Solving ill posed problems (in fluid dynamics): Mathematics and numerics

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Prologue - Lax equivalence principle

Formulation for LINEAR problems

Peter D. Lax

- **Stability** - uniform bounds of approximate solutions
- **Consistency** - vanishing approximation error

$$\Longrightarrow$$

- **Convergence** - approximate solutions converge to exact solution

Euler system of gas dynamics

Leonhard Paul Euler 1707–1783

Equation of continuity – Mass conservation

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0, \ \mathbf{m} = \varrho \mathbf{u}$$

Momentum equation – Newton's second law

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = 0, \ p(\varrho) = a\varrho^\gamma$$

Impermeability and/or periodic boundary condition

$$\mathbf{u} \cdot \mathbf{n}|_{\partial \Omega} = 0, \ \Omega \subset R^d, \ \text{or } \Omega = \mathbb{T}^d$$

Initial state

$$\varrho(0, \cdot) = \varrho_0, \ \mathbf{m}(0, \cdot) = \mathbf{m}_0$$

Classical solutions

- **Local existence.** Classical solutions exist locally in time as long as the initial data are regular and the initial density strictly positive
- **Finite time blow–up.** Classical solutions develop singularity (become discontinuous) in a *finite* time for a fairly generic class of initial data

Mythology concerning Euler equations in several dimensions

- **Existence.** The long time existence of (possibly weak) solutions is not known
- **Uniqueness.** The is no (known) selection criterion to identify a unique solution (semiflow)
- **Computation.** Oscillatory solutions cannot be visualized by numerical simulation (weak convergence)

Weak (distributional) solutions

Jacques Hadamard 1865–1963

Laurent Schwartz 1915–2002

Mass conservation

$$\int_B \Big[\varrho(t_2,\cdot) - \varrho(t_1,\cdot)\Big] \mathrm{d}x = -\int_{t_1}^{t_2}\int_{\partial B} \varrho\mathbf{u}\cdot\mathbf{n}\ \mathrm{dS}_x \mathrm{d}t$$

$$\left[\int_\Omega \varrho\varphi\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \Big[\varrho\partial_t\varphi + \mathbf{m}\cdot\nabla_x\varphi\Big]\ \mathrm{d}x\mathrm{d}t,\ \ \mathbf{m}\equiv\varrho\mathbf{u}$$

Momentum balance

$$\int_B \Big[\mathbf{m}(t_2,\cdot) - \mathbf{m}(t_1,\cdot)\Big]\ \mathrm{d}x$$

$$= -\int_{t_1}^{t_2}\int_{\partial B}\Big[\mathbf{m}\otimes\mathbf{u}\cdot\mathbf{n} + p(\varrho)\mathbf{n}\Big]\ \mathrm{dS}_x\ \mathrm{d}t$$

$$\left[\int_\Omega \mathbf{m}\cdot\boldsymbol{\varphi}\ \mathrm{d}x\right]_{t=0}^{t=\tau}$$

$$= \int_0^\tau \int_\Omega \left[\mathbf{m}\cdot\partial_t\boldsymbol{\varphi} + \frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} : \nabla_x\boldsymbol{\varphi} + p(\varrho)\mathrm{div}_x\boldsymbol{\varphi}\right]\ \mathrm{d}x\mathrm{d}t$$

Time irreversibility – energy dissipation

Energy

$$\mathcal{E} = \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho),\ \ P'(\varrho)\varrho - P(\varrho) = p(\varrho)$$

$$p' \geq 0 \Rightarrow [\varrho, \mathbf{m}] \mapsto \begin{cases} \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \text{ if } \varrho > 0 \\ P(\varrho) \text{ if } |\mathbf{m}| = 0,\ \ \varrho \geq 0 \\ \infty \text{ otherwise} \end{cases} \quad \text{is convex l.s.c}$$

