INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

Perturbations of surjective homomorphisms between algebras of operators on Banach spaces

Bence Horváth
Zsigmond Tarcsay

Preprint No. 18-2020

PRAHA 2020

PERTURBATIONS OF SURJECTIVE HOMOMORPHISMS BETWEEN ALGEBRAS OF OPERATORS ON BANACH SPACES

BENCE HORVÁTH AND ZSIGMOND TARCSAY

Abstract. A remarkable result of Molnár [*Proc. Amer. Math. Soc.*, 126 (1998), 853–861] states that automorphisms of the algebra of operators acting on a separable Hilbert space is stable under "small" perturbations. More precisely, if ϕ, ψ are endomorphisms of $\mathcal{B}(\mathcal{H})$ such that $\|\phi(A) - \psi(A)\| < \|A\|$ and ψ is surjective then so is ϕ. The aim of this paper is to extend this result to a larger class of Banach spaces including ℓ_p and L_p spaces $(1 < p < +\infty)$.

En route to the proof we show that for any Banach space X from the above class all faithful, unital, separable, reflexive representations of $\mathcal{B}(X)$ which preserve rank one operators are in fact isomorphisms.

1. Introduction and statement of main results

It is well known that if ψ, ϕ are endomorphisms of a C^*-algebra (not necessarily involution preserving), and ψ is an automorphism with $\|\psi - \phi\| < 1/\|\psi^{-1}\|$, then ϕ is an automorphism too. Motivated by this fact, Molnár proved in [23] that in fact a shaper version of this result holds for $\mathcal{B}(\mathcal{H})$, the C^*-algebra of bounded linear operators on a separable Hilbert space $\mathcal{H}$. More precisely, he showed in [23, Theorem 1] that if $\phi, \psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ are algebra homomorphisms such that ψ is surjective and $\|\psi(A) - \phi(A)\| < \|A\|$ for each non-zero $A \in \mathcal{B}(\mathcal{H})$, then ϕ is also surjective. Let us remark here that ψ and ϕ are automatically continuous, and their surjectivity implies their injectivity, as shown in the proof of [23, Theorem 1], for example. The main tool in Molnár's proof is a previous, deep result of his from [22].

The purpose of this paper is the extend [23, Theorem 1] for a large class of (non-hilbertian) Banach spaces, see Theorem 1.2. *En route* to this we shall prove a theorem about certain faithful representations of $\mathcal{B}(X)$, where X is a Banach space from the same class (see Theorem 1.1). We believe this result to be of independent interest, since the study of faithful, separable representations of $\mathcal{B}(X)$ goes back to the seminal work of Berkson and Porta in [4]. Our main results are the following:

Theorem 1.1. *Let X and Y be non-zero Banach spaces such that Y is separable and reflexive, and X satisfies one of the following:*

2010 *Mathematics Subject Classification.* Primary 46H10, 47L10; Secondary 46B03, 46B07, 46B10, 47L20.

Key words and phrases. Algebra of operators, Banach space, basis, character, Gowers' space, homomorphism, isomorphism, James space, perturbation, representation, Semadeni space, subsymmetric basis.

The corresponding author B. Horvth acknowledges with thanks the funding received from GAČR project 19-07129Y; RVO 67985840 (Czech Republic).

Zs. Tarcsay was supported by DAAD-TEMPUS Cooperation Project "Harmonic Analysis and Extremal Problems" (grant no. 308015). Project no. ED 18-1-2019-0030 (Application-specific highly reliable IT solutions) has been implemented with the support provided from the National Research, Development and Innovation Fund of Hungary, financed under the Thematic Excellence Programme funding scheme.

(1) $X = L_p[0,1]$, *where* $1 < p < \infty$; *or*
(2) X *is a reflexive Banach space with a subsymmetric Schauder basis.*

Let $\phi : \mathcal{B}(X) \to \mathcal{B}(Y)$ *be a continuous, injective algebra homomorphism. If* $\operatorname{Ran}(\phi)$ *contains an operator with dense range, and* ϕ *maps rank one idempotents into rank one idempotents, then* ϕ *is an isomorphism.*

From the above theorem we will deduce a generalisation of [23, Theorem 1]:

Theorem 1.2. *Let* X *and* Y *be Banach spaces as in Theorem 1.1. Let* $\psi, \phi : \mathcal{B}(X) \to \mathcal{B}(Y)$ *be algebra homomorphisms such that* ψ *is surjective. If*

$$\|\psi(A) - \phi(A)\| < \|A\| \tag{1.1}$$

for each non-zero $A \in \mathcal{B}(X)$, *then* ϕ *is an isomorphism.*

As one might expect, there is no hope for Theorem 1.1 to hold in general for arbitrary Banach spaces X and Y. To be precise, we prove the following:

Proposition 1.3. *Let* X *be a Banach space such that* $\mathcal{B}(X)$ *has a character. Let* Z *be any non-zero Banach space. There is a continuous, injective algebra homomorphism* $\phi : \mathcal{B}(X) \to \mathcal{B}(X \oplus Z)$ *with* $\phi(I_X) = I_{X \oplus Z}$ *which maps rank one operators into rank one operators but* ϕ *is not surjective.*

In particular, let X *be the* p^{th} *James space* $\mathcal{J}_p$ *(where* $1 < p < \infty$*) or the Semadeni space* $C[0, \omega_1]$. *There is a continuous, injective algebra homomorphism* $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ *with* $\phi(I_X) = I_X$ *which maps rank one operators into rank one operators but* ϕ *is not surjective.*

The necessary terminology will be explained in the subsequent sections.

The paper is structured as follows. Section 2 contains a brief overview of the concepts and notations needed to understand the paper. In Section 3 we develop some auxiliary tools which will feature heavily in our arguments later. Section 4 is devoted to the proofs of Theorems 1.1, 1.2 and Proposition 1.3. We conclude Section 4 with some remarks about the possibility of weakening the assumptions in Theorem 1.1.

2. Preliminaries

The notation and terminology used throughout this paper is standard.

2.0.1. *Numbers and sets.* The first infinite cardinal is denoted by $\aleph_0$ and we refer to the cardinal $2^{\aleph_0}$ as the *continuum*. If X is a set then $\mathcal{P}(X)$ denotes its power set, and $|X|$ denotes the cardinality of X. If X, Y are sets then Y^X is the set of functions from X to Y.

Let Γ be a set. A family $\mathcal{F} \subseteq \mathcal{P}(\Gamma)$ is called *almost disjoint* if for any distinct $A, B \in \mathcal{F}$ the set $A \cap B$ is finite. There exists an almost disjoint family of continuum cardinality consisting of infinite subsets of the natural numbers. For a proof we refer the reader to *e.g.* [2, Lemma 2.5.3].

2.0.2. *Ultrafilters, ultralimits.* If $\mathcal{F}$ is a filter on a set X and $\mathcal{U}$ is an ultrafilter on X with $\mathcal{F} \subseteq \mathcal{U}$, then we say that $\mathcal{U}$ *extends* $\mathcal{F}$. As a corollary of Zorn's Lemma any filter can be extended to an ultrafilter.

Let X be a topological space and let $x \in X$. Let $(x_i)_{i \in I}$ be a system in X and let $\mathcal{U}$ be an ultrafilter on I. If $(x_i)_{i \in I}$ converges to x along $\mathcal{U}$ then we will denote this by $x = \lim_{i \to \mathcal{U}} x_i$. Let X be a compact Hausdorff space, and let $(x_i)_{i \in I}$ be a net in X. If $\mathcal{U}$ is an ultrafilter on I then the ultralimit $\lim_{i \to \mathcal{U}} x_i$ exists and it is unique (see *e.g.* [1, Lemma 1.5.9]). If X, Y are topological spaces and $f : X \to Y$ is a continuous function then $\lim_{i \to \mathcal{U}} x_i = x$ implies $\lim_{i \to \mathcal{U}} f(x_i) = f(x)$.

