HIGHER-ORDER DIFFERENTIAL SYSTEMS AND A REGULARIZATION OPERATOR

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(Received May 19, 1997)

Abstract. Sufficient conditions for the existence of solutions to boundary value problems with a Carathéodory right hand side for ordinary differential systems are established by means of continuous approximations.

Keywords: Carathéodory functions, Arzelà-Ascoli theorem, Lebesgue theorem

MSC 2000: 34B10

1. Introduction

In this paper we prove theorems on the existence of solutions to the differential system

$$x^{(k)} = f(t, x, x', \dots, x^{(k-1)}) \tag{1.1}$$

satisfying the boundary condition

$$V(x) = \mathbf{o}, \tag{1.2}$$

where V is a continuous operator of boundary conditions and $\mathbf{o}$ is a zero point of the space $\mathbb{R}^{kn}$, $\mathbf{o} = \overbrace{(0, 0, \dots, 0)}^{kn \text{ times}}$.

We generalize the results of [2] where the second-order differential systems with L^∞-Carathéodory right-hand sides are considered. Here we consider the k-th order differential system (1.1) with a Carathéodory function f. The problem (1.1), (1.2) is approximated by a sequence of problems with continuous right-hand sides. The existence of solutions of (1.1), (1.2) is obtained as a consequence of the existence of solutions of these auxiliary problems.

Let $-\infty < a^* \leqslant a < b \leqslant b^* < \infty$, $I = [a,b]$, $I^* = [a^*, b^*]$, $\mathbb{R} = (-\infty, \infty)$, n, k natural numbers. $\mathbb{R}^n$ denotes the Euclidean n-space as usual and $\|x\|$ denotes the Euclidean norm. $C_n^k(I) = C^k([a,b], \mathbb{R}^n)$ is the Banach space of functions u such that $u^{(k)}$ is continuous on I with the norm

$$\|u\|_k = \max\left\{\|u\|, \|u'\|, \|u''\|, \ldots, \|u^{(k)}\|\right\},$$

where

$$\|u\| = \max\left\{\|u(t)\|,\ t \in I\right\}.$$

Let $C_n(I)$ denote the space $C_n^0(I)$. $C_{nO}^\infty(\mathbb{R}) = C_{nO}^\infty(\mathbb{R}, \mathbb{R}^n)$ is the space of functions φ such that for each $l \in \{1, 2, \ldots\}$ there exists a continuous on $\mathbb{R}$ function $\varphi^{(l)}$ and the support of the function φ is a bounded closed set, $\operatorname{supp}\varphi = \overline{\{x \in \mathbb{R}; \|\varphi(x)\| > 0\}}$. Finally, let $1 \leqslant p < \infty$, let $L_n^p(I) = L_n^p((a,b), \mathbb{R}^n)$ be as usual the space of Lebesgue integrable functions with the norm

$$|u|_p = \left(\int_a^b \|u(t)\|^p \,\mathrm{d}t\right)^{\frac{1}{p}},$$

let us denote $L^p(I) = L_1^p(I)$, $L(I) = L^1(I)$.

Definition 1.1. A function $f\colon I^* \times \mathbb{R}^{kn} \to \mathbb{R}^n$ is a Carathéodory function provided

(i) the map $y \mapsto f(t,y)$ is continuous for almost every $t \in I^*$,

(ii) the map $t \mapsto f(t,y)$ is measurable for all $y \in \mathbb{R}^{kn}$,

(iii) for each bounded subset $B \subset \mathbb{R}^{kn}$ we have

$$l_f(t) = \sup\{\|f(t,y)\|,\ y \in B\} \in L(I^*).$$

Throughout the paper let us assume $f\colon I^* \times \mathbb{R}^{kn} \to \mathbb{R}^n$ is a Carathéodory function and $V\colon C_n^{k-1}(I) \to \mathbb{R}^{kn}$ is a continuous operator.

If f is continuous, by a solution on I to the equation (1.1) we mean a classical solution with a continuous k-th derivative, while if f is a Carathéodory function, a solution will mean a function x which has an absolutely continuous $(k-1)$-st derivative such that x fulfils the equality $x^{(k)}(t) = f(t, x(t), x'(t), \ldots, x^{(k-1)}(t))$ for almost every $t \in I$.

By xy where $x, y \in \mathbb{R}^n$ we mean a scalar product of two vectors from $\mathbb{R}^n$.

2. Regularization operator

Let φ in C^∞_{1O} be such that

$$\varphi(t) \geqslant 0 \quad \forall t \in \mathbb{R}, \quad \operatorname{supp}\varphi = [-1,1], \quad \int_{-1}^{1} \varphi(t)\,\mathrm{d}t = 1.$$

For an example of such a function see [4], page 26.

