



INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

**Approximate modularity: Kalton's  
constant is not smaller than 3**

*Michał Gnacik*

*Marcin Guzik*

*Tomasz Kania*

Preprint No. 14-2020

PRAHA 2020



# APPROXIMATE MODULARITY: KALTON'S CONSTANT IS NOT SMALLER THAN 3

MICHAŁ GNACIK, MARCIN GUZIK, AND TOMASZ KANIA

ABSTRACT. Kalton and Roberts [*Trans. Amer. Math. Soc.*, 278 (1983), 803–816] proved that there exists a universal constant  $K \leq 44.5$  such that for every set algebra  $\mathcal{F}$  and every 1-additive function  $f: \mathcal{F} \rightarrow \mathbb{R}$  there exists a finitely additive signed measure  $\mu$  defined on  $\mathcal{F}$  such that  $|f(A) - \mu(A)| \leq K$  for any  $A \in \mathcal{F}$ . The only known lower bound for the optimal value of  $K$  was found by Pawlik [*Colloq. Math.*, 54 (1987), 163–164], who proved that this constant is not smaller than 1.5; we improve this bound to 3 already on a non-negative 1-additive function. Recently, Feige, Feldman, and Talgam-Cohen decreased an upper estimate for  $K$  to 24 [*SIAM J. Comput.*, 49 (2020), 67–97] and drew a connection between better estimation of Kalton's constant and enhancing various optimisation algorithms; we improve another constant related to approximately modular functions considered *ibid.*

## 1. INTRODUCTION

Let  $\mathcal{F}$  be a set algebra and  $\Delta \geq 0$ . A function  $f: \mathcal{F} \rightarrow \mathbb{R}$  is  $\Delta$ -additive, whenever  $f(\emptyset) = 0$  and

$$|f(A) + f(B) - f(A \cup B)| \leq \Delta \quad (A, B \in \mathcal{F}, A \cap B = \emptyset).$$

Quite clearly, 0-additive maps are nothing but signed, finitely-additive measures on  $\mathcal{F}$ . Kalton and Roberts proved in [4] a rather surprising stability theorem for  $\Delta$ -additive maps, which asserts that there exists a universal constant (we follow Pawlik's convention [8] and refer to it as *Kalton's constant*)  $K \leq 44.5$  (independent of the choice of  $\mathcal{F}$ ) such that for every  $\Delta$ -additive function  $f: \mathcal{F} \rightarrow \mathbb{R}$  there exists a (signed, finitely-additive) measure  $\mu: \mathcal{F} \rightarrow \mathbb{R}$  such that

$$(1.1) \quad \sup_{A \in \mathcal{F}} |f(A) - \mu(A)| \leq K \cdot \Delta.$$

In 2014, Bondarenko, Prymak, and Radchenko decreased the upper bound for  $K$  from 44.5 to 38.8 (see [2, Proof of Corollary 1.2]).

Results of this kind (that is, including ours) are of importance in functional analysis, for example, in the theory of twisted sums of (quasi-)Banach spaces and certain stability problems of vector measures [4, 5]. Improving Kalton's constant may likely fine-tune

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*Date:* March 3, 2020.

*2010 Mathematics Subject Classification.* 28A60 (primary), and 39B82, 90C27, 94C10 (secondary).

*Key words and phrases.* 1-additive set function, Kalton's constant, Ulam–Hyers stability, approximate modularity.

various optimisation algorithms in machine learning and algorithmic game theory (see [3, Section 1.2] and references therein for more details). Moreover, recently there have been efforts to extend the validity of the Kalton–Roberts theorem to lattices [1].

An analogous and closely related stability problem for  $\varepsilon$ -modular ( $\varepsilon > 0$ ) set functions was recently studied by Feige, Felfman, and Talgam-Cohen [3], which led to a further improvement for an upper estimate of Kalton’s constant. In order to present it, we require a piece of notation.