There is a standard way of connecting convergence in a topological space with convergence along certain ultrafilters. Let I be a directed set. For any $i \in I$ we define $A_i := \{j \in I : j \geq i\}$. Then the set

$$\mathcal{F}_{\mathrm{ord}} := \{S \in \mathcal{P}(I) : (\exists i \in I)(A_i \subseteq S)\}$$

is easily seen to be a filter on I, called the *order filter*.

Let X be a topological space and let $(x_i)_{i \in I}$ be a net in X converging to $x \in X$. If $\mathcal{U}$ is an ultrafilter on I with $\mathcal{F}_{\mathrm{ord}} \subseteq \mathcal{U}$ then $x = \lim_{i \to \mathcal{U}} x_i$.

2.1. **Background material on Banach spaces and Banach algebras.** In this paper all Banach spaces and Banach algebras are assumed to be complex.

2.1.1. *The dual space; weak- and weak* topologies.* If X is a Banach space, then for its dual space we write X^*. In the following $\langle \cdot, \cdot \rangle$ denotes the natural duality pairing; that is, $\langle x, f \rangle := f(x)$ whenever $x \in X$ and $f \in X^*$. The weak topology on X is denoted by $\sigma(X, X^*)$, and the weak* topology on X^* is denoted by $\sigma(X^*, X)$.

2.1.2. *Operators on Banach spaces.* The identity operator on a vector space X is denoted by I_X. If X, Y are normed spaces then $\mathcal{B}(X, Y)$ denotes the normed space of bounded linear operators from X to Y. We denote $\mathcal{B}(X, X)$ simply by $\mathcal{B}(X)$. For $T \in \mathcal{B}(X, Y)$ its adjoint is denoted by T^*. If Z is a linear subspace of X, then for $T \in \mathcal{B}(X, Y)$ we denote the restriction of T to Z by $T|_Z$, clearly $T|_Z \in \mathcal{B}(Z, Y)$.

If X, Y are normed spaces and $y \in Y$, $\varphi \in X^*$ then we define $y \otimes \varphi : X \to Y; x \mapsto \langle x, \varphi \rangle y$. It is clear that $y \otimes \varphi \in \mathcal{B}(X, Y)$ is rank one with $\|y \otimes \varphi\| = \|y\| \|\varphi\|$, whenever $y \in Y$ and $\varphi \in X^*$ are non-zero.

Two Banach spaces X and Y are said to be *isomorphic* if there is a linear homeomorphism between X and Y, it will be denoted by $X \simeq Y$.

2.1.3. *Banach algebras, idempotents.* By an *isomorphism of Banach algebras* $\mathcal{A}$ and $\mathcal{B}$ we understand that there is an algebra homomorphism between $\mathcal{A}$ and $\mathcal{B}$ which is also a homeomorphism. This will also be denoted by $\mathcal{A} \simeq \mathcal{B}$.

In an algebra $\mathcal{A}$ an element $p \in \mathcal{A}$ is an *idempotent* if $p^2 = p$. Two idempotents $p, q \in \mathcal{A}$ are *orthogonal* if $qp = 0 = pq$. We say that two idempotents $p, q \in \mathcal{A}$ are *equivalent*, and denote it by $p \sim q$, if there exist $a, b \in \mathcal{A}$ such that $ab = p$ and $ba = q$. For idempotents $p, q \in \mathcal{A}$ we write $p \leq q$ whenever $pq = p = qp$. Clearly $\leq$ is a partial ordering on the set of idempotents of $\mathcal{A}$.

We recall a folklore result, a stronger version of which was proved by Zemánek in [28, Lemma 3.1]. A self-contained elementary proof can be found in [7, Lemma 2.6].

Lemma 2.1. *Let $\mathcal{A}$ be a unital Banach algebra, and let $p, q \in \mathcal{A}$ be idempotents with $\|p - q\| < 1$. Then $p \sim q$.*

Let X be a Banach space. Two idempotents $P, Q \in \mathcal{B}(X)$ are said to be *almost orthogonal* if $PQ, QP \in \mathcal{F}(X)$.

The following lemma is well-known, see for example [17, Lemma 1.4].

Lemma 2.2. *Let X_1, X_2 be Banach spaces and let $P \in \mathcal{B}(X_1)$ and $Q \in \mathcal{B}(X_2)$ be idempotents. Then* $\mathrm{Ran}(P) \simeq \mathrm{Ran}(Q)$ *as Banach spaces if and only if there exist $U \in \mathcal{B}(X_2, X_1)$ and $V \in \mathcal{B}(X_1, X_2)$ with $P = UV$ and $Q = VU$. In particular, for $X := X_1 = X_2$ we have that $P \sim Q$ if and only if* $\mathrm{Ran}(P) \simeq \mathrm{Ran}(Q)$.

2.1.4. *Ideals of operators on Banach spaces.* Let X, Y be Banach spaces, let $T \in \mathcal{B}(X, Y)$. Then T is a *finite-rank operator* if $\operatorname{Ran}(T)$ is finite-dimensional. The symbol $\mathcal{F}(X, Y)$ stands for the set of finite-rank operators on X. It is well-known that $\mathcal{F}$ is the smallest operator ideal in the sense of Pietsch, see for example [24, Theorem 1.2.2]. In an infinite-dimensional Banach space X, $\mathcal{F}(X)$ is always a proper, non-closed, two-sided ideal.

The symbol $\mathcal{A}(X, Y)$ stands for the (operator)norm-closure of $\mathcal{F}(X, Y)$. It is clear that $\mathcal{A}(X)$ is the smallest closed, proper, non-zero, two-sided ideal in $\mathcal{B}(X)$. An element of $\mathcal{A}(X, Y)$ is called an *approximable operator*. The set of compact operators from X to Y is denoted by $\mathcal{K}(X, Y)$. It is known that $\mathcal{K}$ is a closed operator ideal such that $\mathcal{A} \subseteq \mathcal{K}$.

2.1.5. *Schauder bases in Banach spaces.* Let X be a Banach space with Schauder basis $(b_n)_{n\in\mathbb{N}}$. Then $f_n \in X^*$ and $P_n \in \mathcal{B}(X)$ denote the corresponding n^{th} coordinate functional and projection, respectively, for all $n \in \mathbb{N}$. It is standard that $(P_n)_{n\in\mathbb{N}}$ converges to I_X in the strong operator topology. In particular, $(P_n)_{n\in\mathbb{N}}$ is uniformly bounded by the Banach–Steinhaus Theorem. We remark in passing that if a Banach space X has a basis then $\mathcal{A}(X) = \mathcal{K}(X)$.

Recall that if $(b_n)_{n\in\mathbb{N}}$ is an unconditional basis in X then for any $A \subseteq \mathbb{N}$

$$P_A : X \to X; \quad x \mapsto \sum_{n\in A} \langle x, f_n\rangle b_n \tag{2.1}$$

defines a bounded linear idempotent operator on X and the system $(P_A)_{A\in\mathcal{P}(\mathbb{N})}$ is uniformly bounded. A basis $(b_n)_{n\in\mathbb{N}}$ of X is called *subsymmetric* if it is an uncounditional basis and the basic sequence $(b_{\sigma(n)})_{n\in\mathbb{N}}$ is equivalent to $(b_n)_{n\in\mathbb{N}}$ for every strictly monotone increasing function $\sigma : \mathbb{N} \to \mathbb{N}$. We note that the natural bases for c_0 and ℓ_p $(1 \leq p < \infty)$ are subsymmetric, see [2, Section 9.2]. For $p \in [1, \infty)\backslash\{2\}$ the space $L_p[0, 1]$ does not have a subsymmetric basis, see [27, Theorem 21.1]. In fact, $L_1[0, 1]$ does not even have an unconditional basis by [2, Theorem 6.3.3].

The following well known fact can be found, for example, in the monograph of Lindenstrauss and Tzafriri, see the paragraph after [19, Definition 3.a.2].