Instead of problem (1.1), (1.2) we will consider the equation

$$x^{(k)} = f_\varepsilon(t, x, x', \dots, x^{(k-1)}) \tag{2.1_ε}$$

with the boundary condition (1.2), where ε is a positive real number and $\forall y \in \mathbb{R}^{kn}$ we have

$$f_\varepsilon(t,y) = \frac{1}{\varepsilon}\int_{a^*}^{b^*} \varphi\Big(\frac{t-\eta}{\varepsilon}\Big) f(\eta,y)\,\mathrm{d}\eta$$

or equivalently

$$f_\varepsilon(t,y) = \int_{-1}^{1} \overline{f}(t-\varepsilon\eta, y)\varphi(\eta)\,\mathrm{d}\eta,$$

where $\overline{f}(t,y) = \begin{cases} f(t,y) & t \in [a^*,b^*] \\ 0 & t \notin [a^*,b^*] \end{cases}.$

The following theorem is proved in [3] (a simple form for n=1 is presented):

Theorem 2.1. *Let $u \in L^p(I^*)$, where $1 \leqslant p < \infty$, and for $\varepsilon > 0$ let us denote*

$$(R_\varepsilon u)(t) = \frac{1}{\varepsilon}\int_{a^*}^{b^*} \varphi\Big(\frac{t-\eta}{\varepsilon}\Big) u(\eta)\,\mathrm{d}\eta = \int_{-1}^{1} \overline{u}(t-\varepsilon\eta)\varphi(\eta)\,\mathrm{d}\eta,$$

where $\overline{u}(t) = \begin{cases} u(t) & t \in [a^*,b^*] \\ 0 & t \notin [a^*,b^*] \end{cases}.$

Then

(i) $R_\varepsilon u \in C^\infty(\mathbb{R})$ *for* $\varepsilon > 0$,

(ii) $\lim\limits_{\varepsilon\to 0+} |R_\varepsilon u - u|_p = 0.$

Lemma 2.1. *Let B be a bounded subset in $\mathbb{R}^{kn}$. Then the function $f_\varepsilon(t,y)$ is continuous on $I^* \times B$ for every $\varepsilon > 0$.*

Proof. Continuity of f_ε follows from the theorem on continuous dependence of the integral on a parameter. □

Definition 2.1. Let $w\colon I^*\times[0,\infty)\to[0,\infty)$ be a Carathéodory function. We write $w\in M(I^*\times[0,\infty);[0,\infty))$ if w satisfies:

(i) For almost every $t\in I^*$ and for every $d_1,d_2\in[0,\infty)$, $d_1<d_2$ we have

$$w(t,d_1)\leqslant w(t,d_2).$$

(ii) For almost every $t\in I^*$ we have $w(t,0)=0$.

Definition 2.2. Let B be a compact subset of $\mathbb{R}^{kn}$, $\tau\in\mathbb{R}$, $\delta\in[0,\infty)$ and $\varepsilon>0$. Let us denote by $\omega(\tau,\delta)$ the function

$$\begin{aligned}\omega(\tau,\delta)=\max\bigl\{&\|\overline{f}(\tau,x_1,\dots,x_k)-\overline{f}(\tau,y_1,\dots,y_k)\|;\\&(x_1,\dots,x_k),\ (y_1,\dots,y_k)\in B,\ \|x_i-y_i\|\leqslant\delta,\ i=1,\dots,k\bigr\}\end{aligned}$$

and by $\omega_\varepsilon(\tau,\delta)$ the function

$$\omega_\varepsilon(\tau,\delta)=\frac{1}{\varepsilon}\int_{a^*}^{b^*}\varphi\Bigl(\frac{\tau-\eta}{\varepsilon}\Bigr)\omega(\eta,\delta)\,\mathrm{d}\eta$$

or equivalently

$$\omega_\varepsilon(\tau,\delta)=\int_{-1}^{1}\omega(\tau-\varepsilon\eta,\delta)\varphi(\eta)\,\mathrm{d}\eta.$$

Lemma 2.2. *Let B be a compact subset of $\mathbb{R}^{kn}$. Then for every $\varepsilon>0$*

(i) $\omega,\omega_\varepsilon\in M(I^*\times[0,\infty);[0,\infty))$;

(ii) $\lim\limits_{\varepsilon\to0+}f_\varepsilon(t,y)=f(t,y)$ *and* $\lim\limits_{\varepsilon\to0+}\omega_\varepsilon(t,\delta)=\omega(t,\delta)$ *for all $y\in B$, $\delta\geqslant0$ and for almost every $t\in I^*$;*

(iii) *for every $(x_1,\dots,x_k)$, $(y_1,\dots,y_k)\in B$ and for almost every $t\in I^*$ we have*

$$\begin{aligned}&\|f_\varepsilon(t,x_1,\dots,x_k)-f_\varepsilon(t,y_1,\dots,y_k)-f(t,x_1,\dots,x_k)+f(t,y_1,\dots,y_k)\|\\&\quad\leqslant\omega_\varepsilon(t,\max\{\|x_i-y_i\|;\ i=1,2,\dots,k\})+\omega(t,\max\{\|x_i-y_i\|;\ i=1,2,\dots,k\});\end{aligned}$$

(iv) $\lim\limits_{\varepsilon\to0+}\displaystyle\int_a^t(f_\varepsilon(\tau,x)-f(\tau,x))\,\mathrm{d}\tau=0$ *uniformly on $I\times B$.*

P r o o f.