Let  $f: \mathcal{F} \rightarrow \mathbb{R}$  be a function such that  $f(\emptyset) = 0$ . Then,  $f$  is additive (that is, it is a finitely additive signed measure) if and only if it satisfies the *modular identity*:

$$f(A) + f(B) = f(A \cup B) + f(A \cap B) \quad (A, B \in \mathcal{F}).$$

Functions that assume a possibly non-zero value at the empty set and that satisfy the modular identity are for this reason called *modular*. For  $\varepsilon > 0$ , a function  $f: \mathcal{F} \rightarrow \mathbb{R}$  is then termed  $\varepsilon$ -*modular*, whenever

$$(1.2) \quad |f(A) + f(B) - f(A \cup B) - f(A \cap B)| \leq \varepsilon \quad (A, B \in \mathcal{F}).$$

Also,  $f$  is said to be *weakly- $\varepsilon$ -modular*, whenever (1.2) is satisfied for every sets  $A, B$  so that  $A \cap B = \emptyset$ , in particular, if  $f(\emptyset) = 0$ , then the properties of being weakly- $\varepsilon$ -modular and  $\varepsilon$ -additive are equivalent. Moreover, every weakly- $\varepsilon$ -modular function is  $2\varepsilon$ -modular (see [3, Proposition 2.1]).

The main results in [3] state that there are universal constants  $K_s < 12.62$  (*the strong Kalton constant*) and  $K_w < 24$  (*the weak Kalton constant*) so that for every  $\varepsilon$ -modular function  $f$  there is a modular function  $g_1$  so that

$$(1.3) \quad \sup_{A \in \mathcal{F}} |f(A) - g_1(A)| \leq \varepsilon K_s,$$

and for every weakly- $\varepsilon$ -modular function  $h$  there is an additive function  $g_2$  so that

$$(1.4) \quad \sup_{A \in \mathcal{F}} |h(A) - g_2(A)| \leq \varepsilon K_w.$$

It is also worth emphasising the inequalities between  $K_s$  and  $K_w$  [3, Corollary 2.7]), namely

$$(1.5) \quad \frac{1}{2}K_w \leq K_s \leq K_w.$$

*Remark.* Clearly, if  $f$  is  $\varepsilon$ -additive, then it is weakly- $\varepsilon$ -modular and the converse is not true as  $f(\emptyset)$  may be non-zero. However if  $f$  is weakly- $\varepsilon$ -modular and  $f(\emptyset) = a \neq 0$ , then by shifting we get  $g = f - a \cdot \mathbf{1}_{\mathcal{F}}$  which is  $\varepsilon$ -additive, as for any  $A, B \in \mathcal{F}$

$$\begin{aligned} |g(A) + g(B) - g(A \cup B)| &= |f(A) - a + f(B) - a - f(A \cup B) + a| \\ &= |f(A) + f(B) - f(A \cup B) - f(A \cap B)| \\ &\leq \varepsilon. \end{aligned}$$

Similarly, a (signed, finitely-additive) measures are modular, and starting with a modular function we can easily construct a measure (by shifting), thus (1.1) is equivalent to (1.4) Hence,  $K = K_w$ , that is the weak Kalton constant in the sense of [3] is what we call (after Pawlik) Kalton’s constant.

**Lower bounds.** The results concerning estimating  $K = K_w$  and  $K_s$  from below have been so far rather scarce. In 1987, Pawlik published a paper [8], where Kalton's constant  $K_w$  was estimated from below by  $3/2$ . Recently, his result has been reviewed in [3, Appendix A, Appendix C]. Moreover, Feige *et al.* have proved that  $K_s \geq 1$  [3, Theorem 1.2]. In our paper, we improve the lower estimate to  $K_s \geq 1.5$ .

## 2. MAIN RESULTS

The aim of the present paper is to improve known lower bounds on Kalton's constants by obtaining the following inequality.

**Main Theorem.**  $K = K_w \geq 3$ .

Our Main Theorem and relation (1.5) immediately let us improve a lower bound on the strong Kalton constant too, namely:

**Corollary.**  $K_s \geq \frac{3}{2}$ .

In order to prove the Main Theorem, we require the following fact. Let  $\mathcal{F}_m$  be the power set of an  $m$ -element set (so that  $\mathcal{F}_m$  has  $2^m$  elements) and denote by  $K(m)$  the optimal Kalton constant for 1-additive functions defined on  $\mathcal{F}_m$  only. Then the sequence  $(K(m))_{m=1}^\infty$  is increasing and

$$K = \lim_{m \rightarrow \infty} K(m) = \sup_{m \in \mathbb{N}} K(m)$$

(This follows from a standard compactness argument; see the first paragraph of the proof of [4, Theorem 4.1] for details.) In other words, it is sufficient to work with finite set algebras.

