Dissipative solutions to models of compressible viscous fluids

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Prologue – Navier-Stokes system

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{u})$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = 2\mu \left(\mathbb{D}_x \mathbf{u} - \frac{1}{d} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \lambda \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mathbb{D}_x \mathbf{u} \equiv \frac{1}{2}(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^t)$$

Boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0$$

Energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right) \ \mathrm{d}x + \int_\Omega \boxed{\mathbb{S}(\mathbb{D}_x \mathbf{u}) : \mathbb{D}_x \mathbf{u}} \ \mathrm{d}x \leq 0$$

$$P'(\varrho)\varrho - P(\varrho) = p(\varrho), \ p \text{ increasing, convex}$$

“Implicit” rheological law

Fenchel–Young inequality

$$\boxed{\mathbb{S} : \mathbb{D}} \leq F(\mathbb{D}) + F^*(\mathbb{S})$$

$$\mathbb{S} : \mathbb{D} = F(\mathbb{D}) + F^*(\mathbb{S}) \ \Leftrightarrow\ \mathbb{S} \in \partial F(\mathbb{D}) \ \Leftrightarrow\ \mathbb{D} \in \partial F^*(\mathbb{S})$$

Reformulation of the Navier–Stokes system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right) \ \mathrm{d}x + \int_\Omega \Big(F(\mathbb{D}_x \mathbf{u}) + F^*(\mathbb{S}) \Big) \ \mathrm{d}x \leq 0$$

Dissipation potential

$$F : R^{d\times d}_{\mathrm{sym}} \to [0,\infty] \text{ convex l.s.c.}$$

Example - isothermal case

Constitutive relations

$$\boxed{\text{linear pressure}}\ p(\varrho) = a\varrho,\ \ P(\varrho) = a\varrho \log(\varrho)$$

Dissipation potential

$$F(|\mathbb{D}|) \approx |\mathbb{D}|^q,\ \ q > d$$

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + a\nabla_x \varrho = \mathrm{div}_x \mathbb{S}$$

$$\left[\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + a\varrho \log(\varrho) \right) \ \mathrm{d}x \right]_{t=0}^{\tau} + \int_0^\tau \int_\Omega \Big(F(\mathbb{D}_x \mathbf{u}) + F^*(\mathbb{S}) \Big) \ \mathrm{d}x \ \mathrm{d}t \leq 0$$

Dissipative solutions

Basic properties of generalized solutions

- **Existence.** Generalized solutions exist and represent limits of *consistent* approximations
- **Compatibility.** Smooth generalized solutions are classical solutions
- **Weak–strong uniqueness.** Generalized solution coincides with a smooth solution emanating from the same initial data as long as the latter solution exists
- **Semigroup selection.** The class of generalized solution admits a (Borel measurable) semigroup selection
- **Statistical solution.** Semigroup selection ⇒ existence of statistical solutions. Markovian a.a. semigroup:

$$M_t : \mathfrak{P}[\mathrm{data}] \to \mathfrak{P}[\mathrm{data}],\ M_{t+s} = M_t \circ M_s \text{ for a.a. } s \geq 0$$

$\mathfrak{P}[\mathrm{data}]$ - the set of Borel probability measures on the set of initial/boundary data

Problem formulation

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}$$

$$\mathbb{S} \in \partial F(\mathbb{D}_x \mathbf{u}),\ \ F : R^{d\times d}_{\mathrm{sym}} \to [0, \infty] \text{ convex l.s.c.}$$

Boundary conditions

$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_B,\ \ \Gamma_{\mathrm{in}} = \left\{ x \in \partial\Omega \ \middle|\ \mathbf{u}_B(x) \cdot \mathbf{n} < 0 \right\}$$

$$\varrho|_{\Gamma_{\mathrm{in}}} = \varrho_B$$

Initial conditions

$$\varrho(0, \cdot) = \varrho_0,\ \ (\varrho \mathbf{u})(0, \cdot) = \mathbf{m}_0$$

Dissipative solutions

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S} - \boxed{\mathrm{div}_x \mathfrak{R}} \equiv \mathrm{div}_x \mathbb{S}_{\rm eff}$$