(i) Since $f(\tau,.)$ is a Carathéodory function and B is a compact set, for almost every $\tau\in I^*$ we have $0\leqslant\omega(\tau,\delta)\leqslant2l_f(\tau)$, $\omega(\tau,.)$ is nondecreasing and continuous, $\omega(.,\delta)$ is measurable and

$$\lim_{\delta\to0+}\omega(\tau,\delta)=0.$$

It means that $\omega(\tau,0)=0$ for almost every $\tau\in I^*$. Therefore we can see that $\omega\in M(I^*\times[0,\infty);[0,\infty))$.

By the theorem on continuous dependence of the integral on a parameter, ω_ε is a continuous function for arbitrary $\varepsilon>0$. Therefore ω_ε is a Carathéodory function such that $\omega_\varepsilon(\tau,0)=0$ for almost every $\tau\in I^*$. If $\delta_1<\delta_2$, then for almost every $\tau\in I^*$

$$0\leqslant\omega(\tau,\delta_1)\leqslant\omega(\tau,\delta_2) \tag{2.2}$$

hence for almost every $\eta\in I^*$

$$0\leqslant\frac{1}{\varepsilon}\varphi\Big(\frac{\tau-\eta}{\varepsilon}\Big)\omega(\eta,\delta_1)\leqslant\frac{1}{\varepsilon}\varphi\Big(\frac{\tau-\eta}{\varepsilon}\Big)\omega(\eta,\delta_2)$$

and therefore

$$0\leqslant\omega_\varepsilon(\tau,\delta_1)\leqslant\omega_\varepsilon(\tau,\delta_2). \tag{2.3}$$

It means that $\omega_\varepsilon\in M(I^*\times[0,\infty);[0,\infty))$.

(ii) This statement is a consequence of Theorem 2.1 which asserts that our assumption implies for every $\delta>0$, $y\in B$ and $i=1,2,\dots,n$

$$\lim_{\varepsilon\to0+}\int_{-1}^{1}|\omega_\varepsilon(\tau,\delta)-\omega(\tau,\delta)|\,\mathrm{d}\tau=0,$$

$$\lim_{\varepsilon\to0+}\int_{-1}^{1}|f_{\varepsilon i}(\tau,y)-f_i(\tau,y)|\,\mathrm{d}\tau=0,$$

where f_i, $f_{\varepsilon i}$ are the i-th components of the functions f, f_ε, respectively.

(iii) Obviously for $\|x_i-y_i\|\leqslant\delta$, $i=1,\dots,k$

$$\begin{aligned}
&\|f_\varepsilon(t,x_1,\dots,x_k)-f_\varepsilon(t,y_1,\dots,y_k)\|\\
&\quad=\Big\|\int_{-1}^{1}\varphi(\eta)\big(\overline{f}(t-\varepsilon\eta,x_1,\dots,x_k)-\overline{f}(t-\varepsilon\eta,y_1,\dots,y_k)\big)\,\mathrm{d}\eta\Big\|\\
&\quad\leqslant\int_{-1}^{1}\|\overline{f}(t-\varepsilon\eta,x_1,\dots,x_k)-\overline{f}(t-\varepsilon\eta,y_1,\dots,y_k)\|\varphi(\eta)\,\mathrm{d}\eta\\
&\quad\leqslant\int_{-1}^{1}\omega(t-\varepsilon\eta,\delta)\varphi(\eta)\,\mathrm{d}\eta=\omega_\varepsilon(t,\delta).
\end{aligned}$$

Now it is easy to see that the statement (iii) of the above lemma holds.

(iv) We will prove that for every $(t,x) \in I \times B$, $x = (x_1, \ldots, x_k)$, and every $e > 0$ there exist $\varepsilon_0 > 0$ and a neighbourhood $O_{(t,x)}$ of (t,x) in the set $I \times B$ such that for every $0 < \varepsilon < \varepsilon_0$ and for every $(t', y) \in O_{(t,x)}$, $y = (y_1, \ldots, y_k)$,

$$\left\| \int_a^{t'} (f_\varepsilon(\tau, y) - f(\tau, y)) \, \mathrm{d}\tau \right\| < e.$$

By (ii) and by the Lebesgue dominated convergence theorem there exists $\varepsilon_1 > 0$ such that for every $0 < \varepsilon < \varepsilon_1$

$$\int_a^b \| f_\varepsilon(\tau, x) - f(\tau, x) \| \, \mathrm{d}\tau < \tfrac{e}{4}.$$