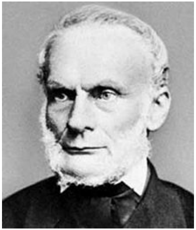
Energy balance (conservation)

$$\partial_t \mathcal{E} + \mathrm{div}_x \left(\mathcal{E}\frac{\mathbf{m}}{\varrho} \right) + \mathrm{div}_x \left(p\frac{\mathbf{m}}{\varrho} \right) = 0$$

Energy dissipation

$$\partial_t \mathcal{E} + \mathrm{div}_x \left(\mathcal{E}\frac{\mathbf{m}}{\varrho} \right) + \mathrm{div}_x \left(p\frac{\mathbf{m}}{\varrho} \right) \boxed{\leq} 0$$

$$E = \int_\Omega \mathcal{E}\ \mathrm{d}x,\ \ \partial_t E \leq 0,\ \ E(0+) = \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right]\ \mathrm{d}x$$



Rudolf Clausius 1822–1888

Wild solutions?

Charles Hermite [1822-1901]

In a letter to Stieltjes

I turn with terror and horror from this lamentable scourge of continuous functions with no derivatives

Known facts concerning global solvability

- Existence of infinitely many weak solution for any continuous initial data (Chiodaroli, DeLellis–Széhelyhidi, EF...)
- Existence of "many" initial data that give rise to infinitely many weak solutions satisfying the energy inequality (Chiodaroli, EF, Luo, Xie, Xin...)
- Existence of smooth initial data that ultimately give rise to infinitely many weak solutions satisfying the energy inequality (Kreml et al)
- Weak–strong uniqueness in the class of admissible weak solutions (Dafermos)

Ill posedness

Theorem [A.Abbatiello, EF 2019]

Anna Abbatiello (TU Berlin)

Let $d = 2, 3$. Let ϱ_0, $\mathbf{m}_0$ be given such that

$$\varrho_0 \in \mathcal{R},\ 0 \leq \underline{\varrho} \leq \varrho_0 \leq \overline{\varrho},$$

$$\mathbf{m}_0 \in \mathcal{R},\ \mathrm{div}_x \mathbf{m}_0 \in \mathcal{R},\ \mathbf{m}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Let $\{\tau_i\}_{i=1}^{\infty} \subset (0, T)$ be an arbitrary (countable dense) set of times.
Then the Euler problem admits infinitely many weak solutions ϱ, $\mathbf{m}$ with a strictly decreasing total energy profile such that

$$\varrho \in C_{\rm weak}([0, T]; L^{\gamma}(\Omega)),\ \mathbf{m} \in C_{\rm weak}([0, T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d))$$

but

$$t \mapsto [\varrho(t, \cdot), \mathbf{m}(t, \cdot)]\ \boxed{\text{is not}}\ \text{strongly continuous at any } \tau_i$$

FV numerical scheme

$$(\varrho_h^0, \mathbf{u}_h^0) = (\Pi_{\mathcal{T}} \varrho_0, \Pi_{\mathcal{T}} \mathbf{u}_0)$$

$$D_t \varrho_K^k + \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{|K|} F_h(\varrho_h^k, \mathbf{u}_h^k) = 0$$

$$D_t (\varrho_h^k \mathbf{u}_h^k)_K + \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{|K|} \left(\mathbf{F}_h(\varrho_h^k \mathbf{u}_h^k, \mathbf{u}_h^k) + \overline{p(\rho_h^k)} \mathbf{n} - h^\beta \left[\left[\mathbf{u}_h^k \right]\right] \right) = 0.$$

Mária Lukáčová (Mainz)

Hana Mizerová (Bratislava)

Discrete time derivative

$$D_t r_K^k = \frac{r_K^k - r_K^{k-1}}{\Delta t}$$

Upwind, fluxes

$$\mathrm{Up}[r, \mathbf{v}] = \overline{r}\ \overline{\mathbf{v}} \cdot \mathbf{n} - \frac{1}{2} |\overline{\mathbf{v}} \cdot \mathbf{n}|\, [[r]]$$

$$F_h(r, \mathbf{v}) = \mathrm{Up}[r, \mathbf{v}] - h^\alpha\, [[r]]$$

Consistent approximation

Equation of continuity

$$\int_0^T \int_\Omega \left[\varrho_n \partial_t \varphi + \mathbf{m}_n \cdot \nabla_x \varphi \right] \mathrm{d}x\mathrm{d}t = e_{1,n}[\varphi]$$