## 3. PROOF OF THE MAIN RESULT

Let  $\Omega_{k,n}$  be a set of cardinality  $n \cdot k$ ; we write  $\Omega_{k,n}$  as the disjoint union of sets  $X_1, \dots, X_n$  each set having cardinality  $k$ , where  $n, k \geq 2$ . We define  $f_{k,n}$  by setting

- $f_{k,n}(\emptyset) = 0$ ;
- $f_{k,n}(A) = 3$  for every set  $A$  with  $A \cap X_j \neq \emptyset$  for all  $j \leq n$  and  $A \cap X_j = X_j$  for at least one  $j$ ;
- $f_{k,n}(B) = 1$  for all other sets  $B$ .

In particular,  $f_{k,n}(\Omega_{k,n}) = 3$ . It is a matter of direct verification that each function  $f_{k,n}$  is 1-additive and so weakly-1-modular, which yields also that  $f_{k,n}$  is 2-modular.

*Proof of the Main Theorem.* Let  $\mu_{k,n}$  be a measure that minimises the distance from  $f_{k,n}$  to the space of measures on  $\Omega_{k,n}$ . Choose indices  $i_k^1, \dots, i_k^n$  that realise  $\gamma_{k,n}^1, \dots, \gamma_{k,n}^n$ , where

$$\gamma_{k,n}^j = \min_{i \in X_j} |\mu_{k,n}(\{i\})| \quad (j = 1, \dots, n).$$

We *claim* that for all  $j$  and  $n$  we have  $\gamma_{k,n}^j \rightarrow 0$  as  $k \rightarrow \infty$ . Assume not. Then  $\gamma_{k,n}^j \geq \gamma$  for some  $\gamma > 0$  and almost all  $k$ . If the sequence  $(\mu_k(\Omega_{k,n}))_{k=4}^\infty$  were unbounded, the theorem would have been proved, so we may assume it is bounded. Let

$$M = \sup_{k,n} |\mu_{k,n}(\Omega_{k,n})|$$

(of course, it follows from the Kalton–Roberts theorem that  $M \leq 89/2 + 3$ , but there is no need to invoke such a deep result here.) As  $k$  increases, the number of those  $i \in X_j$  for which  $\mu_k(\{i\})$  are either all positive or all negative increases to infinity; let  $A_k$  denote the subset of  $X_k$  comprising such elements of the same sign. In particular,

$$|\mu_{k,n}(A_k)| \geq \gamma \cdot |A_k| \rightarrow \infty$$

as  $k \rightarrow \infty$ ; a contradiction.

Let us note that

$$\begin{aligned} n \cdot K(k \cdot n) &\geq n \cdot \sup_{A \in \mathcal{F}_k} |f_{k,n}(A) - \mu_{k,n}(A)| \\ &\geq \sum_{j=1}^n |f_{k,n}(X_j \cup \{i_k^\ell : \ell \neq j\}) - \mu_{k,n}(X_j) - \mu_{k,n}(\{i_k^\ell : \ell \neq j\})| \\ &= \sum_{j=1}^n |3 - \mu_{k,n}(X_j) - \mu_{k,n}(\{i_k^\ell : \ell \neq j\})| \\ &\geq \sum_{j=1}^n (3 - \mu_{k,n}(X_j) - \mu_{k,n}(\{i_k^\ell : \ell \neq j\})) \\ &= 3n - \mu_{k,n}(\Omega_{k,n}) - \sum_{j=1}^n \mu_{k,n}(\{i_k^\ell : \ell \neq j\}). \end{aligned}$$

We have

$$K(k \cdot n) \geq 3 - \frac{1}{n} \mu_k(\Omega_{k,n}) - \frac{1}{n} \sum_{j=1}^n \mu_{k,n}(\{i_k^\ell : \ell \neq j\}),$$

which shows that

$$\begin{aligned} K &\geq 3 - \frac{1}{n} \limsup_{k \rightarrow \infty} \mu_k(\Omega_{k,n}) - \frac{1}{n} \limsup_{k \rightarrow \infty} \sum_{j=1}^n \mu_{k,n}(\{i_k^\ell : \ell \neq j\}) \\ &= 3 - \frac{1}{n} \limsup_{k \rightarrow \infty} \mu_k(\Omega_{k,n}) - \frac{1}{n} \limsup_{k \rightarrow \infty} \sum_{j=1}^n \sum_{\ell \neq j} \gamma_{k,n}^\ell \\ &= 3 - \frac{1}{n} \limsup_{k \rightarrow \infty} \mu_k(\Omega_{k,n}) \\ &\geq 3 - \frac{M}{n} \end{aligned}$$

because

$$\sum_{j=1}^n \sum_{\ell \neq j} \gamma_{k,n}^\ell \rightarrow 0$$

as  $k \rightarrow \infty$  (and  $n$  is fixed).