Proposition 2.3. *Let X be a Banach space with a subsymmetric basis $(b_n)_{n\in\mathbb{N}}$. For any strictly monotone increasing function $\sigma : \mathbb{N} \to \mathbb{N}$ the map*

$$S_\sigma : X \to X; \quad x \mapsto \sum_{n\in\mathbb{N}} \langle x, f_n\rangle b_{\sigma(n)} \tag{2.2}$$

is an isomorphism onto its range.

We recall that a Schauder basis $(b_n)_{n\in\mathbb{N}}$ for a Banach space X is *shrinking* if the sequence of coordinate functionals $(f_n)_{n\in\mathbb{N}}$ associated with $(b_n)_{n\in\mathbb{N}}$ is a Schauder basis for X^*. Any Schauder basis in a reflexive Banach space is shrinking (see [27, Example 4.3]). Clearly ℓ_1 and $L_1[0, 1]$ cannot have shrinking bases since their dual spaces are non-separable.

2.2. **Dual Banach algebras and approximate identities.** A Banach algebra $\mathcal{B}$ is a *dual Banach algebra with predual* $(\mathcal{B}_*, \varphi)$, if $\mathcal{B}_*$ is a Banach $\mathcal{B}$-bimodule and $\varphi : \mathcal{B} \to (\mathcal{B}_*)^*$ is an isomorphism of Banach $\mathcal{B}$-bimodules such that the maps

$$\begin{aligned} l_a &:= \varphi \circ \lambda_a \circ \varphi^{-1} \qquad (a \in \mathcal{B}) \\ r_a &:= \varphi \circ \rho_a \circ \varphi^{-1} \qquad (a \in \mathcal{B}) \end{aligned} \tag{2.3}$$

are $\sigma((\mathcal{B}_*)^*, \mathcal{B}_*)$ - to - $\sigma((\mathcal{B}_*)^*, \mathcal{B}_*)$ continuous; here λ_a and ρ_a denote the multiplication on $\mathcal{B}$ by the element a from the left and right, respectively.

If X is a Banach space, then the projective tensor product $X\hat{\otimes}_\pi X^*$ is easily seen to be a Banach $\mathcal{B}(X)$-bimodule with the multiplication defined pointwise for $A \in \mathcal{B}(X)$, $x \in X$, and $\varphi \in X^*$ as

$$A \cdot (x \otimes \varphi) := (Ax) \otimes \varphi, \quad (x \otimes \varphi) \cdot A := x \otimes (A^*\varphi) \tag{2.4}$$

and then extended by linearity and continuity. For background information on the projective tensor products of Banach spaces we refer the reader to [8] and [26].

The following result is taken from [25, Example 5.1.4]:

Lemma 2.4. *If X is a reflexive Banach space then there is an isometric isomorphism $\varphi : \mathcal{B}(X) \to (X\hat{\otimes}_\pi X^*)^*$ such that for any $x \in X, f \in X^*$ and $A \in \mathcal{B}(X)$:*

$$\langle x \otimes f, \varphi(A)\rangle = \langle Ax, f\rangle,$$

and $(X\hat{\otimes}_\pi X^*, \varphi)$ is a predual for $\mathcal{B}(X)$.

Let $\mathcal{A}$ be a Banach algebra. A net $(e_\gamma)_{\gamma\in\Gamma}$ in $\mathcal{A}$ is a *bounded left (respectively, right) approximate identity* if $\sup_\gamma \|e_\gamma\| < \infty$ and $\lim_\gamma e_\gamma a = a$ (respectively, $\lim_\gamma a e_\gamma = a$) for every $a \in \mathcal{A}$. A net $(e_\gamma)_{\gamma\in\Gamma}$ is a *bounded approximate identity (b.a.i.)* if it is a bounded left- and right approximate identity.

The following is an immediate consequence of [6, Theorem 2.9.37].

Corollary 2.5. *Let X be a Banach space with a Schauder basis. Then the sequence of coordinate projections $(P_n)_{n\in\mathbb{N}}$ is a bounded left approximate identity for $\mathcal{K}(X)$. If X has a shrinking basis then $(P_n)_{n\in\mathbb{N}}$ is a bounded approximate identity for $\mathcal{K}(X)$.*

3. Some auxiliary results

In the following, if X is a Banach space, $(f_i)_{i\in I}$ is a system in the topological space $(X^*, \sigma(X^*, X))$ and $\mathcal{U}$ is an ultrafilter on I such that the ultralimit of $(f_i)_{i\in I}$ along $\mathcal{U}$ with respect to the topology $\sigma(X^*, X)$ exists in X^*, then this limit will be denoted by $\text{w}^*\text{-}\lim_{i\to\mathcal{U}} f_i$.

Lemma 3.1. *Let $\mathcal{B}$ be a dual Banach algebra. Let $(q_\gamma)_{\gamma\in\Gamma}$ be a bounded net in $\mathcal{B}$ such that $\lim_\omega q_\omega q_\gamma = q_\gamma$ in norm for any $\gamma \in \Gamma$. Then for any ultrafilter $\mathcal{U}$ on Γ extending the order filter, $p := \varphi^{-1}(\text{w}^*\text{-}\lim_{\gamma\to\mathcal{U}} \varphi(q_\gamma)) \in \mathcal{B}$ exists and defines an idempotent.*

Proof. Let $\mathcal{U}$ be an ultrafilter on Γ extending the order filter. By the Banach–Alaoglu Theorem $p := \varphi^{-1}(\text{w}^*\text{-}\lim_{\gamma\to\mathcal{U}} \varphi(q_\gamma)) \in \mathcal{B}$ is well-defined. We show that $p \in \mathcal{B}$ is idempotent. Recall that for any $b \in \mathcal{B}$ the maps $\varphi \circ \lambda_b \circ \varphi^{-1}$ and $\varphi \circ \rho_b \circ \varphi^{-1}$ are weak*-continuous on $(\mathcal{B}_*)^*$ and therefore for any $\gamma \in \Gamma$

$$\varphi(pq_\gamma) = (\varphi \circ \rho_{q_\gamma} \circ \varphi^{-1})(\text{w}^*\text{-}\lim_{\omega\to\mathcal{U}} \varphi(q_\omega)) = \text{w}^*\text{-}\lim_{\omega\to\mathcal{U}} \varphi(q_\omega q_\gamma) = \varphi(q_\gamma)$$

because $\lim_{\omega\to\mathcal{U}} q_\omega q_\gamma = q_\gamma$. Consequently,

$$\begin{aligned}\varphi(p^2) &= \varphi\left(p\varphi^{-1}(\text{w}^*\text{-}\lim_{\gamma\to\mathcal{U}} \varphi(q_\gamma))\right) = (\varphi \circ \lambda_p \circ \varphi^{-1})(\text{w}^*\text{-}\lim_{\gamma\to\mathcal{U}} \varphi(q_\gamma)) \\ &= \text{w}^*\text{-}\lim_{\gamma\to\mathcal{U}} \varphi(pq_\gamma) = \text{w}^*\text{-}\lim_{\gamma\to\mathcal{U}} \varphi(q_\gamma) = \varphi(p).\end{aligned}$$

This shows that $p^2 = p$, proving the claim. ☐

The following lemma has many "folk" variations (see *e.g.* [4, Lemma 2.23]). Rather than hunt for a reference which states Lemma 3.2 exactly in the form suitable for our purpose, we shall prove the result here.