Momentum equation

$$\int_0^T \int_\Omega \left[\mathbf{m}_n \cdot \partial_t \boldsymbol{\varphi} + \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} : \nabla_x \varphi + p(\varrho_n) \mathrm{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x\mathrm{d}t = e_{2,n}[\boldsymbol{\varphi}]$$

Stability - bounded energy

$$\mathcal{E}(\varrho_n, \mathbf{m}_n) \equiv \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_n|^2}{\varrho_n} + P(\varrho_n) \right] \mathrm{d}x \lesssim 1$$

Consistency

$$e_{1,n}[\varphi] \to 0, \ \ e_{2,n}[\boldsymbol{\varphi}] \to 0 \text{ as } n \to \infty$$

Weak vs strong convergence

Weak convergence

$$\varrho_n \to \varrho \text{ weakly-(*) } L^\infty(0,T;L^\gamma(\Omega))$$

$$\mathbf{m}_n \to \mathbf{m} \text{ weakly-(*) } L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega;R^d))$$

Strong convergence (Theorem EF, M.Hofmanová)

- Suppose

$$\Omega \subset R^d \text{ bounded}$$

$$\varrho_n \to \varrho,\ \mathbf{m}_n \to \mathbf{m} \text{ strongly a.a. pointwise in } \mathcal{U} \text{ open, } \partial\Omega \subset \mathcal{U}$$

- Then the following is equivalent:

$$\varrho, \mathbf{m} \text{ weak solution to the Euler system}$$

$$\Leftrightarrow$$

$$\varrho_n \to \varrho,\ \mathbf{m}_n \to \mathbf{m} \text{ strongly (pointwise) in } \Omega$$

Martina Hofmanová (Bielefeld)

Dissipative solutions – limits of numerical schemes

Dominic Breit (Edinburgh)

Martina Hofmanová (Bielefeld)

Equation of continuity

$$\partial_t \boxed{\varrho} + \mathrm{div}_x \mathbf{m} = 0$$

Momentum balance

$$\partial_t \boxed{\mathbf{m}} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = -\mathrm{div}_x \mathfrak{R}$$

Energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) \leq 0, \ \ E(t) \leq E_0, \ \ E_0 = \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \ \mathrm{d}x$$

$$\boxed{E} \equiv \left(\int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] \ \mathrm{d}x + d \int_{\overline{\Omega}} \mathrm{trace}[\mathfrak{R}] \right)$$

Reynolds stress

$$\mathfrak{R} \in L^\infty(0, T; \mathcal{M}^+(\overline{\Omega}; R^{d \times d}_{\mathrm{sym}}))$$

Basic properties of dissipative solutions

Well posedness, weak strong uniqueness

- **Existence.** Dissipative solutions exist globally in time for any finite energy initial data
- **Limits of consistent approximations** Limits of consistent approximations are dissipative solutions, in particular limits of consistent numerical schemes.
- **Compatibility.** Any C^1 dissipative solution $[\varrho, \mathbf{m}]$, $\varrho > 0$ is a classical solution of the Euler system
- **Weak–strong uniqueness.** If $[\widetilde{\varrho}, \widetilde{\mathbf{m}}]$ is a classical solution and $[\varrho, \mathbf{m}]$ a dissipative solution starting from the same initial data, then $\mathfrak{R} = 0$ and $\varrho = \widetilde{\varrho}$, $\mathbf{m} = \widetilde{\mathbf{m}}$.
- **Maximal dissipation principle.** The exists a solution maximizing the dissipation rate. Any such solution satisfies

$$\|\mathfrak{R}(t)\|_{\mathcal{M}^+(\overline{\Omega}; R^{d\times d}_{\rm sym})} \to 0 \text{ as } t \to \infty.$$