Since $\omega \in M(I^* \times [0,\infty); [0,\infty))$ there exists such a $\delta > 0$ that

$$\int_a^b \omega(\tau, \delta) \, \mathrm{d}\tau < \tfrac{e}{4}.$$

By (ii) and the Lebesgue dominated convergence theorem there exists $\varepsilon_2 > 0$ such that for every $0 < \varepsilon < \varepsilon_2$

$$\int_a^b \omega_\varepsilon(\tau, \delta) \, \mathrm{d}\tau < \tfrac{e}{2}.$$

Let us denote $O_{(t,x)} = \{(t', y) \in I \times B; \|x_i - y_i\| < \delta, \ i = 1, 2, \ldots, k\}$ and $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$. Now for every $0 < \varepsilon < \varepsilon_0$ and for every $(t', y) \in O_{(t,x)}$ we have

$$\begin{aligned}
&\left\| \int_a^{t'} (f_\varepsilon(\tau, y) - f(\tau, y)) \, \mathrm{d}\tau \right\| \\
&\qquad \leqslant \left\| \int_a^{t'} \Big(f_\varepsilon(\tau, x) - f(\tau, x) \Big) \, \mathrm{d}\tau \right\| \\
&\qquad\quad + \left\| \int_a^{t'} (f_\varepsilon(\tau, x) - f_\varepsilon(\tau, y) - f(\tau, x) + f(\tau, y)) \, \mathrm{d}\tau \right\| \\
&\qquad \leqslant \int_a^b \| f_\varepsilon(\tau, x) - f(\tau, x) \| \, \mathrm{d}\tau + \int_a^b \omega_\varepsilon(\tau, \delta) + \omega(\tau, \delta) \, \mathrm{d}\tau \\
&\qquad < \tfrac{e}{4} + \tfrac{e}{2} + \tfrac{e}{4} \leqslant e.
\end{aligned}$$

This means that the system of the sets $\{O_{(t,x)}\}_{(t,x) \in I \times B}$ covers the compact set $I \times B$ and therefore there exists a finite subsystem which covers the set $I \times B$ and therefore the statement of (iv) holds. $\square$

Lemma 2.3. *Let $B \subset \mathbb{R}^{kn}$ be a compact set. Let $\mathfrak{E}$ be a set of $\varepsilon > 0$ such that the system of functions $\{x_\varepsilon\}_{\varepsilon\in\mathfrak{E}}$, $x_\varepsilon\colon I \to B$, is equi-continuous and $0 \in \overline{\mathfrak{E}}$.*

Then $\lim\limits_{\varepsilon\to 0+} \displaystyle\int_a^t f_\varepsilon(\tau, x_\varepsilon(\tau)) - f(\tau, x_\varepsilon(\tau))\,\mathrm{d}\tau = 0$ *uniformly on I.*

Proof. This proof is a modification of the proof of Lemma 3.1 in [6].

For $\varepsilon \in \mathfrak{E}$ let us denote

$$\alpha_\varepsilon = \sup\Bigl\{\Bigl\|\int_s^t f_\varepsilon(\tau, y) - f(\tau, y)\,\mathrm{d}\tau\Bigr\|;\ a \leqslant s < t \leqslant b,\ y \in B\Bigr\},$$

$$\beta_\varepsilon = \max\Bigl\{\Bigl\|\int_a^t f_\varepsilon(\tau, x_\varepsilon(\tau)) - f(\tau, x_\varepsilon(\tau))\,\mathrm{d}\tau\Bigr\|; a \leqslant t \leqslant b\Bigr\}.$$

By (iv) of Lemma 2.2

$$\lim_{\varepsilon\to 0} \alpha_\varepsilon = 0.$$

We want to prove

$$\lim_{\varepsilon\to 0} \beta_\varepsilon = 0.$$

Let $e > 0$ be an arbitrary real number. Then by (i) of Lemma 2.2 there exists such a $\delta > 0$ that

$$\int_a^b \omega(\tau, \delta)\,\mathrm{d}\tau < \tfrac{e}{3},$$

and by (i), (ii) of Lemma 2.2 such an $\varepsilon_1 > 0$ that for every $\varepsilon \in \mathfrak{E}$, $\varepsilon < \varepsilon_1$ we have

$$\int_a^b \omega_\varepsilon(\tau, \delta)\,\mathrm{d}\tau < \tfrac{2e}{3}.$$

Since $\{x_\varepsilon\}_{\varepsilon\in\mathfrak{E}}$, $x_\varepsilon = (x_{\varepsilon 1}, \ldots, x_{\varepsilon k})$ is equi-continuous there exists $\delta_0 > 0$ such that

$$\|x_{\varepsilon i}(t) - x_{\varepsilon i}(\tau)\| < \delta \text{ for } \ t, \tau \in I,\ i = 1, \ldots, k,\ |t - \tau| \leqslant \delta_0,\ \varepsilon \in \mathfrak{E}.$$

Let l be such an integer that $l \leqslant \frac{b-a}{\delta_0} < l + 1$. Let us denote $t_j = a + j\delta_0$ and $\overline{x_\varepsilon}(t) = x_\varepsilon(t_j)$ for $t_j \leqslant t < t_{j+1}$, where $j = 0, 1, \ldots, l$. Then

$$\|x_{\varepsilon i}(t) - \overline{x_{\varepsilon i}}(t)\| < \delta$$

for $t \in I$, $i = 1, \ldots, k$ and $\varepsilon \in \mathfrak{E}$ and

$$\Bigl\|\int_a^t f_\varepsilon(\tau, \overline{x_\varepsilon}(\tau)) - f(\tau, \overline{x_\varepsilon}(\tau))\,\mathrm{d}\tau\Bigr\| \leqslant (l + 1)\alpha_\varepsilon$$

for $a < t < b$ and $\varepsilon < \varepsilon_0$, $\varepsilon \in \mathfrak{E}$.