Velocity boundary condition and Reynolds stress

$$(\mathbf{u} - \mathbf{u}_B) \in W^{1,q}_0(\Omega; R^d)$$

$$\boxed{\mathfrak{R} \in L^\infty(0,T; \mathcal{M}^+(\overline{\Omega}; R^{d \times d}_{\rm sym}))}$$

Energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u} - \mathbf{u}_B|^2 + P(\varrho) + d \ \boxed{\mathrm{tr}[\mathfrak{R}]} \right] \ \mathrm{d}x + \int_\Omega \Big[F(\mathbb{D}_x \mathbf{u}) + F^*(\mathbb{S}) \Big] \ \mathrm{d}x$$

$$+ \int_{\partial \Omega} P(\varrho) \mathbf{u}_B \cdot \mathbf{n} \ \mathrm{d}S_x \leq - \int_\Omega \Big[p(\varrho) \mathbb{I} + \varrho \mathbf{u} \otimes \mathbf{u} \Big] : \mathbb{D}_x \mathbf{u}_B \ \mathrm{d}x$$

$$- \int_\Omega \varrho \mathbf{u} \cdot \Big(\mathbf{u}_B \cdot \nabla_x \mathbf{u}_B \Big) \ \mathrm{d}x + \int_\Omega \mathbb{S} : \mathbb{D}_x \mathbf{u}_B \ \mathrm{d}x - \int_{\overline{\Omega}} \boxed{\mathfrak{R}} : \mathbb{D}_x \mathbf{u}_B \ \mathrm{d}x$$

Relative energy

Relative energy

$$E\left(\varrho,\mathbf{u}\middle|\widetilde{\varrho},\widetilde{\mathbf{u}}\right) = \left[\frac{1}{2}\varrho|\mathbf{u}-\widetilde{\mathbf{u}}|^2 + P(\varrho) - P'(\widetilde{\varrho})(\varrho-\widetilde{\varrho}) - P(\widetilde{\varrho})\right]$$

Integrated relative energy

$$\mathcal{E}\left(\varrho,\mathbf{u}\middle|\widetilde{\varrho},\widetilde{\mathbf{u}}\right) = \int_{\Omega}\left[\frac{1}{2}\varrho|\mathbf{u}-\widetilde{\mathbf{u}}|^2 + P(\varrho) - P'(\widetilde{\varrho})(\varrho-\widetilde{\varrho}) - P(\widetilde{\varrho})\right]\,\mathrm{d}x$$

Augmented relative energy

$$\mathcal{E}\left(\varrho,\mathbf{u}\middle|\widetilde{\varrho},\widetilde{\mathbf{u}}\right) + d\int_{\overline{\Omega}}\mathrm{d}\ \mathrm{tr}[\mathfrak{R}]$$