□

#### 4. THE CONSTANTS $K(m)$

The proof of the Main Theorem has an asymptotic nature as it involves all constants  $K(m)$  at once. It would be thus desirable to find lower (and upper) estimates for  $K(m)$ s as well. This can be achieved by estimating the distances of the functions appearing in the proof of the Main Theorem to the space of measures.

We start with the following lemma, which asserts that it is always possible to find a measure minimising the distance to  $f_{k,n}$  that is constant on singletons from the respective partitions. (As the supremum norm that we consider here is not strictly convex, there is no guarantee for the uniqueness of the element that minimises a distance to a subspace.)

For  $n, k \in \mathbb{N}$ , denote by  $\mathcal{F}_{k,n}$  the power-set of  $\Omega_{k,n}$  and let  $\mathcal{S}_{k,n}$  be the set of all self-bijections of  $\Omega_{k,n}$  that leave each set  $X_j$  invariant ( $j \leq n$ ). Then  $\mathcal{S}_{k,n}$  has exactly  $(k!)^n$  elements.

**Lemma 1.** *Let  $n, k \in \mathbb{N}$ . Then there exists a measure  $\nu$  that minimises the distance from  $f_{k,n}$  to the space of measures on  $\Omega_{k,n}$  with the property for every  $j \leq n$  the function  $x \mapsto \nu(\{x\})$  is constant on  $X_j$ .*

*Proof.* Let  $\mu$  be any measure that minimises the distance from  $f_{k,n}$  to the space of measures. For any self-bijection  $\sigma$  of  $\Omega_{k,n}$ , the composition  $\mu \circ \sigma$  defines a measure again. Let us observe that the measure

$$\nu = \frac{1}{(k!)^n} \sum_{\sigma \in \mathcal{S}_{k,n}} \mu \circ \sigma$$

has the desired properties. Indeed, it is clear that the function  $x \mapsto \nu(\{x\})$  is constant on the respective sets  $X_j$  ( $j \leq n$ ) as we consider only bijections that leave each set  $X_j$  invariant. Let then prove that  $\nu$  also minimises the distance to the space of measures. Indeed, we have

$$\begin{aligned} \sup_{A \in \mathcal{F}_{k,n}} \left| f_{k,n}(A) - \frac{1}{(k!)^n} \sum_{\sigma \in \mathcal{S}} (\mu \circ \sigma)(A) \right| &\leq \sup_{A \in \mathcal{F}_{k,n}} \frac{1}{(k!)^n} \sum_{\sigma \in \mathcal{S}} |f_{k,n}(A) - (\mu \circ \sigma)(A)| \\ &\leq \frac{1}{(k!)^n} \sum_{\sigma \in \mathcal{S}} \sup_{A \in \mathcal{F}_{k,n}} |f_{k,n}(A) - (\mu \circ \sigma)(A)| \\ &= \sup_{A \in \mathcal{F}_{k,n}} |f_{k,n}(A) - \mu(A)|. \end{aligned}$$

As  $\mu$  was chosen to minimise the distance, the proof is complete.

□

4.1. **The case  $n = 2$ .** Let  $n = 2$ ,  $k \in \mathbb{N}$ , and let  $\nu$  be a measure as in the statement of Lemma 1. In that case,  $\Omega_{k,2} = X_1 \cup X_2$ . Denote  $x = \nu(X_1)$  and  $y = \nu(X_2)$ . In this case, we have essentially three types of sets to consider:

- $X_1 \cup \{\omega_2\}$ , where  $\omega_2 \in X_2$ ,
- $X_2 \cup \{\omega_1\}$ , where  $\omega_1 \in X_1$ ,
- $\Omega_{k,2} \setminus \{\omega_1, \omega_2\}$ , where  $\omega_1 \in X_1$  and  $\omega_2 \in X_2$ .

