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Flows in Graphs and Related Problems

název disertace

Komise pro obhajoby doktorských disertací v oboru Matematické struktury

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I. A Survey of The Topic

Nowhere-zero flows in (finite) graphs have been introduced by Tutte [Tu1, Tu2, Tu3]. Primarily he showed that a planar graph is k-colorable if and only if its dual admits a nowhere-zero k-flow (its edges can be oriented and assigned values $\pm 1, \ldots, \pm(k-1)$ so that the sum of the incoming values equals the sum of the outcoming ones for every vertex of the graph). Tutte also proved the classical equivalence result that a graph admits a nowhere-zero k-flow if and only if it admits a flow whose values are the nonzero elements of a finite Abelian group of order k.

There are three celebrated unsolved conjectures dealing with nowhere-zero flows in bridgeless graphs, all due to Tutte. The first is the 5-Flow Conjecture of [Tu1], that every such graph admits a nowhere-zero 5-flow. The 4-Flow Conjecture of [Tu3] suggests that if the graph does not contain a subgraph contractible to the Petersen graph, then it has a nowhere-zero 4-flow. Finally the 3-Flow Conjecture of Tutte is that if the graph does not contain an edge cut of cardinality 1 and 3, then it has a nowhere-zero 3-flow. The Weak 3-Flow Conjecture, introduced by Jaeger [J4], is that there exists $k \geq 4$ such that every k-edge-connected graph has a nowhere-zero 3-flow (it is well-known that for $k = 4$ this is an equivalent form of the 3-Flow Conjecture). The 5- and 3-Flow Conjectures generalize two known properties of planar graphs, namely the classic results of Heawood [He] and Grötzsch [Gr], that every planar graph is 5-colorable and every planar graph without triangles is 3-colorable, respectively. Jaeger [J1] and Kilpatrick [Ki] proved that every bridgeless graph admits a nowhere-zero 8-flow. Seymour [Se2] has improved this result showing that every bridgeless graph admits a nowhere-zero 6-flow. Jaeger [J1] also proved that every 4-edge-connected graph has a nowhere-zero 4-flow. Certain generalizations of the last two results were obtained by Jaeger, Linial, Payan, and Tarsi [JLPT].

By a cycle we mean a graph where every vertex has even valency. A circuit is a 2-regular connected graph. A cycle double covering (in abbreviation CDC) of a graph is a family $\mathcal{S}$ of cycles such that every edge of the graph is contained in exactly two cycles from $\mathcal{S}$. If the number of cycles is k, then $\mathcal{S}$ is called a *k-CDC*. A graph has a 2-CDC if and only if it has a nowhere-zero 2-flow. The following statements are pairwise equivalent for any graph G.

- G has a 3-CDC,
- G has a 4-CDC,
- G has a nowhere-zero 4-flow.

Thus an equivalent form of the 4-Flow Conjecture is that if a bridgeless graph does not contain a subgraph contractible to the Petersen graph, then it has a 4-CDC (or, equivalently, a 3-CDC). Alspach, Goddyn, and Zhang [AGZ] proved that such graphs have a CDC. The CDC (resp. 5-CDC) Conjecture is that every bridgeless graph has a CDC (resp. 5-CDC). This conjecture was explicitly formulated by Seymour [Se1] and Szekeres [Sz] (resp. Celmins [C] and Preismann [Pr]). Jaeger [J3] pointed out that the CDC conjecture holds true if the following two conjectures are satisfied. The first one, formulated by Fleischner [Fl1], is that every cyclically 4-edge connected cubic graph has either an edge-3-coloring or a dominating circuit (i. e., a circuit whose vertices are incident to every edge of the graph). The second one is the Sabidussi's Compatibility Conjecture [Sa] that given an eulerian trail T in an eulerian graph G without 2-valent vertices, there exists a decomposition $\mathcal{S}$ of G into circuits such that consecutive edges in T belong to different circuits in $\mathcal{S}$. By Fleischner [Fl2], the Sabidussi's Compatibility Conjecture is equivalent to the conjecture that given a dominating circuit C in a cubic graph G, there exists a cycle double cover of G which includes C. A stronger conjecture, so called Fixed Circuit CDC Conjecture, was introduced by Goddyn [Go2]. It says that given a circuit C in a bridgeless graph G, there exists a CDC of G which includes C.

A survey about nowhere-zero flows and cycle double coverings in graphs can be found in the books of Zhang [Z], Jensen and Toft [JT] and the survey papers of Jaeger [J3, J4], Jackson [Jac], and Seymour [Se3].

It is well-known that the smallest counterexample to the 5-Flow and CDC Conjectures must be a snark, which is a nontrivial cubic (3-regular) graph without an edge-3-coloring. By nontrivial we mean cyclically 4-edge-connected and with girth (the length of the shortest circuit) at least 5. The term snark was introduced by Gardner [Ga] borrowing this name from Lewis Caroll's ballad "The Hunting of the Snarks". By Tait [Tai], the four-color theorem (proved by Appel and Haken [AH] and Robertson, Sanders, Seymour, and Thomas [RSST]) is equivalent to the statement that there exists no planar snark.

Construction of snarks is not an easy task. For instance the first nontrivial infinite family of them was constructed in 1975 by Isaacs [I], though the first snark, the Petersen graph, was known late in the 19th century (see Petersen [P] and Kempe [Ke]). From this experience we might concede that snarks are not common among cubic graphs. A result of Robinson and Wormald [RW], stating that almost every cubic graph is hamiltonian, thus also edge-3-colorable, supports this observation. Snarks are studied very intensively and several methods have been developed for their constructions. The most interesting were introduced in Adelson-Velskij and Titov [AVT], Goldberg [Gol], Holyer [Hol], and Isaacs [I]. Some further constructions of snarks are surveyed in Chetwynd and Wilson [CW], Watkins and Wilson [WW], Watkins [Wa], and Cavicchioli, Meschiari, Ruini, and Spaggiari [CMRS].