Basic tool

Relative energy inequality

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega E\left(\varrho,\mathbf{u}\ \middle|\ \widetilde{\varrho},\widetilde{\mathbf{u}}\right)\ \mathrm{d}x + d\int_{\overline{\Omega}} \mathrm{tr}[\mathfrak{R}]\right) + \int_\Omega \Big[F(\mathbb{D}_x\mathbf{u}) + F^*(\mathbb{S})\Big]\ \mathrm{d}x \\
&- \int_\Omega \mathbb{S} : \nabla_x \widetilde{\mathbf{u}}\ \mathrm{d}x + \int_{\Gamma_{\rm out}} \left[P(\varrho) - P'(\widetilde{\varrho})(\varrho-\widetilde{\varrho}) - P(\widetilde{\varrho})\right] \mathbf{u}_B\cdot\mathbf{n}\ \mathrm{d}S_x \\
&+ \int_{\Gamma_{\rm in}} \left[P(\varrho_B) - P'(\widetilde{\varrho})(\varrho_B-\widetilde{\varrho}) - P(\widetilde{\varrho})\right] \mathbf{u}_B\cdot\mathbf{n}\ \mathrm{d}S_x \\
\leq &- \int_\Omega \varrho(\widetilde{\mathbf{u}}-\mathbf{u})\cdot(\widetilde{\mathbf{u}}-\mathbf{u})\cdot\nabla_x\widetilde{\mathbf{u}}\ \mathrm{d}x \\
&- \int_\Omega \Big[p(\varrho) - p'(\widetilde{\varrho})(\varrho-\widetilde{\varrho}) - p(\widetilde{\varrho})\Big]\mathrm{div}_x\widetilde{\mathbf{u}}\ \mathrm{d}x \\
&+ \int_\Omega \frac{\varrho}{\widetilde{\varrho}}(\widetilde{\mathbf{u}}-\mathbf{u})\cdot\Big[\partial_t(\widetilde{\varrho}\widetilde{\mathbf{u}}) + \mathrm{div}_x(\widetilde{\varrho}\widetilde{\mathbf{u}}\otimes\widetilde{\mathbf{u}}) + \nabla_x p(\widetilde{\varrho})\Big]\ \mathrm{d}x \\
&+ \int_\Omega \left(\frac{\varrho}{\widetilde{\varrho}}(\mathbf{u}-\widetilde{\mathbf{u}})\cdot\widetilde{\mathbf{u}} + p'(\widetilde{\varrho})\left(1-\frac{\varrho}{\widetilde{\varrho}}\right)\right)\Big[\partial_t\widetilde{\varrho} + \mathrm{div}_x(\widetilde{\varrho}\widetilde{\mathbf{u}})\Big]\ \mathrm{d}x \\
&- \int_{\overline{\Omega}} \nabla_x\widetilde{\mathbf{u}} : \mathrm{d}\ \mathfrak{R}
\end{aligned}
$$

Maximal solutions

Comparison relation

$$\mathcal{E} \equiv \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u} - \mathbf{u}_B|^2 + P(\varrho) + d \ \mathrm{tr}[\mathfrak{R}] \right] \ \mathrm{d}x$$

$$[\varrho_1, \mathbf{u}_1, \mathcal{E}_1] \prec [\varrho_2, \mathbf{u}_2, \mathcal{E}_2] \ \Leftrightarrow \ \mathcal{E}_1 \leq \mathcal{E}_2 \text{ for all } t > 0$$

Maximal solutions

A solution $[\varrho, \mathbf{u}, \mathcal{E}]$ is maximal if it is minimal with respect to $\prec$

Asymptotic behavior of maximal solutions

Suppose $\mathbf{u}_B = 0$ - no-slip boundary conditions.
If $[\varrho, \mathbf{u}]$ is maximal, then

$$\|\mathfrak{R}(t)\|_{\mathcal{M}^+} \to 0 \text{ as } t \to \infty$$

Semigroup selection

Set of data

$$\mathcal{D} = \left\{ \varrho, \mathbf{m}, E \ \Big| \ \int_\Omega \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \ \mathrm{d}x \leq \mathcal{E} \right\}, \ \mathcal{E} \text{ càglàd}$$

Set of trajectories

$$\mathcal{T} = \left\{ \varrho(t, \cdot), \mathbf{m}(t, \cdot), \mathcal{E}(t, \cdot) \Big| t \in (0, \infty) \right\}$$

Solution set

$$\mathcal{U}[\varrho_0, \mathbf{m}_0, \mathcal{E}_0] = \left\{ [\varrho, \mathbf{m}, \mathcal{E}] \ \Big| [\varrho, \mathbf{m}, \mathcal{E}] \text{ dissipative solution} \right.$$

$$\left. \varrho(0, \cdot) = \varrho_0, \ \mathbf{m}(0, \cdot) = \mathbf{m}_0, \ \mathcal{E}(0+) \leq \mathcal{E}_0 \equiv \mathcal{E}(0-) \right\}$$

Semiflow selection – semigroup

$$U[\varrho_0, \mathbf{m}_0, \mathcal{E}_0] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, \mathcal{E}_0], \ [\varrho_0, \mathbf{m}_0, \mathcal{E}_0] \in \mathcal{D}$$

$$U(t_1 + t_2)[\varrho_0, \mathbf{m}_0, \mathcal{E}_0] = U(t_1) \circ \Big[U(t_2)[\varrho_0, \mathbf{m}_0, \mathcal{E}_0] \Big], \ t_1, t_2 > 0$$

Andrej Markov (1856–1933)