Semiflow selection

Set of data

$$\mathcal{D} = \left\{ \varrho, \mathbf{m}, E \ \Big| \ \int_\Omega \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \ \mathrm{d}x \leq E \right\}$$

Set of trajectories

$$\mathcal{T} = \left\{ \varrho(t, \cdot), \mathbf{m}(t, \cdot), E(t-, \cdot) \Big| t \in (0, \infty) \right\}$$

Solution set

$$\mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] = \Big\{ [\varrho, \mathbf{m}, E] \ \Big| [\varrho, \mathbf{m}, E] \text{ dissipative solution}$$

$$\varrho(0, \cdot) = \varrho_0, \ \mathbf{m}(0, \cdot) = \mathbf{m}_0, \ E(0+) \leq E_0 \Big\}$$

Semiflow selection – semigroup

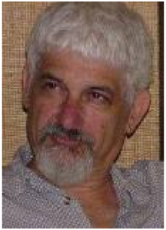
$$U[\varrho_0, \mathbf{m}_0, E_0] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0], \ [\varrho_0, \mathbf{m}_0, E_0] \in \mathcal{D}$$

$$U(t_1 + t_2)[\varrho_0, \mathbf{m}_0, E_0] = U(t_1) \circ \Big[U(t_2)[\varrho_0, \mathbf{m}_0, E_0] \Big], \ t_1, t_2 > 0$$

Andrej Markov (1856–1933)

N. V. Krylov

Strong instead of weak (numerics)



Janos Komlos (Rutgers Univ.)

Komlos theorem (a variant of Strong Law of Large Numbers)

$$\{U_n\}_{n=1}^{\infty} \text{ bounded in } L^1(Q)$$

$$\Rightarrow$$

$$\frac{1}{N}\sum_{k=1}^{N} U_{n_k} \to \overline{U} \text{ a.a. in } Q \text{ as } N \to \infty$$

Convergence of numerical solutions - EF, M.Lukáčová, H.Mizerová 2018

$$\frac{1}{N}\sum_{k=1}^{N} \varrho_{n_k} \to \varrho \text{ in } L^1((0,T)\times\Omega) \text{ as } N \to \infty$$

$$\frac{1}{N}\sum_{k=1}^{N} \mathbf{m}_{n_k} \to \mathbf{m} \text{ in } L^1((0,T)\times\Omega) \text{ as } N \to \infty$$

$$\frac{1}{N}\sum_{k=1}^{N}\left[\frac{1}{2}\frac{|\mathbf{m}_{n,k}|^2}{\varrho_{n,k}} + P(\varrho_{n,k})\right] \to \overline{\mathcal{E}} \in L^1((0,T)\times\Omega) \text{ a.a. in } (0,T)\times\Omega$$

Computing defect – Young measure

Generating Young measure

$$\mathbf{U}_n = [\varrho_n, \mathbf{m}_n] \in R^{d+1} \text{ phase space}$$

$$\{\mathbf{U}_n\}_{n=1}^{\infty} \text{ bounded in } L^1(Q; R^d) \approx \nu^n_{t,x} = \delta_{\mathbf{U}_n(t,x)}$$

$$\Rightarrow$$

$$\frac{1}{N}\sum_{k=1}^{N} \nu^{n_k}_{t,x} \to \nu_{t,x} \text{ narrowly } \boxed{a.a.} \text{ in } Q \text{ as } N \to \infty$$

Young measure

$$(t,x) \in Q \mapsto \nu_{t,x} \in \mathcal{P}[R^{d+1}] \text{ weakly-(*) measurable mapping}$$

$$\mathfrak{R} \approx \left\langle \nu; \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right\rangle - \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \langle \nu; p(\varrho) \rangle - p(\varrho)$$

Erich J. Balder (Utrecht)