Therefore by (iii) of Lemma 2.2 we obtain

$$\begin{aligned}
\Big\| \int_a^t \big(f_\varepsilon(\tau, x_\varepsilon(\tau)) - f(\tau, x_\varepsilon(\tau))\big)\,\mathrm{d}\tau \Big\|
&\leqslant \int_a^t \|f_\varepsilon(\tau, x_\varepsilon(\tau)) - f(\tau, x_\varepsilon(\tau)) - f_\varepsilon(\tau, \overline{x_\varepsilon}(\tau)) + f(\tau, \overline{x_\varepsilon}(\tau))\|\,\mathrm{d}\tau \\
&\quad + \Big\| \int_a^t \big(f_\varepsilon(\tau, \overline{x_\varepsilon}(\tau)) - f(\tau, \overline{x_\varepsilon}(\tau))\big)\,\mathrm{d}\tau \Big\| \\
&\leqslant \int_a^b \big(\omega_\varepsilon(\tau, \delta) + \omega(\tau, \delta)\big)\,\mathrm{d}\tau + (l+1)\alpha_\varepsilon < e + (l+1)\alpha_\varepsilon
\end{aligned}$$

for $t \in I$, $\varepsilon < \varepsilon_1$, $\varepsilon \in \mathfrak{E}$.

Therefore $\beta_\varepsilon < e + (l+1)\alpha_\varepsilon$ for $\varepsilon < \varepsilon_1$, $\varepsilon \in \mathfrak{E}$. Since $\lim\limits_{\varepsilon \to 0} \alpha_\varepsilon = 0$ and e is arbitrary we conclude that $\lim\limits_{\varepsilon \to 0} \beta_\varepsilon = 0$. $\square$

Theorem 2.2. *Let $f\colon I^* \times \mathbb{R}^{kn} \to \mathbb{R}^n$ be a Carathéodory function. Denote by $\mathfrak{E}$ the set of positive ε such that for each $\varepsilon \in \mathfrak{E}$ there exists a solution $x_\varepsilon\colon I \subseteq I^* \to \mathbb{R}^n$ to the problem (2.1_ε), (1.2). Suppose that $0 \in \overline{\mathfrak{E}}$ and that there exists a compact subset $B \subset \mathbb{R}^{kn}$ independent of ε such that $(x_\varepsilon(t), x'_\varepsilon(t), \ldots, x_\varepsilon^{(k-1)}(t)) \in B$ is satisfied for each $\varepsilon \in \mathfrak{E}$ and for each $t \in I$.*

Then there exist a sequence $\{\varepsilon_s\}_{s=1}^\infty$ and a solution $x\colon I \to \mathbb{R}^n$ to the given boundary value problem (1.1), (1.2) *such that $\varepsilon_s \in \mathfrak{E}$ for all $s \in \mathbb{N}$, $\lim\limits_{s\to\infty} \varepsilon_s = 0$, $\big(x(t), x'(t), \ldots, x^{(k-1)}(t)\big) \in B$ for all $t \in I$, $\lim\limits_{s\to\infty} x_{\varepsilon_s}^{(i)}(t) = x^{(i)}(t)$ uniformly on I for any $i = 1, 2, \ldots, k-1$, and $\lim\limits_{s\to\infty} x_{\varepsilon_s}^{(k)}(t) = x^{(k)}(t)$ on I.*

P r o o f. First let us prove that the set $\{x_\varepsilon\}_{\varepsilon \in \mathfrak{E}}$ is relatively compact in $C_n^{k-1}(I)$. Really, for the assumptions of the Arzelà-Ascoli theorem to be satisfied, it is necessary to prove equi-continuity of the set $\{x_\varepsilon^{(k-1)}\}_{\varepsilon \in \mathfrak{E}}$.

Let $e > 0$ be an arbitrary real number, suppose $t_1, t_2 \in I$ and compute

$$\begin{aligned}
\|x_\varepsilon^{(k-1)}(t_1) - x_\varepsilon^{(k-1)}(t_2)\| &= \Big\| \int_{t_1}^{t_2} x_\varepsilon^{(k)}(t)\,\mathrm{d}t \Big\| \\
&= \Big\| \int_{t_1}^{t_2} f_\varepsilon(t, x_\varepsilon(t), x'_\varepsilon(t), \ldots, x_\varepsilon^{(k-1)}(t))\,\mathrm{d}t \Big\| \\
&= \Big\| \int_{t_1}^{t_2} \int_{-1}^{1} \overline{f}(t - \varepsilon\eta, x_\varepsilon(t), x'_\varepsilon(t), \ldots, x_\varepsilon^{(k-1)}(t))\varphi(\eta)\,\mathrm{d}\eta\,\mathrm{d}t \Big\| \\
&\leqslant \Big| \int_{t_1}^{t_2} \int_{-1}^{1} l_{\overline{f}}(t - \varepsilon\eta)\varphi(\eta)\,\mathrm{d}\eta\,\mathrm{d}t \Big|,
\end{aligned}$$