By Celmins [Ce] (resp. Goddyn [Go1]), the smallest counterexample to the 5-Flow (resp. CDC) Conjecture must be a cyclically 5-edge-connected snark with girth at least 7 (resp. a snark with girth at least 8). This is very interesting, because until recently, there have been known only snarks with girth at most 6. Jaeger and Swart [JS] conjectured that there is no snark with girth greater than 6 (see also [CW], [WW], [Wa]).

By Holyer [Hol], the problem to determine whether a graph admits a nowhere-zero 4-flow is NP-complete. By Garey, Johnson and Stockmeyer [GJS] the problem to determine whether a planar graph has a 3-coloring is NP-complete. Since, by Tutte [Tu1], a graph is 3-colorable if and only if its dual has a nowhere-zero 3-flow, we get that the problem to determine whether a graph admits a nowhere-zero 3-flow is also NP-complete.

Nowhere-zero flows in graphs have some common features with the classical flow model on directed graphs of Ford and Fulkerson [FF]. It is known that a graph G has a nowhere-zero k-flow if and only if it has such a flow with values $1, \dots, k-1$ (with respect to a special orientation of G). But the theorem of Hoffman [Hof] and Minty [Mi] (see also Zhang [Z]) says that a directed graph G has a flow with values $1, \dots, k-1$ if and only if G has the output-input ratio at most $k-1$ (the maximum ratio between the number of edges directed out from W and into W for a subset W of the set of vertices of G). Thus a graph has a nowhere-zero k-flow if and only if it has an orientation with the output-input ratio at most $k-1$.

Generalizations of the classical flow model on directed graphs have been introduced by Hassin [Has] and Lawler and Martel [LM1]. In these models, capacity constraints are imposed by polymatroid rank functions on the sets of edges directed into and out of each edge. A variety of classical optimization problems can be formulated and solved in terms of these models, for example the max-flow min-cut theorem of Ford and Fulkerson [FF], the abovementioned theorem of Hoffman [Hof] and Minty [Mi], Hall's theorem on distinct representatives [Ha1], Menger's theorem [Me] on maximal cuts in graphs, and Dilworth's theorem [Di] on maximal antichains in partially ordered sets (see Lawler and Martel [LM2] and Schrijver [Sc2]). It is also known that these theorems can be presented as special cases of the Edmonds' intersection theorem [Ed2], giving a necessary and sufficient condition for two matroids to have a common independent set of cardinality at least d. More sophisticated form of this theorem states that the linear system describing the intersection polytope of two polymatroids is totally dual integral (see Schrijver [Sc2, Sc3]). Note that polymatroids (introduced by Edmonds [Ed2]) are polyhedra of nonnegative vectors bounded by submodular functions. Generalized polymatroids (introduced independently by Frank [Fr] and Kovalev [Ko]) are polyhedra bounded by sub- and supermodular functions satisfying an additional condition. They are generalizations of matroids and satisfy majority of their nice properties. A flow model using generalized polymatroids as set constraints was introduced by Lawler and Martel [LM3]. Another general framework generalizing several constructions of matroids are linking systems of Schrijver [Sc1] (introduced independently by Kung [Ku]).

A result similar to the Edmonds' intersection theorem was received by Davies and McDiarmid [DM], who gave a necessary and sufficient condition for two strongly base orderable matroids to have k disjoint common independent sets of cardinality at least d. Strongly base orderable matroids form a proper subclass of matroids and were introduced by Brualdi [Bru], Brualdi and Scrimger [BS], and Mason [Ma] (see also Welsh [We]).

More details about matroids, (generalized) polymatroids and submodular functions are in [Ai], Frank and Tardos [FT], Fujishige [Fj], Lovász [Lo], Lovász and Plummer [LP], Nemhauser and Wolsey [NW], Oxley [O], Schrijver [Sc2, Sc3], and Welsh [We].

II. A Survey of The Results

In [1] we build general methods on constructing graphs without nowhere-zero k-flows. We call such graphs k-snarks. First we study flows in multi-terminal networks and generalize some classical results which have been known for flows in graphs. This enables us to develop several methods on constructing k-snarks, some of them having roots in constructions of snarks. This has many applications. We show that the 3-Flow Conjecture is equivalent to the statement that every graph with at most three edge cuts of cardinality 3 has a nowhere-zero 3-flow. Furthermore, in [3] we show that the 3-Flow Conjecture suffices to verify for 5-edge-connected graphs.

In [2] (resp. [1]) we prove that if the 5-Flow (resp. Weak 3-Flow) Conjecture is not true, then the problem to determine whether a bridgeless (resp. k-edge-connected) graph has a nowhere-zero 5-flow (resp. 3-flow) is NP-complete.

In [4] we deal with class of graphs with maximum degree four. Let $\mathcal{H}$ be the set of nonisomorphic simple graphs on four vertices (depicted in Fig. 2). For every $\mathcal{H}' \subseteq \mathcal{H}$, consider the family of graphs $\mathcal{X}(\mathcal{H}')$ such that a graph G belongs to $\mathcal{X}(\mathcal{H}')$ if and only if each vertex of G has degree at most four, and the neighborhood of each 4-degree vertex induces a graph isomorphic to a member of the set $\mathcal{H}'$. The main result of [4] is that, for a graph from $\mathcal{X}(\mathcal{H}')$, the 3-coloring problem either is NP-complete, or can be solved in linear time.