Lemma 3.2. *Let X be a reflexive Banach space and let $(Q_n)_{n\in\mathbb{N}}$ be a bounded sequence of monotone increasing idempotents in $\mathcal{B}(X)$. Then there exists an idempotent $Q \in \mathcal{B}(X)$ such that $(Q_n)_{n\in\mathbb{N}}$ converges to Q in the strong operator topology.*

Proof. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. We show first that there exists an idempotent $Q \in \mathcal{B}(X)$ such that (Q_n) converges to Q along $\mathcal{U}$ in the weak operator topology. Let $(X\hat{\otimes}_\pi X^*, \varphi)$ be the canonical predual of $\mathcal{B}(X)$ as in Lemma 2.4. According to Lemma 3.1, $Q := \varphi^{-1}(\text{w*-}\lim_{n\to\mathcal{U}} \varphi(Q_n))$ is a well-defined idempotent operator in $\mathcal{B}(X)$. It remains to show that $\lim_{n\to\mathcal{U}}\langle Q_n x, f\rangle = \langle Qx, f\rangle$ for any $x \in X$ and $f \in X^*$. This is a simple calculation:

$$\begin{aligned}\langle Qx, f\rangle &= \langle \varphi^{-1}(\text{w*-}\lim_{n\to\mathcal{U}} \varphi(Q_n))x, f\rangle = \langle x\otimes f, \text{w*-}\lim_{n\to\mathcal{U}} \varphi(Q_n)\rangle \\ &= \lim_{n\to\mathcal{U}}\langle x\otimes f, \varphi(Q_n)\rangle = \lim_{n\to\mathcal{U}}\langle Q_n x, f\rangle.\end{aligned}$$

We show that (Q_n) converges to Q in the strong operator topology. Firstly let us observe that $Q_n Q = Q_n$ for any $n \in \mathbb{N}$. Indeed, for any $z \in X$ and $f \in X^*$ we have that

$$\langle Q_n Q z, f\rangle = \langle Qz, Q_n^* f\rangle = \lim_{i\to\mathcal{U}}\langle Q_i z, Q_n^* f\rangle = \lim_{i\to\mathcal{U}}\langle Q_n Q_i z, f\rangle = \langle Q_n z, f\rangle,$$

thus proving $Q_n Q = Q_n$. A similar argument shows that also $QQ_n = Q_n$. Let us now fix $x \in X$. Clearly $Q_n x \in \operatorname{conv}\{Q_m x : m \in \mathbb{N}\}$ for any $n \in \mathbb{N}$. Therefore $Qx = \text{w-}\lim_{n\to\mathcal{U}} Q_n x$ with Mazur's Theorem (see *e.g.* [21, Theorem 2.5.16]) imply that $Qx \in \overline{\operatorname{conv}}\{Q_m x : m \in \mathbb{N}\}$, where the closure is taken with respect to the norm topology of X. Let us fix $\varepsilon > 0$. There exist a finite set $\Gamma \subseteq \mathbb{N}$ and $(\lambda_j)_{j\in\Gamma}$ in $[0, 1]$ such that $\|Qx - \sum_{j\in\Gamma}\lambda_j Q_j x\| < \varepsilon/(K + 1)$. Let $N := \max(\Gamma)$, then $Q_n(\sum_{j\in\Gamma}\lambda_j Q_j) = \sum_{j\in\Gamma}\lambda_j Q_j$ for any $n \geq N$. Consequently for each $n \geq N$:

$$\begin{aligned}\|Qx - Q_n x\| &\leq \Big\|Qx - \sum_{j\in\Gamma}\lambda_j Q_j x\Big\| + \Big\|Q_n\Big(\sum_{j\in\Gamma}\lambda_j Q_j x - Qx\Big)\Big\| \\ &< \frac{\varepsilon}{K+1} + K\frac{\varepsilon}{K+1} = \varepsilon.\end{aligned}$$

This shows that $(Q_n x)$ converges to Qx in X as required. □

Lemma 3.3. *Let $\mathcal{A}$ be a Banach algebra, let $\mathcal{J} \trianglelefteq \mathcal{A}$ be a closed, two-sided ideal with a b.a.i. $(e_\gamma)_{\gamma\in\Gamma}$. Let $\mathcal{B}$ be a unital, dual Banach algebra. Suppose $\psi : \mathcal{A} \to \mathcal{B}$ is a continuous algebra homomorphism. If $\mathcal{U}$ is an ultrafilter on Γ which extends the order filter, then:*

1. *$p := \varphi^{-1}(\text{w*-}\lim_{\gamma\to\mathcal{U}} \varphi(\psi(e_\gamma))) \in \mathcal{B}$ is an idempotent;*
2. *For any $c \in \mathcal{J}$, $p\psi(c) = \psi(c) = \psi(c)p$;*
3. *For any $a \in \mathcal{A}$, $p\psi(a) = p\psi(a)p = \psi(a)p$;*
4. *The map*

$$\theta : \mathcal{A} \to \mathcal{B}; \quad a \mapsto (1_\mathcal{B} - p)\psi(a)(1_\mathcal{B} - p) \tag{3.1}$$

is a continuous algebra homomorphism with $\theta|_\mathcal{J} = 0$.

Proof. (1) Since we have $(e_\gamma)_{\gamma\in\Gamma}$ is a bounded approximate identity in $\mathcal{J}$, it follows that $\lim_\gamma \psi(e_\gamma)\psi(e_\omega) = \lim_\gamma \psi(e_\gamma e_\omega) = \psi(e_\omega)$ and similarly, $\lim_\gamma \psi(e_\omega)\psi(e_\gamma) = \lim_\gamma \psi(e_\omega e_\gamma) = \psi(e_\omega)$ for any $\omega \in \Gamma$, the statement follows from Lemma 3.1.

Before we proceed we observe that for any $a \in \mathcal{A}$

$$\varphi(p\psi(a)) = \text{w*-}\lim_{\gamma\to\mathcal{U}} \varphi(\psi(e_\gamma a)), \tag{3.2}$$

as it follows from the calculation

$$\begin{aligned}\varphi(p\psi(a)) &= \varphi\left(\varphi^{-1}\left(\text{w*-}\lim_{\gamma\to\mathcal{U}}\varphi(\psi(e_\gamma))\right)\psi(a)\right) = (\varphi\circ\rho_{\psi(a)}\circ\varphi^{-1})(\text{w*-}\lim_{\gamma\to\mathcal{U}}\varphi(\psi(e_\gamma)))\\ &= \text{w*-}\lim_{\gamma\to\mathcal{U}}\varphi(\psi(e_\gamma)\psi(a)) = \text{w*-}\lim_{\gamma\to\mathcal{U}}\varphi(\psi(e_\gamma a)).\end{aligned}$$

(2) Let us fix $c \in \mathcal{J}$. Then from (3.2) and the fact that (e_γ) is a b.a.i. for $\mathcal{J}$ we obtain

$$\varphi(p\psi(c)) = \text{w*-}\lim_{\gamma\to\mathcal{U}}\varphi(\psi(e_\gamma c)) = \varphi(\psi(c)),$$

proving $p\psi(c) = \psi(c)$. An analogous argument shows $\psi(c)p = \psi(c)$.

(3) Let us fix $a \in \mathcal{A}$. Since $e_\gamma a \in \mathcal{J}$ for any $\gamma \in \Gamma$, it follows from (2) that $\psi(e_\gamma a) = \psi(e_\gamma a)p = \psi(e_\gamma)\psi(a)p = \rho_{\psi(a)p}\psi(e_\gamma)$. From this and (3.2) we obtain

$$\begin{aligned}\varphi(p\psi(a)) &= \text{w*-}\lim_{\gamma\to\mathcal{U}}\varphi(\psi(e_\gamma a)) = \text{w*-}\lim_{\gamma\to\mathcal{U}}(\varphi\circ\rho_{\psi(a)p}\circ\varphi^{-1})(\varphi(\psi(e_\gamma)))\\ &= (\varphi\circ\rho_{\psi(a)p}\circ\varphi^{-1})\left(\text{w*-}\lim_{\gamma\to\mathcal{U}}\varphi(\psi(e_\gamma))\right) = \varphi(p\psi(a)p).\end{aligned}$$

Consequently $p\psi(a) = p\psi(a)p$ holds. A similar argument shows $\psi(a)p = p\psi(a)p$.