Thus, we seek to minimise the following expressions simultaneously:

$$\left| x + \frac{y}{k} - 3 \right|, \left| y + \frac{x}{k} - 3 \right|, \left| x + y - \frac{x+y}{k} - 1 \right|$$

with respect to  $(x, y)$ . We then arrive at the following set of equations:

$$\begin{cases} \frac{k-1}{k}(x+y) - 1 = 3 - x - \frac{y}{k} \\ \frac{k-1}{k}(x+y) - 1 = 3 - y - \frac{x}{k}, \end{cases}$$

which has the unique solution:

$$\begin{cases} x = \frac{4k}{3k-1} \\ y = \frac{4k}{3k-1}. \end{cases}$$

In that case, the lower estimates for  $K(m)$  are suboptimal as asymptotically they yield the inequality  $K \geq 5/3$ .

4.2. **The case  $n \geq 3$ .** Analogously to the case  $n = 2$ , let us denote  $x_i = \nu(X_i)$  for  $i = 1, \dots, n$ . For every  $j \leq n$ , let us pick  $\omega_j \in X_j$  ( $j \leq n$ ). By Lemma 1, we may restrict our attention to measures that assume equal values on singletons from the respective sets  $X_j$ . In other words, it is enough to consider the following sets:

- $\Omega_{k,n} \setminus \{\omega_\ell : \ell \leq n\}$ ;
- $X_j \cup \{\omega_\ell : \ell \leq n, \ell \neq j\}$  ( $j \leq n$ ).

Thus, this time, we seek to minimise the following expressions simultaneously:

$$\left| \sum_{\ell \leq n} x_\ell - \frac{\sum_{\ell \leq n} x_\ell}{k} - 1 \right|, \left| x_j + \sum_{\ell \neq j} \frac{x_\ell}{k} - 3 \right| \quad (j \leq n)$$

with respect to  $(x_1, x_2, \dots, x_n)$ . In particular, for  $j \leq n$ , we have

$$(4.1) \quad \frac{k-1}{k} \sum_{\ell \leq n} x_\ell - 1 = 3 - x_j - \sum_{\ell \neq j} \frac{x_\ell}{k}.$$

The sum  $t = \sum_{\ell \leq n} x_\ell$  may be then computed by adding these equations together. More specifically,  $t = \frac{4nk}{(n+1)k-1}$ . It follows from (4.1) that for any  $j \leq n$  we have

$$\frac{k-1}{k} t - 1 = 3 - \frac{k-1}{k} x_j - \frac{t}{k}.$$



Finally, for every  $j \leq n$ , we have

$$x_j = \frac{k}{k-1} \left( 4 - \frac{4nk}{(n+1)k-1} \right) = \frac{4k}{(n+1)k-1}.$$

Since the double sequence

$$a_{k,n} = 3 - \frac{4k}{(n+1)k-1} - \frac{n-1}{k} \frac{4k}{(n+1)k-1}$$

converges to 3 as  $k, n \rightarrow \infty$ , we may estimate  $K(m)$  from below by  $a_{k,n}$ , where  $k, n$  are such that  $m = k \cdot n$ .

In particular, we could restrict, for example, to  $n = k$ . In this case, for  $n = k = 10$ , we get approximately 2.305, for  $n = k = 20$ , it is approximately 2.628, and for  $n = k = 200$ , we arrive at 2.96.

## 5. CLOSING REMARKS

Feige, Feldman, and Talgam-Cohen remarked that *obtaining good lower bounds on  $K_s$  is also not easy. Part of the difficulty is that even if one comes up with a function  $f$  that is a candidate to yield the lower bound, verifying that it is  $\varepsilon$ -modular involves checking roughly  $2^{2^n}$  approximate modularity equations* [3, p. 69].

For this reason, we have found a suitable candidate for the function(s)  $f_{k,2}$  using a Python script, which gave us a lower estimate of  $5/3$  for  $K$ . Subsequently, we added more degrees of freedom (by defining  $f_{k,n}$ ) in an analogous manner. Let us briefly explain our approach, which would probably make the proof of the main result less *ad hoc*.