In [5] we show that cubic graphs not containing a subgraph homeomorphic to the Petersen graph have nowhere-zero 5-flows.

In [7] is indicated a construction of a new infinite family of cyclically 6-edge-connected snarks.

In [8] we prove that the smallest counterexample to the 5-Flow Conjecture must be cyclically 6-edge-connected snark.

Cyclically 5-edge-connected snarks with arbitrary large girth are constructed in [5], thereby obtaining a counterexample to the conjecture of Jaeger and Swart [JS].

Several families of snarks with some special properties are constructed in [1, Chapter 10].

In [9] we construct snarks with a (dominating) circuit C so that no other circuit C' satisfies $V(C) \subseteq V(C')$. These snarks are interesting because the Fixed Circuits CDC Conjecture and the conjecture of Fleischner [Fl2], that every bridgeless graph has a CDC containing a fixed dominating circuit (the equivalent form of the Sabidussi's Compatibility Conjecture [Sa]), suffices to verify for them.

In [10] is proved that the conjecture of Fleischner [Fl1] (that every cyclically 4-edge-connected cubic graph has either an edge-3-coloring or a dominating cycle) is equivalent to a conjecture of Thomassen [Th1] (that every 4-connected line graph is hamiltonian).

In [11] is introduced oddness of a graph (a parameter expressing how far a graph is from admitting a nowhere-zero 4-flow; for cubic graphs it is the minimal number of odd circuits in a 2-factor). Oddness of a graph is always even and is zero if and only if the graph has a nowhere-zero 4-flow. We show that if a graph has oddness 2, then it has a 5-CDC.

In [12] is proved that every cyclically 4-edge-connected cubic graph has a dominating circuit if and onlu if any two edges of a cyclically 4-edge-connected cubic graph are contained in a dominating circuit.

In [13] we show that many conjectures and theorems about graphs could be proved if one could show that they are true apart from some "errors", provided that the number of these errors grows asymptotically slower than the order of these graphs. This study we begun in [1], where we introduce k-reluctance of graphs, a parameter expressing how far a graph is from admitting a nowhere-zero k-flow. We show that the 5-flow (resp. 3-flow) conjecture is equivalent to the statement that 5-reluctance (resp. 3-reluctance) of bridgeless (resp. 4-edge-connected) graphs growth assymtotically slower than the order of graphs. Similar property holds for the conjectures of Fleischner [Fl1] and Thomassen [Th1] (which we have mentioned by describing the results from [10]).

In [14] we introduce a flow model from combinatorial optimization using quasi polymatroids as constraints for the flow conservation. Quasi polymatroids are polyhedra arising from generalized polymatroids after reflexions of some coordinates. They are introduced in [14]. Abstract network flow model is introduced in [15]. This has many common features with the flows used in [1, 2, 3, 13] and is only formally different from the model of [14]. Using abstract networks we introduce a general framework for various results regarding constructions of matroids and (generalized) polymatroids – for instance, the basic operations on (generalized) polymatroids and constructions of transversal matroids, gammoids, and their generalizations in [15].

In the last three papers [16, 17, 18] we study certain generalizations of the Hall's theorem [Ha1] of distinct representatives. Let G be a bipartite graph and assume that for any vertex v of G a strongly base orderable matroid is given on the set of edges incident to v. Call a subgraph of G a system of representatives of G if the edge neighborhood of each vertex of this subgraph is independent in the corresponding matroid. Two systems of representatives we call compatible if they have no common edge. In [16] we give a necessary and sufficient condition for G to have k pairwise compatible systems of representatives with at least d edges. Unfortunately, this condition is not sufficient if we deal with arbitrary matroids. In [17] and [18] are constructed latin $(n \times n \times (n-d)$-parallelepipeds that cannot be extended to a latin cube of order n for any pair of integers d, n where $d \geq 2$ and $n \geq 2d+1$.

III. Methods

We use methods from combinatorics and discrete mathematics, primarily from graph theory.

Majority of the constructions of snarks and graphs without nowhere-zero k-flows are based on superposition. This method is introduced in [1] and partially also in [6, 7]. We apply this method and its variants in [1, 2, 3, 6, 7, 9, 10, 13].

A new linear algebra approach to nowhere-zero flow problems on graphs is introduces in [8].

In papers [4, 5, 10, 11, 12, 13] are used methods from classical graph theory, in particular we deal with colorings, paths, cycles, and 2-factors.

In [14, 15, 16] we apply combinatorial optimization, in particular theory of matroids, polymatroids, generalized polymatroids, and submodular functions.

In [17, 18] are used methods of latin squares, in particular theorems about extensions of incomplete latin squares.

In [2] and partially in [1] we use methods from complexity theory.

IV. Description of The Results

1. Nowhere-Zero Flows

1.1. Graphs, networks and flows. The notation introduced here more or less coincides with the one from [1-15]. We use finite and unoriented graphs with multiple edges and loops, unless stated otherwise. We use standard graph theoretical terms which can be found in Bondy and Murty [BM].

If G is a graph, then $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. By a *multi-terminal network*, briefly a *network*, we mean a pair (G, U) where G is a graph and $U = \{u_1, \ldots, u_n\}$ is a set of pairwise distinct vertices of G.