(4) It is clear that θ is a bounded linear map. Let us first fix $a \in \mathcal{A}$. From (3) we have $\psi(a)p = p\psi(a)p$ and hence

$$\theta(a) = \psi(a) - \psi(a)p - p\psi(a) + p\psi(a)p = \psi(a) - p\psi(a). \tag{3.3}$$

From the above, another application of (3), and the fact that ψ is an algebra homomorphism it follows that θ is multiplicative. Finally, it is straightforward from (2) that $\theta|_{\mathcal{J}} = 0$. □

Before we proceed let us recall some basic probability-theoretic background and terminology. In the brief exposition below we follow Fremlin's book [10, Sections 254J-254R].

Remark 3.4. We consider the the probability space $(\{0,1\}, \mathcal{P}(\{0,1\}), \mu)$ where μ is the "fair-coin" probability measure, i.e., $\mu(\{0\}) = 1/2 = \mu(\{1\})$. Let $(\{0,1\}^{\mathbb{N}}, \Lambda, \nu)$ denote the product of the system $\big((\{0,1\}, \mathcal{P}(\{0,1\}), \mu)\big)_{n\in\mathbb{N}}$ of probability spaces. The measure space $(\{0,1\}^{\mathbb{N}}, \Lambda, \nu)$ is isomorphic to $([0,1], \mathcal{A}, \lambda)$, where λ is the Lebesgue measure restricted to $[0,1]$. Consequently for all $1 \le p < \infty$ the spaces $L_p(\{0,1\}^{\mathbb{N}}, \Lambda, \nu)$ and $L_p([0,1], \mathcal{A}, \lambda)$ are isometrically isomorphic as Banach spaces (see also [2, page 125]).

For any $S \subseteq \mathbb{N}$ let us define

$$\pi_S : \{0,1\}^{\mathbb{N}} \to \{0,1\}^S; \quad (x_n)_{n\in\mathbb{N}} \mapsto (x_n)_{n\in S}$$

and

$$\Lambda_S := \Big\{A \in \Lambda : A = \pi_S^{-1}[\pi_S[A]]\Big\}.$$

The set Λ_S is a σ-subalgebra of Λ. In the case when S is an infinite subset of $\mathbb{N}$, it follows that $(\{0,1\}^{\mathbb{N}}, \Lambda_S, \nu|_{\Lambda_S})$ is isomorphic to $([0,1], \mathcal{A}, \lambda)$, thus for any $1 \le p < \infty$ the Banach spaces $L_p(\{0,1\}^{\mathbb{N}}, \Lambda_S, \nu|_{\Lambda_S})$ and $L_p([0,1], \mathcal{A}, \lambda)$ are isometrically isomorphic. On the other hand, if S is a finite subset of $\mathbb{N}$ then $L_p(\{0,1\}^{\mathbb{N}}, \Lambda_S, \nu|_{\Lambda_S})$ is a finite-dimensional Banach space; this follows easily from the fact that Λ_S is a finite set in that case.

The above technique is well know among experts in Banach space theory, we refer the interested reader to [15] for a more sophisticated approach.

Part (2) of the following result we learned from William B. Johnson, and it forms part of ongoing joint work between W. B. Johnson, Ch. Phillips and G. Schechtman. With their kind permission we give our version of the proof here.

Proposition 3.5. *Let X be a Banach space such that one of the following two conditions is satisfied.*

(1) *X has a subsymmetric Schauder basis; or*
(2) *$X = L_p[0,1]$ where $1 \leq p < \infty$.*

Then $\mathcal{B}(X)$ admits a family $\mathcal{Q}$ of commuting, almost orthogonal idempotents such that $|\mathcal{Q}| = 2^{\aleph_0}$ and $\operatorname{Ran}(P) \simeq X$ for every $P \in \mathcal{Q}$.

Proof. We take an almost disjoint family $\mathcal{D}$ of continuum cardinality consisting of infinite subsets of $\mathbb{N}$.

(1) Suppose X has a subsymmetric Schauder basis (b_n) with coordinate functionals (f_n). Let $\mathcal{Q} := \{P_N\}_{N \in \mathcal{D}}$, where for $N \in \mathcal{D}$

$$P_N x := \sum_{n \in N} \langle x, f_n \rangle b_n \qquad (x \in X)$$

defines an idempotent in $\mathcal{B}(X)$. Clearly $P_N P_M = P_{N \cap M} = P_M P_N$ has finite rank for distinct $N, M \in \mathcal{D}$ and $\operatorname{Ran}(P_N) \simeq X$ for every $N \in \mathcal{D}$ due to Proposition 2.3.

(2) In the notation of Remark 3.4, for every $N \in \mathcal{D}$ we consider the conditional expectation operator

$$\mathbb{E}(\cdot|\Lambda_N) : \ L_p(\{0,1\}^{\mathbb{N}}, \Lambda, \mu) \to L_p(\{0,1\}^{\mathbb{N}}, \Lambda, \mu); \quad f \mapsto \mathbb{E}(f|\Lambda_N). \tag{3.4}$$

By [2, Lemma 6.1.1], for any $N \in \mathcal{D}$ the bounded linear operator $\mathbb{E}(\cdot|\Lambda_N)$ is an idempotent with range $L_p(\{0,1\}^{\mathbb{N}}, \Lambda_N, \mu|_{\Lambda_N})$, so in particular $\operatorname{Ran}(\mathbb{E}(\cdot|\Lambda_N))$ is isomorphic to $L_p([0,1], \mathcal{A}, \lambda)$. It follows from [10, Theorem 254Ra] that for any two distinct $N, M \in \mathcal{D}$

$$\mathbb{E}(\cdot|\Lambda_N)\mathbb{E}(\cdot|\Lambda_M) = \mathbb{E}(\cdot|\Lambda_{N \cap M}),$$

where $\operatorname{Ran}\left(\mathbb{E}(\cdot|\Lambda_{N \cap M})\right) = L_p(\{0,1\}^{\mathbb{N}}, \Lambda_{N \cap M}, \mu|_{\Lambda_{N \cap M}})$ is finite dimensional.

Let $T : L_p([0,1], \mathcal{A}, \lambda) \to L_p(\{0,1\}^{\mathbb{N}}, \Lambda, \mu)$ be an isomorphism. Let $P_N := \mathbb{E}(\cdot|\Lambda_N)$ and $Q_N := T^{-1} P_N T$ for all $N \in \mathcal{D}$. Then $Q_N \in \mathcal{B}(L_p[0,1])$ is an idempotent with $\operatorname{Ran}(Q_N) \simeq \operatorname{Ran}(P_N)$ and thus

$$\operatorname{Ran}(Q_N) \simeq \operatorname{Ran}(P_N) = L_p(\{0,1\}^{\mathbb{N}}, \Lambda_N, \mu|_{\Lambda_N}) \simeq L_p([0,1], \mathcal{A}, \lambda).$$

Since $\operatorname{Ran}(Q_N Q_M)$ is finite-dimensional for distinct $N, M \in \mathcal{D}$ we obtain that the system $\mathcal{Q} := \{Q_N\}_{N \in \mathcal{D}}$ satisfies all of our requirements. ☐

The following fact is standard, we leave its proof to the reader.

Lemma 3.6. *Let X be a Banach space and let $(Q_i)_{i \in I}$ be a bounded system of mutually orthogonal, non-zero idempotents in $\mathcal{B}(X)$. Then for the density of X we have $\operatorname{dens}(X) \geq |I|$.*

Remark 3.7. Let $\mathcal{A}$ be an algebra, let $\mathcal{J} \trianglelefteq \mathcal{A}$ be a two-sided ideal. If $p, q \in \mathcal{A}$ are idempotents with $p \sim q$ then $p \in \mathcal{J}$ if and only if $q \in \mathcal{J}$. Indeed, let $a, b \in \mathcal{A}$ be such that $ab = p$ and $ba = q$. Hence $p = p^2 = abab = aqb$ and similarly $q = bpa$.

The following proposition is a dichotomy result about separable representations of $\mathcal{B}(X)$ for certain Banach spaces X, in the sense of Berkson and Porta [4]. In particular, Proposition 3.8 generalises their result [4, Corollary 6.16].