where $l_{\bar{f}}(t) = \begin{cases} l_f(t) & t \in I^* \\ 0 & t \notin I^* \end{cases}$. Now for ε close to 0 ($\varepsilon < \varepsilon_1$, where ε_1 is defined below) we have

$$\Big|\int_{t_1}^{t_2}\int_{-1}^{1} l_{\bar{f}}(t-\varepsilon\eta)\varphi(\eta)\,\mathrm{d}\eta\,\mathrm{d}t\Big| \\ \leqslant \Big|\int_{t_1}^{t_2} l_f(t)\,\mathrm{d}t\Big| + \Big|\int_{t_1}^{t_2}\Big(\int_{-1}^{1} l_{\bar{f}}(t-\varepsilon\eta)\varphi(\eta)\,\mathrm{d}\eta - l_f(t)\Big)\,\mathrm{d}t\Big|.$$

Since $l_f(t) \in L(I^*)$ then $\int_a^t l_f(\tau)\,\mathrm{d}\tau$ is a continuous function, every continuous function on a compact interval is uniformly continuous on that interval, and therefore there exists $\delta_1 > 0$ such that for all $|t_1 - t_2| < \delta_1$ we have

$$\Big|\int_{t_1}^{t_2} l_f(t)\,\mathrm{d}t\Big| < \tfrac{e}{2}.$$

By Theorem 2.1 there exists ε_1 such that for each $\varepsilon \in \mathfrak{E}$, $0 < \varepsilon < \varepsilon_1$,

$$\int_a^b \Big|\int_{-1}^{1} l_{\bar{f}}(t-\varepsilon\eta)\varphi(\eta)\,\mathrm{d}\eta - l_f(t)\Big|\,\mathrm{d}t < \tfrac{e}{2},$$

and therefore for $\forall\varepsilon \in \mathfrak{E}$, $0 < \varepsilon < \varepsilon_1$, we have

$$\Big|\int_{t_1}^{t_2}\int_{-1}^{1} l_{\bar{f}}(t-\varepsilon\eta)\varphi(\eta)\,\mathrm{d}\eta\,\mathrm{d}t\Big| < e.$$

Now for $\varepsilon \in \mathfrak{E}$, $\varepsilon_1 \leqslant \varepsilon$,

$$\Big|\int_{t_1}^{t_2}\int_{-1}^{1} l_{\bar{f}}(t-\varepsilon\eta)\varphi(\eta)\,\mathrm{d}\eta\,\mathrm{d}t\Big| = \frac{1}{\varepsilon}\Big|\int_{t_1}^{t_2}\int_a^b l_f(\eta)\varphi\Big(\frac{t-\eta}{\varepsilon}\Big)\,\mathrm{d}\eta\,\mathrm{d}t\Big|.$$

Let $\Phi = \max\{\varphi(t), t \in I\}$. Then

$$\frac{1}{\varepsilon}\Big|\int_{t_1}^{t_2}\int_a^b l_f(\eta)\varphi\Big(\frac{t-\eta}{\varepsilon}\Big)\,\mathrm{d}\eta\,\mathrm{d}t\Big| \\ \leqslant \frac{1}{\varepsilon_1}\Big|\int_{t_1}^{t_2}\int_a^b l_f(\eta)\Phi\,\mathrm{d}\eta\,\mathrm{d}t\Big| \leqslant \frac{1}{\varepsilon_1}|t_1 - t_2|\Phi\int_a^b l_f(\eta)\,\mathrm{d}\eta.$$

Let $\delta_2 = \frac{e\varepsilon_1}{\Phi\int_a^b l_f(\eta)\,\mathrm{d}\eta}$, then for $|t_1 - t_2| < \delta_2$ we obtain

$$\Big|\int_{t_1}^{t_2}\int_{-1}^{1} l_f(t-\varepsilon\eta)\varphi(\eta)\,\mathrm{d}\eta\,\mathrm{d}t\Big| < e.$$

Let $\delta = \min\{\delta_1, \delta_2\}$ then for $|t_1 - t_2| < \delta$ we have

$$\|x_\varepsilon^{(k-1)}(t_1) - x_\varepsilon^{(k-1)}(t_2)\| < e.$$

This means that the set $\{x_\varepsilon\}_{\varepsilon\in\mathfrak{E}}$ is relatively compact in $C_n^{k-1}(I)$. Therefore there exist a sequence $\{\varepsilon_s\}$, $\varepsilon_s \in \mathfrak{E}$, $\varepsilon_s \to 0$ and a function $x\colon I \to \mathbb{R}^n$ such that $(x(t), x'(t), \ldots, x^{(k-1)}(t)) \in B$, $\forall t \in I$, $x_{\varepsilon_s} \to x$ in $C_n^{k-1}(I)$.