We postulate that with each edge of G there are associated two distinct *arcs*. Arcs on distinct edges are distinct. If an arc on an edge is denoted by x the other is denoted by x^{-1}. If the ends of an edge e of G are vertices u and v, one of the arcs on e is said to be *directed* from u to v and the other one is directed from v to u. The two arcs on a loop, though distinct, are directed to the same vertex. (In other words, each edge of G is duplicated and the two resulting edges are directed oppositely.)

Let $D(G)$ denote the set of arcs of G. Then $|D(G)| = 2|E(G)|$. If $X \subseteq D(G)$, then denote by $X^{-1} = \{x^{-1}; x \in X\}$. By an *orientation* of G we mean every $X \subseteq D(G)$ such that $X \cup X^{-1} = D(G)$ and $X \cap X^{-1} = \emptyset$. If $W \subseteq V(G)$, then $\omega_G^+(W)$ denotes the set of the arcs of G directed from W to $V(G) \setminus W$. We write $\omega_G^+(v)$ instead of $\omega_G^+(\{v\})$.

Every Abelian group considered here is additive and has order at least two. If G is a graph and A is an Abelian group, then an *A-chain* in G is a mapping $\varphi : D(G) \to A$ such that $\varphi(x^{-1}) = -\varphi(x)$ for every $x \in D(G)$. Furthermore, the mapping $\partial\varphi : V(G) \to A$ such that

$$\partial\varphi(v) \;=\; \sum_{x\in\omega^+_G(v)} \varphi(x) \qquad (v \in V(G))$$

is called the *boundary* of φ. The set of edges associated with the arcs of G having nonzero values in φ is called the *support* of φ. An A-chain φ in G is called *nowhere-zero* if its support equals $E(G)$. If (G,U) is a network, then an A-chain φ in G is called an *A-flow* in (G,U) if $\partial\varphi(v) = 0$ for every inner vertex v of (G,U).

By a (*nowhere-zero*) *A-flow* in a graph G we mean a (nowhere-zero) A-flow in the network $(G,\emptyset)$ (i. e., the graph G is identified with the network $(G,\emptyset)$). If k is an integer ≥ 2, then by a (*nowhere-zero*) *k-flow* φ in a network (G,U) we mean a (nowhere-zero) $\mathbb{Z}$-flow in (G,U) such that $|\varphi(x)| < k$ for every $x \in D(G)$. Our concept of nowhere-zero flows in graphs coincides with the usual definition of nowhere-zero flows as presented in Jaeger [J4], Younger [Y], and Zhang [Z]. The only difference is that instead of a fixed (but arbitrary) orientation of a graph G we use the set $D(G)$ as a domain for a flow.

With every A-flow (resp. k-flow) in a network (G,U), where $U = \{u_1,\ldots, u_n\}$, is associated a *characteristic vector* $\chi(\varphi) = \langle z_1,\ldots,z_n\rangle$ so that $z_i = 0$ if $\partial\varphi(u_i) = 0$ (resp. $\partial\varphi(u_i) \equiv 0 \mod k$) and $z_i = 1$ otherwise. The *A-characteristic set* $\chi_A(G,U)$ (resp. *k-characteristic set* $\chi_k(G,U)$) of the network (G,U) is the set of all characteristic vectors $\chi(\varphi)$ where φ is a nowhere-zero A-flow in (G,U) (resp. a nowhere-zero integral k-flow in (G,U)). In the following statement, the classical equivalence results of Tutte [Tu1, Tu2, Tu3] are generalized.

Theorem 1.1. [1] *Let (G,U) be a network and $k \geq 2$ be an integer. Then the following statements are satisfied.*

(1) *(G,U) has a nowhere-zero k-flow if and only if (G,U) has a nowhere-zero A-flow for any Abelian group A of order k.*
(2) *If (G,U) admits a nowhere-zero k-flow, then it admits a nowhere-zero $(k+1)$-flow.*
(3) *$\chi_k(G,U) = \chi_A(G,U)$ for any Abelian group A of order k.*
(4) *$\chi_k(G,U) \subseteq \chi_{k+1}(G,U)$.*

By a *k-snark* we mean every network without a nowhere-zero k-flow ($k \geq 2$). We say that a graph G is a *k-snark* if $(G,\emptyset)$ is a k-snark.

Networks (G,U) and (G',U) are *homeomorphic* if they arise from the same network after applying finitely many subdivisions (subdivision vertices are always assumed to be inner). In this case $\chi_k(G,U) = \chi_k(G',U)$ for every $k \geq 2$.

1.2. Superposition. Suppose that we get a network (G',U'), $U' = \{u'_1,\ldots, u'_n\}$, from a network (G,U), $U = \{u_1,\ldots,u_n\}$, by the following process. Take a vertex w of G and replace it by a graph H disjoint from G so that each edge of G with one end (or two ends) in w gets a new end (or two new ends) from $V(H)$. Moreover, assume that $u'_i = u_i$ if $u_i \neq w$ and $u'_i \in V(H)$ if $u_i = w$. Then (G', U') is called a *w-superposition* (or a *vertex superposition*) of (G, U).

Suppose that we get a network (G', U'), $U' = \{u'_1, \dots, u'_n\}$, from a network (G, U), $U = \{u_1, \dots, u_n\}$, by the following process. Take a network $(H, \{v_1, v_2\})$ disjoint from (G, U), delete from G an edge e with ends w_1, w_2 and identify the sets of vertices $\{w_1, v_1\}$ and $\{w_2, v_2\}$ to new vertices w'_1 and w'_2, respectively. Furthermore, let $u'_i = u_i$ if $u_i \neq w_1, w_2$, and $u'_i = w'_1$ (or w'_2) if $u_i = w_1$ (or w_2). Then (G', U') is called an *e-superposition* (or an *edge superposition*) of (G, U). This superposition is *k-strong* if H is a k-snark.