Proposition 3.8. *Let X be a Banach space such that one of the following two conditions is satisfied.*

(1) *X has a subsymmetric Schauder basis; or*
(2) *$X = L_p[0,1]$ where $1 \leq p < \infty$.*

Let Y be a separable Banach space and let $\theta : \mathcal{B}(X) \to \mathcal{B}(Y)$ be a continuous algebra homomorphism. Then θ is either injective or $\theta = 0$.

Proof. Assume towards a contradiction that θ is not injective and $\theta \neq 0$. In particular $\mathcal{K}(X) \subseteq \mathrm{Ker}(\theta)$. By Lemma 3.5, $\mathcal{B}(X)$ admits a family $(P_i)_{i \in I}$ of commuting, almost orthogonal idempotents such that $|I| = 2^{\aleph_0}$ and $\mathrm{Ran}(P_i) \simeq X$ for each $i \in I$. We *claim* that $(\theta(P_i))_{i \in I}$ is a bounded family of mutually orthogonal, non-zero idempotents of continuum cardinality in $\mathcal{B}(Y)$. To see this, we observe first that $\theta(P_i)\theta(P_j) = \theta(P_i P_j) = 0$ for each distinct $i, j \in I$, as $P_i P_j \in \mathcal{K}(X) \subseteq \mathrm{Ker}(\theta)$. Now observe that $\theta(P_i)$ is non-zero for each $i \in I$. Indeed, $I_X \notin \mathrm{Ker}(\theta)$ as θ is non-zero. Since $\mathrm{Ran}(P_i) \simeq X$ for all $i \in I$, in view of Lemma 2.2 this means $P_i \sim I_X$ thus by Remark 3.7 we obtain $P_i \notin \mathrm{Ker}(\theta)$. This shows the claim. But now with Lemma 3.6 we obtain $\mathrm{dens}(Y) \geq 2^{\aleph_0}$, a contradiction. □

4. Proof of the main results

4.1. **The proof of Theorem 1.1.** We are now in position to prove our main result. Before we get to it, let us mention that the techniques below are akin to those employed by Molnár to Hilbert spaces in [22], some of which techniques go back to at least Johnson's seminal work on approximately multiplicative maps between Banach algebras, see [14].

Proof of Theorem 1.1. Since Y is reflexive, in view of Lemma 2.4 we can take the canonical, isometric predual $(Y \hat{\otimes}_\pi Y^*, \varphi)$ of $\mathcal{B}(Y)$.

If X has a subsymmetric basis, let this be denoted by (b_n). If $X = L_p[0,1]$, where $1 < p < \infty$, then (b_n) denotes the Haar basis. In both cases (f_n) stands for the sequence of coordinate functionals associated to (b_n). As X is reflexive, it follows from Corollary 2.5 that the sequence of coordinate projections (P_n) is a b.a.i. for $\mathcal{K}(X)$.

Since $(\phi(P_n))$ is a bounded, increasing sequence of idempotents in $\mathcal{B}(Y)$ it follows from Lemma 3.2 that there exists an idempotent $P \in \mathcal{B}(Y)$ such that $(\phi(P_n))$ converges to P in the strong operator topology. We show that in fact $P = I_Y$. To this end we consider the map

$$\theta : \mathcal{B}(X) \to \mathcal{B}(Y); \quad A \mapsto (I_Y - P)\phi(A)(I_Y - P),$$

which is a continuous algebra homomorphism with $\theta|_{\mathcal{K}(X)} = 0$ by Lemma 3.3. Due to separability of Y, Proposition 3.8 yields $\theta = 0$. By the assumption, we can take $T \in \mathcal{B}(X)$ such that $\phi(T)$ has dense range. Consequently

$$0 = \theta(T) = (I_Y - P)\phi(T)(I_Y - P) = (I_Y - P)\phi(T)$$

by equation (3.3). So $(I_Y - P)|_{\mathrm{Ran}(\phi(T))} = 0$ and $\mathrm{Ran}(\phi(T))$ is dense in Y, hence $P = I_Y$.

Let $x_0 \in X$ be such that $\|x_0\| = 1$, and choose $f_0 \in X^*$ such that $\langle x_0, f_0 \rangle = 1 = \|f_0\|$. As ϕ is injective, we can pick $y_0 \in Y^*$ with $\|y_0\| = 1$ such that $\phi(x_0 \otimes f_0)y_0 \neq 0$. Thus we can define the non-zero map

$$S : X \to Y; \quad x \mapsto \phi(x \otimes f_0)y_0$$

which is easily seen to be linear and bounded. We observe that

$$SA = \phi(A)S \qquad (A \in \mathcal{B}(X)). \tag{4.1}$$

Indeed, fix $A \in \mathcal{B}(X)$ and $x \in X$. Then

$$\phi(A)Sx = \phi(A)\phi(x \otimes f_0)y_0 = \phi(A(x \otimes f_0))y_0 = \phi(Ax \otimes f_0)y_0 = SAx.$$

In the following we show that S is an isomorphism.

We observe that S is injective. For assume in search of a contradiction it is not; let $x \in X$ be such that $Sx = 0$ and $\|x\| = 1$. Let $f \in X^*$ be such that $\langle x, f\rangle = 1 = \|f\|$. Then in view of equation (4.1) we have that

$$0 = \phi(z \otimes f)Sx = S(z \otimes f)x = S(\langle x, f\rangle z) = Sz \qquad (z \in X).$$

Thus $S = 0$, a contradiction.

We show that S has closed range. To this end, let (x_n) be a sequence in X such that (Sx_n) converges to some $y \in Y$. Let $x \in X$ be non-zero. As S is injective, we have $Sx \neq 0$; thus we can choose $h \in Y^*$ with $\langle Sx, h\rangle = 1$. Let $f \in X^*$ be arbitrary fixed, then

$$\langle x_n, f\rangle Sx = S(x \otimes f)x_n = \phi(x \otimes f)Sx_n \qquad (n \in \mathbb{N}),$$

hence $\langle x_n, f\rangle Sx \to \phi(x \otimes f)y \in Y$ and therefore $\langle x_n, f\rangle \to \langle \phi(x \otimes f)y, h\rangle$. As $f \in X^*$ was arbitrary, this shows that (x_n) is a weak Cauchy sequence in X. Since X is reflexive, it is weakly sequentially complete (see *e.g.* [5, Chapter V, Corollary 4.4]), hence (x_n) converges weakly to some $x' \in X$. As S is weakly continuous, $Sx_n \to Sx'$ weakly in Y. But (Sx_n) converges in norm to $y \in Y$, so it also converges to y weakly, thus by uniqueness of the weak limit $Sx' = y$.

It remains to show that that S has dense range. Clearly $b_n \otimes f_n \in \mathcal{B}(X)$ is a rank one idempotent, hence by the assumption $\phi(b_n \otimes f_n) \in \mathcal{B}(Y)$ is a rank one idempotent too for each $n \in \mathbb{N}$. Let $u_n \in Y$ and $h_n \in Y^*$ be such that $\phi(b_n \otimes f_n) = u_n \otimes h_n$ and $\langle u_n, h_n\rangle = 1$. Recall that $(\phi(P_n))$ converges to I_X in the strong operator topology, consequently

$$x = \lim_{n\to\infty} \phi(P_n)x = \sum_{i=1}^{\infty} \phi(b_i \otimes f_i)x = \sum_{i=1}^{\infty}(u_i \otimes h_i)x = \sum_{i=1}^{\infty}\langle x, h_i\rangle u_i \qquad (x \in X).$$