Now, since x_{ε_s} is the solution to the equation (2.1_ε) for $\varepsilon = \varepsilon_s$, we have

$$(2.4) \qquad x_{\varepsilon_s}^{(k-1)}(t) = x_{\varepsilon_s}^{(k-1)}(a) + \int_a^t f_{\varepsilon_s}(\tau, x_{\varepsilon_s}(\tau), x'_{\varepsilon_s}(\tau), \ldots, x_{\varepsilon_s}^{(k-1)}(\tau))\,\mathrm{d}\tau,\ \forall t \in I.$$

Using Lemma 2.3 we get

$$x^{(k-1)}(t) = x^{(k-1)}(a) + \int_a^t f(\tau, x(\tau), x'(\tau), \ldots, x^{(k-1)}(\tau))\,\mathrm{d}\tau,$$

which means that x is a solution to the equation (1.1).

Since x_{ε_s} uniformly converges to x in $C_n^{k-1}(I)$, V is a continuous operator $V\colon C_n^{k-1}(I) \to \mathbb{R}^{kn}$ and x_{ε_s} is a solution to the problem (2.1_{ε_s}), (1.2), we can see that

$$V(x_{\varepsilon_s}) = \mathbf{o},$$

and therefore for $\varepsilon_s \to 0$ we have

$$V(x) = \mathbf{o}.$$

It means that x is a solution to the problem (1.1), (1.2). □

R e m a r k 2.1. When $l_f(t) \in L^p(I^*)$ in Definition 1.1, where $1 \leqslant p < \infty$ (in this case we speak about an L^p-Carathéodory function) we can prove that the convergence of $x_{\varepsilon_s}^{(k)}$ to $x^{(k)}$ is in the norm of $L^p(I^*)$. To prove it we need only to assume in Definition 2.2

$$\omega(\tau, \delta) = \max\bigl\{\|\overline{f}(\tau, x_1, \ldots, x_k) - \overline{f}(\tau, y_1, \ldots, y_k)\|^p\bigr\}.$$

3. An application

As an example how to use Theorem 2.2 we may consider the equation

$$x'' = f(t, x, x') \tag{3.1}$$

with the four point boundary conditions

$$x(0) = x(c), \qquad x(d) = x(1), \tag{3.2}$$

where $0 < c \leqslant d < 1$. In [1] the following result is proved.

Theorem 3.1. *Let $f\colon [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be a continuous function and let us consider the problem* (3.1), (3.2). *Assume*

(i) *there is a constant $M \geqslant 0$ such that $uf(t,u,p) \geqslant 0$ for $\forall t \in [0,1]$, $\forall u \in \mathbb{R}^n$, $\|u\| > M$ and $\forall p \in \mathbb{R}^n$, $pu = 0$,*

(ii) *there exist continuous positive functions A_j, B_j, $j \in \{1,2,\dots,n\}$,*

$$A_j\colon [0,1] \times \mathbb{R}^{n+j-1} \to \mathbb{R}, \qquad B_j\colon [0,1] \times \mathbb{R}^{n+j-1} \to \mathbb{R}$$

such that

$$|f_j(t,u,p)| \leqslant A_j(t,u,p_1,p_2,\dots,p_{j-1})p_j^2 + B_j(t,u,p_1,p_2,\dots,p_{j-1}),$$

where $f = (f_1, f_2, \dots, f_n)$, $u \in \mathbb{R}^n$, $p \in \mathbb{R}^n$, $p = (p_1, p_2, \dots, p_n)$ and for $j = 1$, A_1 and B_1 are independent of p functions.

Then the problem (3.1), (3.2) *has a solution.*

Remark 3.1. From the proof of this theorem and from the topological transversality theorem in [4] it follows that the solution to the problem (3.1), (3.2) is bounded in $C^1_n([0,1])$ by a constant $\mathfrak{M}$ which depends only on M, A_j, B_j.

Now we can extend the results of Theorem 3.1 to the Carathéodory case similarly to [2]. We allow discontinuities of functions A_j, B_j in contrast to [2].

Definition 3.1. Let k, l be natural numbers. A function $f\colon I \times \mathbb{R}^k \to \mathbb{R}^l$ is an L^∞-Carathéodory function provided $f = f(t,u)$ satisfies

(i) the map $u \mapsto f(t,u)$ is continuous for almost every $t \in I$,

(ii) the map $t \mapsto f(t,u)$ is measurable for all $(u,p) \in \mathbb{R}^k$,

(iii) for each bounded subset $B \subset \mathbb{R}^k$,

$$l_f(t) = \sup\{\|f(t,u)\|, u \in B\} \in L^\infty(I),$$

where L^∞ is the space of Lebesgue integrable functions with the norm

$$\|f\|_\infty = \operatorname*{ess\,sup}_{t\in I} \|f\|.$$

Theorem 3.2. *Let $f\colon [0,1]\times\mathbb{R}^{2n}\to\mathbb{R}^n$ be a Carathéodory function and let us consider the problem* (3.1), (3.2). *Assume*