A network (G', U'), $U' = \{u'_1, \dots u'_n\}$, is a (*$k$-strong*) *superposition* of (G, U), $U = \{u_1, \dots u_n\}$, if there exist a sequence $(G_1, U_1) = (G, U)$, (G_2, U_2), $\dots$, $(G_r, U_r) = (G', U')$ such that (G_{j+1}, U_{j+1}) is a vertex or (k-strong) edge superposition of (G_j, U_j) for $j = 1, \dots, r-1$. The following statement is a cornerstone of our constructions.

Lemma 1.2. [1] *Let (G', U') be a k-strong superposition of (G, U), $k \geq 2$. Then $\chi_k(G', U') \subseteq \chi_k(G, U)$. In particular, if (G, U) is a k-snark, then so is (G', U').*

Every superposition arising so that edges are replaced by k-snarks and vertices by arbitrary graphs is k-strong. By Lemma 1.2, this technique produces an infinite class of bridgeless k-snarks if we have at least one such a k-snark.

1.3. Snarks. An edge cut of a graph is called *cyclic* if after deleting its edges we get at least two components having cycles. A graph is called *cyclically k-edge-connected* if it does not have a cyclic edge cut of cardinality smaller than k. *Snarks* are cyclically 4-edge-connected cubic graphs without an edge-3-coloring and with girth (the length of the shortest circuit) at least 5. Note that a cubic graph G has an edge-3-coloring if and only if it has a nowhere-zero 4-flow (because nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flows in G coincides with an edge-3-coloring of G by nonzero elements of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$). Similarly a cubic graph G has an edge-3-coloring if and only if it has a 3-CDC. Using these facts we can show that the 5-Flow and CDC Conjectures suffices to verify for snarks.

In [1, 6, 7] we have applied superposition and constructed new families of snarks. For example take a circuit of length 6 in the Petersen graph P and replace its edges by 6 copies of the flower snark I_5 of Isaacs [I]. We get a graph G_1 indicated in Fig. 1. Replacing in G_1 each vertex of valency 7 by vertices of valencies 3, 2, and 2, we get a graph G_2. By Lemma 1.2, G_1 and G_2 are 4-snarks. Thus the graph G_{118} (homeomorphic with G_2) is a cyclically 6-edge-connected snark of order 118.

Theorem 1.3. [1, 7] *For every even $n \geq 118$, there exists a cyclically 6-edge-connected snark of order n.*

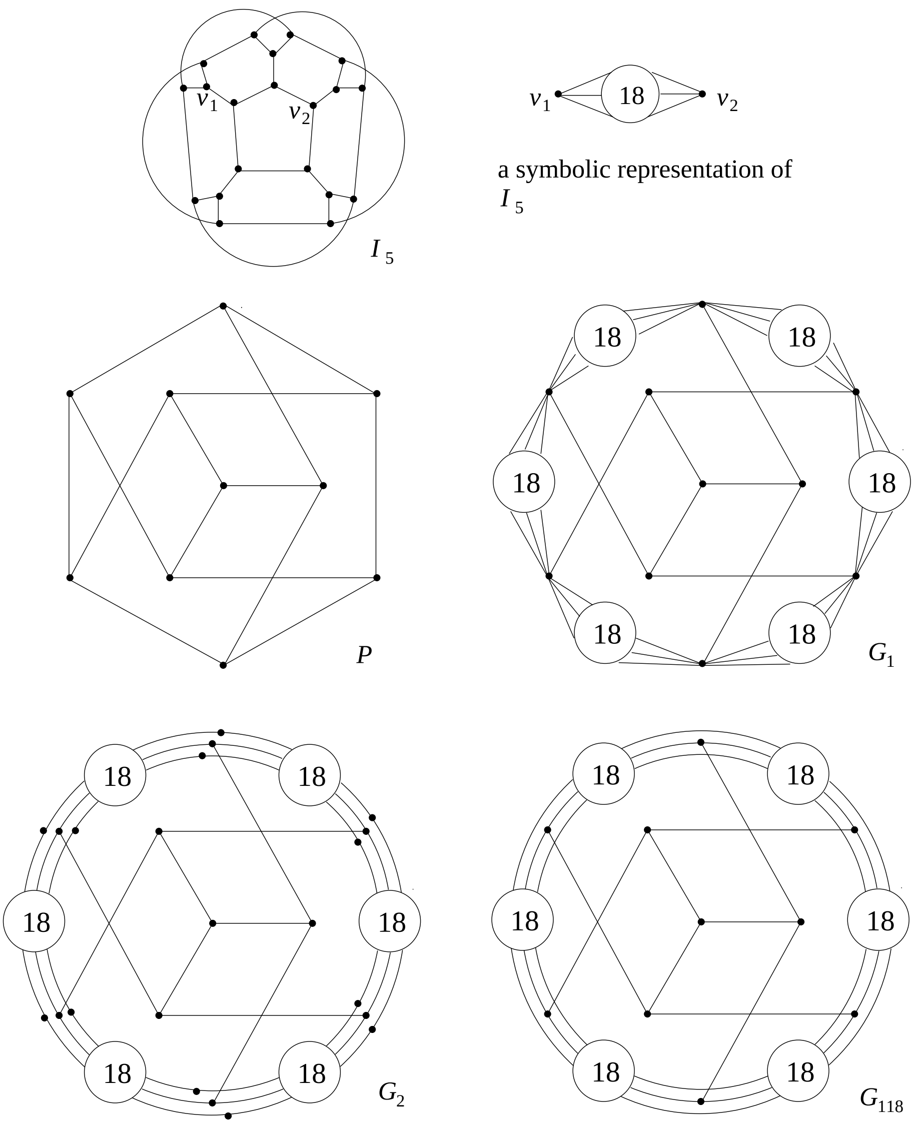

FIG. 1

By a *simple* 5-*cut snark* we mean a cyclically 5-edge-connected snark with girth 5 such that deleting any cyclic 5-edge cut we get at least one component equal to a circuit. These snarks are interesting because, by Birkhoff [Bi], the smallest counterexample to the four-color theorem must be a simple 5-cut planar snark.