This shows $X = \overline{\operatorname{span}}\{u_n : n \in \mathbb{N}\}$. To conclude the proof, it suffices to show that $u_n \in \operatorname{Ran}(S)$ for each $n \in \mathbb{N}$. This essentially follows from equation (4.1), as

$$Sb_n = S(b_n \otimes f_n)b_n = \phi(b_n \otimes f_n)Sb_n = (u_n \otimes h_n)Sb_n = \langle Sb_n, h_n\rangle u_n$$

for each $n \in \mathbb{N}$. Injectivity of S implies that Sb_n is non-zero, hence $\langle Sb_n, h_n\rangle \neq 0$ by the equation above, which yields $u_n \in \operatorname{Ran}(S)$ indeed. Thus equation (4.1) amounts to

$$\phi(A) = SAS^{-1} \qquad (A \in \mathcal{B}(X)),$$

which proves that ϕ is an isomorphism. □

Example 4.1. Each of the following spaces are reflexive and have a subsymmetric basis, hence satisfy the conditions of Theorem 1.1 (2):

(1) The sequence spaces ℓ_p $(1 < p < \infty)$, see Section 2.1.5;
(2) Every reflexive Orlicz sequence space l_M with Orlicz function M satisfying the Δ_2-condition $\limsup_{t\to 0} M(2t)/M(t) < +\infty$, by [18, Propositions 4.a.4 and 3.a.3];
(3) A Lorentz sequence space $d(w, p)$ with $p > 1$ and non-increasing $w = (w_n)_{n\in\mathbb{N}}$, $w_1 = 1$, $\lim\limits_{n\to\infty} w_n = 0$ and $\sum_{n=1}^{\infty} w_n = +\infty$, by [18, Propositions 4.e.3 and 1.c.12].

Remark 4.2. In the proof of Theorem 1.1 the cornerstone of our argument is that X has *uncountably many* complemented subspaces, each of which are isomorphic to X itself, but any two have a finite-dimensional intersection. We hope that Remark 4.6 at the end of the paper sheds some light on why this phenomenon might be essential.

4.2. **The proof of Theorem 1.2.** In the following let X and Y be arbitrary non-zero Banach spaces, and let $\psi, \phi : \mathcal{B}(X) \to \mathcal{B}(Y)$ be algebra homomorphisms such that $\|\psi(A) - \phi(A)\| < \|A\|$ for each non-zero $A \in \mathcal{B}(X)$.

With an application of the triangle inequality we arrive to the simple but useful estimate

$$\|\psi(A)\| \leq \|\psi(A) - \phi(A)\| + \|\phi(A)\| < \|A\| + \|\phi(A)\|. \tag{4.2}$$

Similarly we obtain $\|\phi(A)\| < \|A\| + \|\psi(A)\|$. In particular, these estimates immediately yield that ϕ is continuous if and only if ψ is.

Lemma 4.3. *Let $P \in \mathcal{B}(X)$ be a norm one idempotent. Then $P \in \operatorname{Ker}(\phi)$ if and only if $P \in \operatorname{Ker}(\psi)$. Consequently, ψ is injective if and only if ϕ is injective.*

Proof. Assume $P \in \operatorname{Ker}(\phi)$. Then it follows from equation (4.2) that $\|\psi(P)\| < \|P\| = 1$. As $\psi(P) \in \mathcal{B}(Y)$ is an idempotent, this is equivalent to saying $\psi(P) = 0$. The other direction follows analogously.

In order to show the "consequently" part suppose contrapositively that ψ is not injective. Let $x \in X$ be such that $\|x\| = 1$, pick $f \in X^*$ such that $\langle x, f \rangle = 1 = \|f\|$. Hence $x \otimes f \in \mathcal{F}(X)$ is a norm one idempotent. In particular $x \otimes f \in \operatorname{Ker}(\psi)$, which by the first part of the lemma is equivalent to $x \otimes f \in \operatorname{Ker}(\phi)$. This shows that ϕ is not injective. Similarly, one obtains that injectivity of ψ implies injectivity of ϕ. □

Proposition 4.4. *Let $P \in \mathcal{B}(X)$ be a norm one idempotent. Then $\operatorname{Ran}(\psi(P)) \simeq \operatorname{Ran}(\phi(P))$. If ψ is surjective then $\phi(I_X) = I_Y$. Moreover, if ψ is an isomorphism, then $\operatorname{Ran}(\phi(P)) \simeq \operatorname{Ran}(P)$.*

Proof. As $\|P\| = 1$, the estimate $\|\psi(P) - \phi(P)\| < 1$ and Lemma 2.1 imply $\psi(P) \sim \phi(P)$. In view of Lemma 2.2 this is equivalent to saying $\operatorname{Ran}(\psi(P)) \simeq \operatorname{Ran}(\phi(P))$.

Suppose ψ is surjective, then $\psi(I_X) = I_Y$. Indeed, there is $A \in \mathcal{B}(X)$ such that $\psi(A) = I_Y$, hence $\psi(I_X) = \psi(I_X)I_Y = \psi(I_X)\psi(A) = \psi(I_X A) = \psi(A) = I_Y$. Therefore

$$\|I_Y - \phi(I_X)\| = \|\psi(I_X) - \phi(I_X)\| < 1,$$

which by the Carl Neumann series implies that $\phi(I_X)$ is invertible in $\mathcal{B}(Y)$. As $\phi(I_X)$ is an idempotent, $\phi(I_X) = I_Y$ must hold.

Suppose ψ is an isomorphism. By Eidelheit's Theorem (see *e.g.* [6, Theorem 2.5.7]) there is an isomorphism $S \in \mathcal{B}(X, Y)$ such that $\psi(A) = SAS^{-1}$ for each $A \in \mathcal{B}(X)$. In particular, $(SP)(PS^{-1}) = SPS^{-1} = \psi(P)$ and $(PS^{-1})(SP) = P$ imply (with Lemma 2.2) that $\operatorname{Ran}(P) \simeq \operatorname{Ran}(\psi(P))$. By the first part of the proposition $\operatorname{Ran}(\phi(P)) \simeq \operatorname{Ran}(P)$ follows. □

From this point on, we assume that the properties prescribed by the conditions of Theorem 1.1 stand for the Banach spaces X and Y, and $\psi : \mathcal{B}(X) \to \mathcal{B}(Y)$ is assumed to be surjective. We recall that due to the deep automatic continuity result of B. E. Johnson [13], any surjective homomorphism between algebras of operators of Banach spaces is automatically continuous (see *e.g.* [6, Theorem 5.1.5] for a detailed proof).

Outfitted with Theorem 1.1 and the results above, we are now able to provide the

Proof of Theorem 1.2. We first observe that ψ is automatically injective. Indeed, Y is non-zero, hence ψ is non-zero, since it is surjective. By Proposition 3.8 it follows that ψ is injective.

Thus by Lemma 4.3, ϕ is injective too. Continuity of ψ and equation (4.2) imply that ϕ is continuous. Furthermore, from Proposition 4.4 we conclude that $\phi(I_X) = I_Y$ (which witnesses that $\mathrm{Ran}(\phi)$ contains an operator with dense range), and ϕ preserves rank one idempotents. Hence Theorem 1.1 applies. □

4.3. **The proof of Proposition 1.3.** In each of the following examples, $\mathcal{B}(X)$ has a character. In examples (1)–(3) this character is shown explicitly and in example (4) the character is obtained from a commutative quotient on $\mathcal{B}(X)$. We remark in passing that the list below is not intended to be comprehensive.

Example 4.5. Each of the following spaces X are such that $\mathcal{B}(X)$ has a character:

(1) $X = \mathcal{J}_p$ where $1 < p < \infty$ and $\mathcal{J}_p$ is the p^{th} James space, since by [9, Paragraph 8] $\mathcal{B}(X)$ has a character whose kernel is $\mathcal{W}(X)$, the *ideal of weakly compact operators*, see also [16, Theorem 4.16];

(2) $X = C[0, \omega_1]$, where ω_1 is the first uncountable ordinal, since by [9, Paragraph 9] $\mathcal{B}(X)$ has a character, see also [20, Proposition 3.1];

(3) $X = X_{GM}$ is the *hereditarily indecomposable* Banach space constructed by Gowers and Maurey in [12], since $\mathcal{B}(X)$ has a character whose kernel is $\mathcal{S}(X)$, the ideal of *strictly singular operators*;

(4) $X = \mathcal{G}$, where $\mathcal{G}$ is the Banach space constructed by Gowers in [11], because we have $\mathcal{B}(X)/\mathcal{S}(X) \simeq \ell_\infty/c_0$, as shown in [16, Corollary 8.3].