N. V. Krylov

Convergence to equilibria

Hypotheses

$$p(\varrho) \approx \varrho^{\gamma},\ F(\mathbb{D}) \approx |\mathbb{D}|^{q}, \mathbf{u}_B = 0$$

$$\frac{1}{\gamma} + \frac{1}{q} \leq 1 \text{ if } q > \frac{d}{2},\ \frac{\gamma + 1}{2\gamma} + \frac{d - q}{dq} < 1 \text{ if } q \leq \frac{d}{2}.$$

Long time behavior

$$[\varrho, \mathbf{m}] \text{ maximal}$$

$$\Rightarrow$$

$$\varrho(t, \cdot) \to \varrho_S \text{ in } L^{\gamma}(\Omega),\ \mathbf{m} \to 0\ L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d) \text{ as } t \to \infty$$

Stability of rarefaction waves

1-D Euler system

$$\partial_t \widetilde{\varrho} + \partial_{x_1}(\widetilde{\varrho}\widetilde{u}) = 0,$$
$$\partial_t(\widetilde{\varrho}\widetilde{u}) + \partial_{x_1}(\widetilde{\varrho}\widetilde{u}^2) + \partial_{x_1} p(\widetilde{\varrho}) = 0,$$

Riemann data

$$[\widetilde{\varrho}(0,x_1), \widetilde{u}(0,x_1)] = [\varrho_0, \mathbf{u}_0] := \left\{ \begin{array}{l} [\widetilde{\varrho}_L, \widetilde{u}_L] \text{ if } x_1 < 0, \\ [\widetilde{\varrho}_R, \widetilde{u}_R] \text{ if } x_1 \geq 0, \end{array} \right.$$

Rarefaction waves

$$\left| \int_{\widetilde{\varrho}_L}^{\widetilde{\varrho}_R} \frac{p'(z)}{z} \,\mathrm{d}z \right| \leq \widetilde{u}_R - \widetilde{u}_L \leq \int_0^{\widetilde{\varrho}_L} \frac{p'(z)}{z} \,\mathrm{d}z + \int_0^{\widetilde{\varrho}_R} \frac{p'(z)}{z} \,\mathrm{d}z,$$

$$\widetilde{\varrho} = \widetilde{\varrho}\left(\frac{x_1}{t}\right), \ \widetilde{u} = \widetilde{u}\left(\frac{x_1}{t}\right), \ \text{Lipschitz for } \boxed{t > 0}$$

Stability of rarefaction waves

Spatial domain, relative energy

$$\Omega := \left\{ [x_1, \dots, x_d] \ \middle| \ x_1 \in (-L, L),\ [x_2, \dots, x_d] \in \mathcal{T}^{d-1} \right\},\ d = 2, 3,$$

$$E\left(\varrho, \mathbf{m} \ \middle| \widetilde{\varrho}, \widetilde{\mathbf{u}}\right) = \frac{1}{2}\varrho \left| \frac{\mathbf{m}}{\varrho} - \widetilde{\mathbf{u}} \right|^2 + P(\varrho) - P'(\widetilde{\varrho})(\varrho - \widetilde{\varrho}) - P(\widetilde{\varrho}).$$

Vanishing viscosity limit

$$F_\varepsilon(\mathbb{D}) \approx \varepsilon F(\mathbb{D})$$

Stability of planar rarefaction wave

$\widetilde{\varrho}$, $\widetilde{\mathbf{u}}$ planar rarefaction wave profile

$$\int_\Omega E\left(\varrho_{0,\varepsilon}, \mathbf{m}_{0,\varepsilon} \ \middle| \widetilde{\varrho}_0, \widetilde{\mathbf{u}}_0\right) \ \mathrm{d}x \to 0 \text{ as } \varepsilon \to 0$$

$$\Rightarrow$$

$$\int_\Omega E\left(\varrho_\varepsilon(\tau, \cdot), \mathbf{m}_\varepsilon(\tau, \cdot) \ \middle| \widetilde{\varrho}(\tau, \cdot), \widetilde{\mathbf{u}}(\tau, \cdot)\right) \ \mathrm{d}x \to 0 \text{ as } \varepsilon \to 0,\ \tau > 0$$