Computing defect numerically -EF, M.Lukáčová, B.She

Monge–Kantorowich (Wasserstein) distance

$$\left\| \mathrm{dist} \left(\frac{1}{N} \sum_{k=1}^{N} \nu_{t,x}^{n_k} ; \nu_{t,x} \right) \right\|_{L^q(Q)} \to 0$$

for some $q > 1$

Mária Lukáčová (Mainz)

Convergence in the first variation

$$\frac{1}{N} \sum_{k=1}^{N} \left\langle \nu_{t,x}^{n_k} ; \left| \widetilde{\mathbf{U}} - \frac{1}{N} \sum_{k=1}^{N} \mathbf{U}_n \right| \right\rangle \to \left\langle \nu_{t,x} ; \left| \widetilde{\mathbf{U}} - \mathbf{U} \right| \right\rangle$$

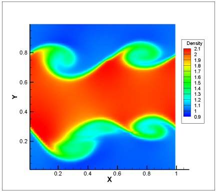
in $L^1(Q)$

Bangwei She (CAS Praha)

Experiment I, density for Kelvin–Helmholtz problem (M. Lukáčová, Yue Wang)

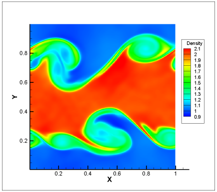
density ϱ

$n = 128,\ T = 2$



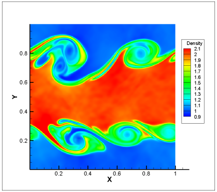
density ϱ

$n = 256,\ T = 2$



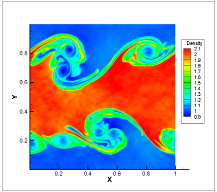
density ϱ

$n = 512,\ T = 2$



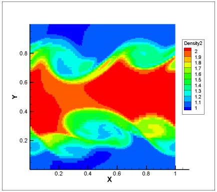
density ϱ

$n = 1024,\ T = 2$



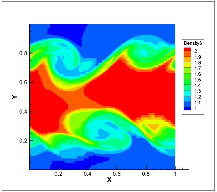
Cèsaro averages density ϱ

$n = 128,\ T = 2$



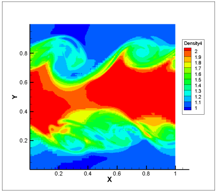
Cèsaro averages density ϱ

$n = 256,\ T = 2$



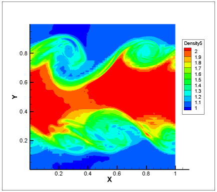
Cèsaro averages density ϱ

$n = 512,\ T = 2$



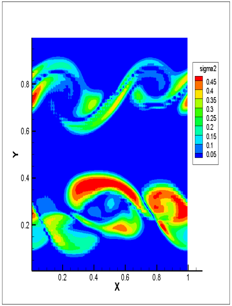
Cèsaro averages density ϱ

$n = 1024,\ T = 2$

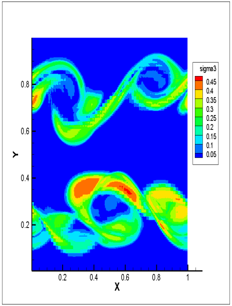


Experiment II, density variations for Kelvin–Helmholtz problem (M. Lukáčová, Yue Wang)

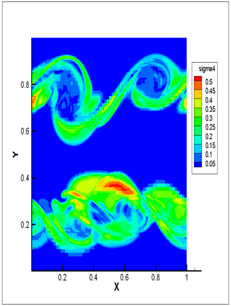
density variation
$n = 128, \ T = 2$



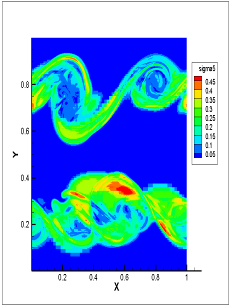
density variation
$n = 256, \ T = 2$



density variation
$n = 512, \ T = 2$



density variation
$n = 1024, \ T = 2$



Yue Wang, Mainz

Mária Lukáčová, Mainz