Theorem 1.4. [1] *For every even $n \geq 90$, there exists a simple* 5*-cut snark of order n.*

In [6] we disprove the conjecture of Jaeger and Swart [JS], that every snark has girth at most 6.

Theorem 1.5. [6] *There exists an infinite family of cyclically* 5*-edge-connect-*

ed snarks such that if a snark of this family has order n then its girth is at least $(4/3 \pm o(1)) \log_2 n$.

The *k-reluctance* of a network (G, U) is the smallest number of inner vertices of (G, U) that can be added to U so that the resulting network admits a nowhere-zero k-flow.

Theorem 1.6. [1] *For every* $r > 0$*, there exists a cyclically* 6*-edge-connected snark of order* $118r$ *and* 4*-reluctance at least* r.

1.4. The 3-Flow Conjecture. It is well known that a cubic graph is a 3-snark if and only if it is not bipartite (see, e. g., [J4]). The 3-Flow Conjecture of Tutte is that every graph without 1- and 3-edge cuts has a nowhere-zero 3-flow. An equivalent form of this conjecture is that there does not exist a 4-edge-connected 3-snark (see, e. g., Jaeger [J4]). As pointed out before, this conjecture is true for planar graphs (see Grötzsch [Gro] or Thomassen [Th2]).

Theorem 1.7. [1] *The following statements are pairwise equivalent.*

(1) *Every graph without* 1*- and* 3*-edge cuts has a nowhere-zero* 3*-flow.*
(2) *Every bridgeless graph with at most three edge cuts of cardinality* 3 *has a nowhere-zero* 3*-flow.*
(3) *Let* G *be a bridgeless graph with vertices* v_1, v_2, v_3 *such that for any* 3*-edge cut* C *of* G *each component of* $G - C$ *has at least one vertex from* v_1, v_2, v_3. *Then* G *admits a nowhere-zero* 3*-flow.*

Note that items (2) and (3) from Theorem 1.7 are satisfied for planar graphs, as follows from results of Aksionov [Ak] and Borodin [Bo]. (As pointed out before, item (1) holds by Grötzsch [Gr]).

Theorem 1.8. [3, 13] *The following statements are pairwise equivalent.*

(1) *Every graph without* 1*- and* 3*-edge cuts has a nowhere-zero* 3*-flow.*
(2) *Every* 5*-edge-connected graph has a nowhere-zero* 3*-flow.*
(3) *Every* 5*-regular* 5*-connected hamiltonian simple graph has a nowhere-zero* 3*-flow.*

1.5. 3-Coloring of Graphs with Maximum Degree Four. By Brooks' Theorem [Bro] every connected graph with the maximum vertex degree at most three has a 3-coloring or is isomorphic to a complete graph on four vertices K_4. Hence, the decision problem, whether a given graph G has a 3-coloring, is trivial for graphs with maximum degree three. On the other hand, the complexity of the problem jumps from triviality to NP-completeness when we turn to the class of graphs with maximum degree four (see [GJS]). To make the complexity gap more precise, we study a family of graph classes that are intermediate between these two extremes. In Fig. 2 are depicted all simple graphs on four vertices and decomposed into two sets $\mathcal{H}_1$ and $\mathcal{H}_2$. For every $\mathcal{H}' \subseteq \mathcal{H}_1 \cup \mathcal{H}_2$, consider the family of graphs $\mathcal{X}(\mathcal{H}')$ such that a graph G belongs to $\mathcal{X}$ if and only if each vertex of G has degree at most four, and the neighborhood of each 4-degree vertex induces a graph isomorphic to a member of the set $\mathcal{H}'$. The main result of the paper is following.

Theorem 1.9. [4] *The problem to decide whether a graph from $\mathcal{X}(\mathcal{H}')$ is 3-colorable is*

(i) NP-*complete if* $\mathcal{H}' \cap \mathcal{H}_1 \neq \emptyset$,
(ii) *solvable in linear time if* $\mathcal{H}' \cap \mathcal{H}_1 = \emptyset$.

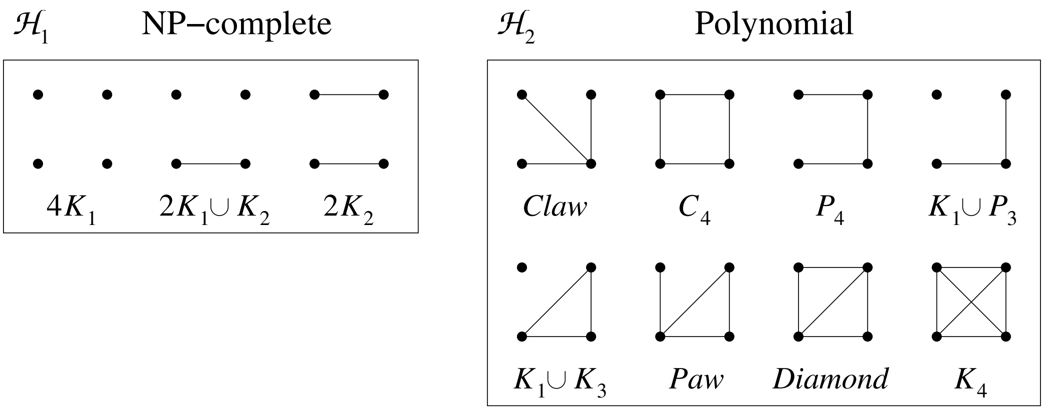

FIG. 2

1.6. Flows and complexity. As pointed out in Chapter 1, the 5-Flow (resp. Weak 3-Flow) Conjecture is that there does not exist a bridgeless 5-snark (resp. no k-edge connected 3-snark for some $k \geq 4$). In [1, 2] we have studied what happens if these conjectures are not true.

