z^{0.5}.$$

The closely related is the $E_z(\nu, a)$ function

(A5) $$E_z(\nu, a) = z^{\nu} e^{az} \gamma^*(\nu, z).$$

The error function is defined as

(A5) $\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-t^2/2\right) dt$

and in term of the incomplete gamma function

(A6) $\operatorname{erf} x = x\gamma^*(1/2, x^2)$.

Since erf ∞ = 1, the complementary error function is defined as

(A7) $\operatorname{erfc} x = 1 - \operatorname{erf} x$.

The Fresnel integrals are

(A8) $S(x) = \int_0^x \sin(\pi t^2/2) dt$.

(A9) $C(x) = \int_0^x \cos(\pi t^2/2) dt$.

The generalized hypergeometric series ${}_pF_q$ is defined as

(A10) $${}_pF_q(a_1,\ldots, a_p, b_1,\ldots, b_q; z) = \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)} \sum_{k=0}^{\infty} \frac{\Gamma(a_1+k)\cdots\Gamma(a_p+k)}{\Gamma(b_1+k)\cdots\Gamma(b_q+k)} \frac{z^k}{k!}.$$

We adopt the abbreviation symbolism

(A10a) $$\left[x \frac{a_1, a_2, \ldots, a_K}{b_1, b_2, \ldots, b_L} \right] = \sum_{i=0}^{\infty} x^i \frac{\prod_{k=1}^{K} \Gamma(i+1+a_k)}{\prod_{l=1}^{L} \Gamma(i+1+b_l)}$$

for what shall term a $\frac{K}{L}$ hypergeometric. Some familiar functions which are instances of such hypergeometrics include

$$\left[x \frac{}{} - \right] = \frac{1}{1-x},$$

$$\left[x \frac{}{0} \right] = \exp(x),$$

$$\left[x\frac{}{c}\right] = \exp(x)\gamma^*(c, x),$$

$$x^{1/2}\left[x\frac{}{1/2}\right] = \exp(x)\mathrm{erf}\,(x^{1/2}),$$

$$0.5\pi^{1/2}\left[-x\frac{}{1/2}\right] = \mathrm{Daw}(x^{1/2}),$$

where Daw(x) is Dawson's integral: Daw(x) = $\exp(-x^2)\int_0^x \exp(t^2)dt$.

The Mittag-Leffler function $E_\alpha(x)$ and generalized Mittag-Leffler function $E_{\alpha,\beta}(x)$

(A11a) $E_\alpha(x) = \sum_0^\infty \frac{x^n}{\Gamma(\alpha n+1)}$, $\alpha > 0$,

(A11b) $E_{\alpha,\beta}(x) = \sum_0^\infty \frac{x^n}{\Gamma(\alpha n+\beta)}$, $\alpha > 0$.

The Laplace transforms of the Mittag-Leffler functions $E_\alpha(x)$ and $E_{\alpha,\beta}(x)$ are

(A11c) $L[E_\alpha(-\lambda x^\alpha)] = s^\alpha/(s^\alpha + \lambda)$,

(A11d) $L[x^{\beta-1}E_{\alpha,\beta}(-\lambda x^\alpha)] = s^{\alpha-\beta}/(s^\alpha + \lambda)$,

The Mittag-Leffler function $E_\alpha(x)$ provides a simple generalization of the exponential function because of the substitution of $n! = \Gamma(n + 1)$ with $(\alpha n)! = \Gamma(\alpha n + 1)$. Particular cases of (A11), from which elementary functions are recovered, are

(A12) $E_2(+x^2) = \cosh x$, $E_2(-x^2) = \cos x$,

(A13) $E_{1/2}(\pm x^{1/2}) = e^x[1 + \mathrm{erf}\,(\pm x^{1/2})]$.

SUPPLEMENT

S1. These fractional integral and derivative nowadays is known as the *Riemann- Liouville* (R-L) ones. On the other hand, the fractional derivative of order α in the *Caputo* sense is defined as the operator ${}_*D^{\alpha}$ such that

(S1.1) $${}_*D^{\alpha}f(t) := I^{m-\alpha}D^{m}f(t),\ m-1 < \alpha \le m,$$

where m is the positive integer.

This implies

(S1.2) $${}_*D^{\alpha}f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t \dfrac{f^{(m)}(\tau)d\tau}{(t-\tau)^{\alpha+1-m}}, & m-1<\alpha<m; \\ \dfrac{d^m}{dt^m}f(t), & \alpha = m. \end{cases}$$

The R-L derivative for $\alpha > 1$ reads

(S1.3) $$D^{\alpha}f(t) = \begin{cases} \dfrac{d^m}{dt^m}\left[\dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t \dfrac{f(\tau)d\tau}{(t-\tau)^{\alpha+1-m}}\right], & m-1<\alpha<m; \\ \dfrac{d^m}{dt^m}f(t), & \alpha = m. \end{cases}$$

It should be point out that the *Caputo* fractional derivative satisfies the relevant property of being zero when applied to a constant.

There exists the essential relationship between the two fractional derivatives for the same non-integer order

(S1.4) $${}_*D^{\alpha}f(t) = D^{\alpha}f(t) - \sum_{k=0}^{m-1}\frac{f^{(k)}(0^+)t^{k-\alpha}}{\Gamma(k-\alpha+1)},\ m-1 < \alpha < m.$$

In particular

(S1.5) $$ {}_*D^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f^{(1)}(\tau)}{(t-\tau)^{\alpha}}d\tau = D^{\alpha}[f(t) - f(0^+)] = $$

$$ = D^{\alpha}f(t) - f(0^+)t^{-\alpha}/\Gamma(1-\alpha),\ 0 < \alpha < 1. $$

In the special case $f^{(k)}(0^+) = 0$ for $k = 0, 1, \ldots, m{-}1$, the two fractional derivatives coincide.

S2. Every function which is formed from the elementary functions by means of a closed expression*, can be differentiated, and its derivative, if it is also a closed expression, formed from the elementary functions. As can be seen from Appendix I and II, formally, introducing the operation of the fractional differentiation supplements the set of elementary functions with the special ones.

*By this we mean a function which can be built up from the elementary functions by repeated application of the rational operations and the processes of compounding and inversion.