Proof of Proposition 1.3. Let $\chi : \mathcal{B}(X) \to \mathbb{C}$ be a character. Let Z be a non-zero Banach space, and consider the map

$$\phi : \mathcal{B}(X) \to \mathcal{B}(X \oplus Z); \quad T \mapsto \begin{bmatrix} T & 0 \\ 0 & \chi(T) I_Z \end{bmatrix}.$$

From $\chi(I_X) = 1$ it is immediate that $\phi(I_X) = I_{X \oplus Z}$. As χ is a norm one algebra homomorphism, it readily follows that ϕ is norm one algebra homomorphism too. The map ϕ is clearly injective. Let $x_0 \in X$ and $f_0 \in X^*$ such that $\langle x_0, f_0 \rangle \neq 0$. As χ is a character of $\mathcal{B}(X)$ and $\mathcal{F}(X)$ is the smallest non-trivial, two-sided ideal of $\mathcal{B}(X)$, we have $x_0 \otimes f_0 \in \mathcal{F}(X) \subseteq \mathrm{Ker}(\chi)$. Thus

$$\phi(x_0 \otimes f_0) = \begin{bmatrix} x_0 \otimes f_0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} f_0 \\ 0 \end{bmatrix},$$

from which it also follows that ϕ maps rank one operators into rank one operators. Finally, it is obvious that ϕ cannot be surjective.

The second part of the proposition is an immediate corollary of Examples 4.5 (1)–(2), the first part of the proposition with the choice $Z := \mathbb{C}$, and the fact that $X \simeq X \oplus \mathbb{C}$. Although the latter is certainly well-known, for completeness we give the details:

(1) *Let* $X := \mathcal{J}_p$, *where* $1 < p < \infty$. Recall that the James space is both one-codimensional and isometrically isomorphic to its bidual (see *e.g* [2, Theorem 3.4.6]). Consequently $X \simeq X^{**} \simeq X \oplus \mathbb{C}$.

(2) *Let* $X := C[0, \omega_1]$. As X has a complemented copy of c_0 (see [3, Proposition 3.2]), and of course $c_0 \simeq c_0 \oplus \mathbb{C}$, we conclude $X \simeq X \oplus \mathbb{C}$. □

Remark 4.6. In light of Proposition 1.3 and Example 4.5 let us make a few remarks about possible weakenings of the conditions in Theorem 1.1. In the following $Y := X \oplus \mathbb{C}$.

- Let $X := X_{GM}$. It is shown [12] that X_{GM} is reflexive and has a Schauder basis, and hence Y is separable and reflexive. This shows that in Theorem 1.1, the conditions on X cannot be weakened to "X is reflexive and has a Schauder basis".

- Let $X := \mathcal{G}$. It is shown in [11] that $\mathcal{G}$ has an unconditional Schauder basis, hence Y is separable. This shows that in Theorem 1.1, the conditions on X and Y cannot be weakened to "X has an unconditional basis and Y is separable".

Acknowledgement. The authors are grateful to W. B. Johnson (Texas A&M) for the useful e-mail exchanges regarding Proposition 3.5. They are also indebted to N. J. Laustsen (Lancaster) for his invaluable suggestions and for carefully reading the first draft of the manuscript.

References

1. Y. A. Abramovich and C. D. Aliprantis, *An Invitation to Operator Theory*, American Mathematical Society, 2002.
2. F. Albiac and N. J. Kalton, *Topics in Banach Space Theory*, Springer-Verlag, USA, 2006.
3. S. A. Argyros, J. F. Castillo, A. S. Granero, M. Jimenéz, and J. P. Moreno, *Complementation and embeddings of $c_0(I)$ in Banach spaces*, Proc. Lond. Math. Soc. **85** (2002), no. 3, 742–768.
4. E. Berkson and H. Porta, *Representations of $\mathcal{B}(X)$*, J. Funct. Anal. **3** (1969), no. 1, 1–34.
5. J. B. Conway, *A Course in Functional Analysis*, Springer-Verlag, New York, 1990.
6. H. G. Dales, *Banach Algebras and Automatic Continuity*, Oxford University Press Inc., New York, 2000.
7. M. Daws and B. Horváth, *Ring-theoretic (in)finiteness in reduced products of Banach algebras*, Preprint. arXiv:1912.07108.
8. A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, North-Holland, Amsterdam, 1993.
9. I. S. Èdel'šteĭn and B. S. Mitjagin, *The homotopy type of linear groups of two classes of Banach spaces. (Russian)*, Funkcional. Anal. i Priloen **4** (1970), no. 3, 61–72.
10. D. H. Fremlin, *Measure Theory, Vol. 2*, Torres Fremlin, 2001.
11. W. T. Gowers, *A solution to Banach's hyperplane problem*, Bull. Lond. Math. Soc. **26** (1994), no. 6, 523–530.
12. W. T. Gowers and B. Maurey, *The unconditional basic sequence problem*, J. Amer. Math. Soc. **6** (1993), no. 4, 851–874.
13. B. E. Johnson, *The uniqueness of the (complete) norm topology*, Bull. Amer. Math. Soc. **73** (1967), no. 4, 537–539.
14. B. E. Johnson, *Approximately multiplicative maps between Banach algebras*, J. Lond. Math. Soc. **37** (1988), no. 2, 294–316.
15. W. B. Johnson and G. Schechtman, *Subspaces of L_p that embed into $L_p(\mu)$ with μ finite*, Israel J. Math. **203** (2014), 211–222.
16. N. J. Laustsen, *Maximal ideals in the algebra of operators on certain Banach spaces*, Proc. Edinb. Math. Soc. **45** (2002), no. 3, 523–546.
17. ———, *On ring-theoretic (in)finiteness of Banach algebras of operators on Banach spaces*, Glasgow Math. J. **45** (2003), no. 1, 11–19.
18. J. Lindenstrauss, *On complemented subspaces of m*, Israel J. Math. **5** (1967), no. 3, 153–156.
19. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer-Verlag, Germany, 1996.
20. R. J. Loy and G. A. Willis, *Continuity of derivations on $\mathcal{B}(E)$ for certain Banach spaces E*, J. Lond. Math. Soc. **40** (1989), no. 2, 327–346.
21. R. E. Megginson, *An Introduction to Banach Space Theory*, Springer-Verlag, New York, 1998.
22. L. Molnár, *The set of automorphisms of $\mathcal{B}(\mathcal{H})$ is topologically reflexive in $\mathcal{B}(\mathcal{B}(\mathcal{H}))$*, Studia Mathematica **122** (1997), no. 2, 183–193.
23. ———, *Stability of the surjectivity of endomorphisms and isometries of $\mathcal{B}(\mathcal{H})$*, Proc. Amer. Math. Soc. **126** (1998), no. 3, 853–861.
24. A. Pietsch, *Operator Ideals*, North-Holland Publishing Company, 1979.
25. V. Runde, *Amenable Banach Algebras — a Panorama*, Springer-Verlag, Germany, 2020.
26. R. A. Ryan, *Introduction to Tensor Products of Banach Spaces*, Springer-Verlag, London, 2002.
27. I. Singer, *Bases in Banach Spaces I*, Springer-Verlag, Germany, 1970.
28. J. Zemánek, *Idempotents in Banach algebras*, Bull. London Math. Soc. **11** (1979), 177–183.

Institute of Mathematics, Czech Academy of Sciences, Žitná 25, 115 67 Prague 1, Czech Republic
E-mail address: horvath@math.cas.cz, hotvath@gmail.com

Department of Applied Analysis and Computational Mathematics, Eötvös L. University, Pázmány Péter sétány 1/c., Budapest H-1117, Hungary
E-mail address: tarcsay@cs.elte.hu