Theorem 1.10. [2] *If there exists a bridgeless graph without a nowhere-zero* 5-*flow, then the problem to determine whether a (cubic) graph admits a nowhere-zero* 5-*flow is* NP-*complete.*

Theorem 1.11. [1] *If there exists a k-edge-connected graph without a nowhere-zero* 3-*flow for some $k \geq 4$, then the problem to determine whether a k-edge-connected graph admits a nowhere-zero* 3-*flow is* NP-*complete.*

1.7. Related topics. A *circular k-flow* φ in a network (G, U) (resp. a graph G) is a nowhere-zero $\mathbb{Z}_k$-flow in (G, U) (resp. G) such that $\varphi(x) \in \{\pm 1\}$ for every $x \in D(G)$. The Circular Flow Conjecture of Jaeger [J2] is that every $4t$-edge-connected graph has a circular $(2t+1)$-flow. For $t = 1$ this is equivalent to the 3-Flow Conjecture. For $t = 2$ Jaeger [J2] proved that the 5-Flow Conjecture holds true if every 9-edge-connected graph has a circular 5-flow (see also Zhang [Z]). Thus the 3- and 5-Flow Conjectures are implied by the following one.

Conjecture 1.12. [3] Every $(4t+1)$-edge-connected graph has a circular $(2t+1)$-flow.

Let A be an Abelian group. We say that a graph G is *A-connected* if for any mapping $h : V(G) \to A$ satisfying $\sum_{v \in V(G)} h(v) = 0$ there exists a nowhere-zero A-chain φ in G such that $\partial\varphi(v) = h(v)$ for every vertex v of G. This concept was introduced in Jaeger, Linial, Payan, and Tarsi [JLPT], where is proved that every bridgeless (resp. 4-edge-connected) graph is $\mathbb{Z}_6$-connected (resp. $\mathbb{Z}_4$-connected), thereby generalizing the result of Seymour [Se2] (resp. Jaeger [J1]). Jaeger et al. [JLPT] also found a 4-connected graph that is not $\mathbb{Z}_3$-connected and conjectured that every 5-edge-connected graph is $\mathbb{Z}_3$-connected. By Theorem 1.8, this conjecture, if true, would imply the 3-Flow Conjecture.

In [8] is proved the following theorem.

Theorem 1.13. [8] *the smallest counterexample to the 5-Flow Conjecture is a cyclically 6-edge-connected snark.*

Since all cyclically 6-edge-connected snarks known until now have girth 6, such a counterexample belongs to a class of graphs for which we do not know whether it is empty.

2. Paths and Cycles

2.1. Cycle double covering. Let H be a cycle of a cubic graph G. Components of H are circuits or isolated vertices. A component of H we shall call *odd* if it is a circuit of odd length or an isolated vertex.

If G is a bridgeless cubic graph, then the minimal number of odd components of a spanning cycle of G is called the *oddness* of G and denoted by $\xi(G)$. Since the number of vertices of a cubic graph is even then also the oddness of a bridgeless cubic graph is always even. Finally, if G has a bridge, then define $\xi(G) = \infty$. Oddness of a cubic graph G is 0 if and only if it is edge-3-colorable. Thus the first nontrivial case is if $\xi(G) = 2$.

Theorem 2.1. [11] *Let G be a bridgeless cubic graph with oddness at most* 2. *Then G has a* 5*-CDC.*

Tarsi [Ta] and Goddyn [Go3] proved that every bridgeless graph with a Hamilton path has a 6-CDC. Using Theorem 2.1 we can improve this result.

Theorem 2.2. [11] *Every bridgeless graph with a Hamilton path has a* 5*-CDC.*

We can construct snarks with arbitrarily large oddness.

Theorem 2.3. [1] *For every $r > 0$, there exists a cyclically* 6*-edge-connected snark of order* $118(2r-1)$ *and oddness at least* $2r$*.*

2.2. Dominating and hamiltonian circuits. A subgraph H of a graph G is called *dominating* in G if every edge of G is incident to a vertex of H. We have mentioned that the following conjecture of Fleischner [Fl1] is an interesting approach towards a solution of the CDC Conjecture.

(a) Every cyclically 4-edge-connected cubic graph has either an edge-3-coloring or a dominating circuit.

A formally weaker conjecture was formulated by Ash and Jackson [AJ].

(b) Every cyclically 4-edge-connected cubic graph has a dominating cycle.

Fleischner and Jackson [FJ] have proved that (b) is equivalent to the following known conjecture due to Thomassen [Th1].

(c) Every 4-connected line graph is hamiltonian.

In [10] we have proved that (a) and (b) are equivalent. Thus the following holds.