(i) *there is a constant $M \geqslant 0$ such that $uf(t,u,p)\geqslant 0$ for almost every t in $[0,1]$, $\forall u\in\mathbb{R}^n$, $\|u\|>M$ and $\forall p\in\mathbb{R}^n$, $pu=0$,*

(ii) *there exist positive L^∞-Carathéodory functions A_j, B_j, where the index j is from $\{1,2,\dots,n\}$,*

$$A_j\colon [0,1]\times\mathbb{R}^{n+j-1}\to\mathbb{R}, \qquad B_j\colon [0,1]\times\mathbb{R}^{n+j-1}\to\mathbb{R},$$

such that for almost every $t\in[0,1]$

$$|f_j(t,u,p)| \leqslant A_j(t,u,p_1,p_2,\dots,p_{j-1})p_j^2 + B_j(t,u,p_1,p_2,\dots,p_{j-1}),$$

where $f=(f_1,f_2,\dots,f_n)$, $u\in\mathbb{R}^n$, $p\in\mathbb{R}^n$, $p=(p_1,p_2,\dots,p_n)$ and for $j=1$, A_1 and B_1 are independent of p functions.

Then the problem (3.1), (3.2) *has a solution.*

Proof. Let f_ε be an approximated function as in Section 2, where $a=a^*=0$, $b=b^*=1$ and $k=2$, that is

$$f_\varepsilon(t,u,p)u = \frac{1}{\varepsilon}\int_0^1 \varphi\Big(\frac{t-\eta}{\varepsilon}\Big) f(\eta,u,p)\,\mathrm{d}\eta,$$

and let $V\colon C_n^1([0,1])\to\mathbb{R}^{2n}$ be a continuous operator of boundary conditions $V(x) = (x(0)-x(a), x(b)-x(1))$. Then

1) for $\forall\varepsilon\in(0,1)$, for $\forall t\in[0,1]$, $\forall u\in\mathbb{R}^n$, $\|u\|>M$ and $\forall p\in\mathbb{R}^n$, $pu=0$ we have

$$f_\varepsilon(t,u,p)u = \Big(\frac{1}{\varepsilon}\int_0^1 \varphi\Big(\frac{t-\eta}{\varepsilon}\Big) f(\eta,u,p)\,\mathrm{d}\eta\Big)u =$$

$$= \frac{1}{\varepsilon}\int_0^1 \varphi\Big(\frac{t-\eta}{\varepsilon}\Big)\big(f(\eta,u,p)u\big)\,\mathrm{d}\eta \geqslant 0$$

by the assumption (i) of this theorem.

2) Let $j\in\{1,2,\dots,n\}$, $u\in\mathbb{R}^n$, $p\in\mathbb{R}^n$, $p=(p_1,p_2,\dots,p_n)$,

$$\mathcal{A}_j(u,p_1,p_2,\dots,p_{j-1}) = \operatorname*{ess\,sup}_{t\in[0,1]}\big\{A_j(t,u,p_1,p_2,\dots,p_{j-1})\big\}$$

and

$$\mathcal{B}_j(u,p_1,p_2,\dots,p_{j-1}) = \operatorname*{ess\,sup}_{t\in[0,1]} \big\{ B_j(t,u,p_1,p_2,\dots,p_{j-1}) \big\}.$$

Since A_j, B_j are L^∞-Carathéodory functions, $\mathcal{A}_j$, $\mathcal{B}_j$ are obviously continuous. Now we have

$$\begin{aligned}
|f_{\varepsilon_j}(t,u,p)| &= \left| \int_{-1}^{1} \overline{f_j}(t-\varepsilon\eta,u,p)\varphi(\eta)\,\mathrm{d}\eta \right| \leqslant \int_{-1}^{1} |\overline{f_j}(t-\varepsilon\eta,u,p)|\varphi(\eta)\,\mathrm{d}\eta \\
&\leqslant \int_{-1}^{1} (\mathcal{A}_j(u,p_1,p_2,\dots,p_{j-1})p_j^2 + \mathcal{B}_j(u,p_1,p_2,\dots,p_{j-1}))\varphi(\eta)\,\mathrm{d}\eta \\
&\leqslant \int_{-1}^{1} \mathcal{A}_j(u,p_1,p_2,\dots,p_{j-1})p_j^2\varphi(\eta)\,\mathrm{d}\eta + \int_{-1}^{1} \mathcal{B}_j(u,p_1,p_2,\dots,p_{j-1})\varphi(\eta)\,\mathrm{d}\eta \\
&= \mathcal{A}_j(u,p_1,p_2,\dots,p_{j-1})p_j^2 + \mathcal{B}_j(u,p_1,p_2,\dots,p_{j-1}).
\end{aligned}$$

By Theorem 3.1 and Remark 3.1, for any $\varepsilon > 0$ there exists a solution x_ε to the approximated problem

$$x'' = f_\varepsilon(t,x,x') \tag{3.1_ε}$$

where x satisfies boundary conditions (3.2) such that $\|x_\varepsilon\|_1 \leqslant \mathfrak{M}$.

Now all assumptions of Theorem 2.1 are fullfiled and therefore there exists a solution to the problem (1.1), (3.1). □

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