We consider the sets  $\Omega_{k,2}$  for  $k \in \mathbb{N}$ . Let  $A = [a_{ij}]_{i,j=1}^3$  be a real matrix. We then define a function  $f: \mathcal{F}_{k,2} \rightarrow \mathbb{R}$  by asserting that

$$\begin{aligned} f_k(\emptyset) &= 0, & f_k(Y') &= a_{12}, & f_k(X_2) &= a_{13}, \\ f_k(X') &= a_{21}, & f_k(X' \cup Y') &= a_{22}, & f_k(X' \cup X_2) &= a_{23}, \\ f_k(X_1) &= a_{31}, & f_k(X_1 \cup Y') &= a_{32}, & f_k(X_1 \cup X_2) &= a_{33} \end{aligned}$$

as long as  $X', Y'$  are proper, non-empty subsets of  $X_1$  and  $X_2$ , respectively.

**Lemma 2.** *The function  $f$  is 1-additive (weakly-1-modular) if and only if the following conditions are satisfied:*

- (i)  $|a| \leq 1$  for  $a \in \{a_{12}, a_{21}, a_{22}\}$ .
- (ii)  $|2a_{12} - a_{13}| \leq 1$ ,
- (iii)  $|2a_{21} - a_{31}| \leq 1$ ,
- (iv)  $|a_{13} + a_{31} - a_{33}| \leq 1$ ,
- (v)  $|a_{12} + a_{21} - a_{22}| \leq 1$ ,
- (vi)  $|2a_{22} - b| \leq 1$  for  $b \in \{a_{23}, a_{32}, a_{33}\}$ ,
- (vii)  $|a_{12} + a_{22} - a_{23}| \leq 1$ ,
- (viii)  $|a_{21} + a_{22} - a_{32}| \leq 1$ ,
- (ix)  $|a_{33} - c| \leq 1$  for  $c \in \{a_{32} + a_{12}, a_{23} + a_{21}\}$ .

*In particular,  $a_{13}, a_{31}, a_{23}, a_{32}, a_{33} \in [-3, 3]$ .*

*Proof.* Straightforward verification. □

Effectively, Pawlik’s construction corresponds to the matrix

$$\begin{bmatrix} 0 & -1 & -3 \\ 1 & 0 & -1 \\ 3 & 1 & 0 \end{bmatrix}.$$

Having implemented the conditions from Lemma 2 in Python, we run a simple script that listed for us *all* 1-additive functions of that form that take values from the list  $(-3, -2.5, -2, \dots, 2, 2.5, 3)$ . (By Lemma 2,  $-3$  and  $3$  are extremal values for the range of such functions.) Overall, we found in total 38,034 such functions that are non-zero (excluding those that differ only by the sign, we had only 19,017 functions to investigate after all). Using a convex optimisation solver SCS ([6, 7]), we filtered out those functions whose distance to the space of measures is at least 1.4 in the case  $k = 4$  (that is, functions on an 8-element set), having found only two:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 3 \\ 1 & 3 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 3 \\ 3 & 3 & 3 \end{bmatrix}.$$

Obviously, the former one corresponds to functions  $f_{k,2}$  that we consider in the present paper.

**Acknowledgements.** We are indebted to Tomasz Kochanek who has introduced to us the problem of improving the bounds for Kalton’s constant. The first-named author’s visit to Prague in December 2019, during which some part of the project was completed, was supported by funding received from GAČR project 19-07129Y; RVO 67985840, which is acknowledged with thanks. The research was partly supported by the Iuventus Plus grant IP2015 035174.

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SCHOOL OF MATHEMATICS AND PHYSICS, LION GATE BUILDING, LION TERRACE, UNIVERSITY OF PORTSMOUTH, PORTSMOUTH, UNITED KINGDOM

*Email address:* `micah.gnacik@port.ac.uk`

UBS BUSINESS SOLUTIONS, KRAKOWSKA 280, 32-080 ZABIERZÓW, POLAND

*Email address:* `marcin.guzik@ubs.com`

INSTITUTE OF MATHEMATICS, CZECH ACADEMY OF SCIENCES, ŽITNÁ 25, 115 67 PRAGUE 1, CZECH REPUBLIC AND INSTITUTE OF MATHEMATICS, JAGIELLONIAN UNIVERSITY, ŁOJASIEWICZA 6, 30-348 KRAKÓW, POLAND

*Email address:* `kania@math.cas.cz`, `tomasz.marcin.kania@gmail.com`