Theorem 2.4. [10] *The following statements are pairwise equivalent.*

(1) *Every cyclically* 4*-edge-connected cubic graph has either an edge-*3*-coloring or a dominating circuit.*
(2) *Every cyclically* 4*-edge-connected cubic graph has a dominating circuit.*
(3) *Every* 4*-connected line graph is hamiltonian.*

Note that by Ryjáček [Ry], (c) is equivalent to the conjecture of Matthews and Sumner [MS], that every 4-connected graph without $K_{1,3}$ as an induced subgraph is hamiltonian.

A circuit C in G is called *stable* if there is no other circuit C' in G satisfying $V(C) \subseteq V(C')$. As pointed out in Chapter 1, the conjecture of Goddyn [Go2] (resp. Fleischner [Fl2]), that given a circuit (resp. a dominating circuit) C in a bridgeless graph G, there exists a CDC of G which includes C, suffices to verify for snarks with a stable circuit (resp. a stable dominating circuit). We have constructed such snarks in [9].

Theorem 2.5. [9] *For every even $k \geq 82$ there exists a snark of order n and with a stable dominating circuit.*

In [12] is proved the following result.

Theorem 2.6. [12] *The following statements are equivalent:*

(1) *Every cyclically* 4*-edge-connected cubic graph has a dominating circuit.*
(2) *Any two edges of a cyclically* 4*-edge-connected cubic graph are contained in a dominating circuit.*

2.3. Sublinear defect property. Many conjectures and theorems about graphs could be proved if one could show that they are true apart from some errors, provided that the number of these errors grows asymptotically slower than the order of these graphs. For example, the four-color theorem (proved in [AH, RSST]) is equivalent to the statement that there are infinitely many values of n such that, for every planar graph G with n vertices, all but $o(n)$ of the vertices of G can be properly colored with 4 colors.

A sequence $\{b_n\}$ of positive integers is called *frequently strongly sublinear* (in abbreviation *sublinear*) if $\liminf_{n\to\infty} b_n/n = 0$.

Theorem 2.7. [13] *The following statements are pairwise equivalent.*

(1) *Every cyclically* 4*-edge-connected cubic graph without an edge-*3*-coloring has a dominating cycle.*
(2) *Every* 4*-connected line graph is hamiltonian.*
(3) *There exists a sublinear sequence* $\{b_n\}$ *such that every cyclically* 4*-edge-connected cubic graph of order* $2n$ *contains a dominating subgraph consisting of at most* b_n *vertex disjoint paths.*
(4) *There exists a sublinear sequence* $\{b_n\}$ *such that the vertices of every* 4*-connected line graph of order* n *can be covered by at most* b_n *vertex disjoint paths.*

Similarly the following statements give formally weaker variants of the 5-Flow and the (Weak) 3-Flow Conjectures (the k-reluctance is defined in Subsection 1.4).

Theorem 2.8. [1, 13] *The following statements are pairwise equivalent.*

(1) *Every bridgeless graph has a nowhere-zero* 5*-flow.*
(2) *There exists a sublinear sequence* $\{b_n\}$ *such that every cyclically* 5*-edge-connected cubic graph of order* $2n$ *has* 5*-reluctance at most* b_n*.*

Theorem 2.9. [13] *Let* $k \geq 4$ *and* $l = 2\lfloor k/2 \rfloor + 1$*. Then the following statements are pairwise equivalent.*

(1) *Every* k*-edge-connected graph has a nowhere-zero* 3*-flow.*
(2) *Every* l*-regular* k*-connected hamiltonian simple graph has a nowhere-zero* 3*-flow.*
(3) *There exists a sublinear sequence* $\{b_n\}$ *such that every* l*-regular* k*-connected hamiltonian simple graph of order* $2n$ *has* 3*-reluctance at most* b_n*.*

The *circular* k*-reluctance* of a network (G, U) is the smallest number of inner vertices of (G, U) that can be added to U so that the resulting network admits a circular k-flow (defined in Subsection 1.6). The following statement gives formally weaker variants of the Circular Flow Conjecture of Jaeger [J2] and Conjecture 1.12.

Theorem 2.10. [13] *Let* $k \geq 4$ *and* $l = 2\lfloor k/2 \rfloor + 1$*. Then the following statements are pairwise equivalent.*

(1) *Every* k*-edge-connected graph has a circular* $(2t + 1)$*-flow.*
(2) *Every* l*-regular* k*-connected hamiltonian simple graph has a circular* $(2t + 1)$*-flow.*
(3) *There exists a sublinear sequence* $\{b_n\}$ *such that every* l*-regular* k*-connected hamiltonian simple graph of order* $2n$ *has circular* $(2t+1)$*-reluctance at most* b_n*.*

3.1. Partial intersection of generalized polymatroids. If S is a finite set, then denote by $\mathbb{R}^S$ (resp. $\mathbb{Z}^S$) the set of real- (resp. integer)-valued functions on S. If $\mathbf{u} \in \mathbb{R}^S$ and $s \in S$, then the sth coordinate of $\mathbf{u}$ is denoted by $\mathbf{u}(s)$. Furthermore, if $S' \subseteq S$, then the vector $\mathbf{u}' \in \mathbb{R}^{S'}$ such that $\mathbf{u}'(s) = \mathbf{u}(s)$ for all $s \in S'$ is called the *restriction* of $\mathbf{u}$ to S'. For two vectors $\mathbf{u} \in \mathbb{R}^S$ and $\mathbf{v} \in \mathbb{R}^T$ with $S \cap T = \emptyset$, their *direct sum* $\mathbf{u} \oplus \mathbf{v} \in \mathbb{R}^{S \cup T}$ is defined by

$$(\mathbf{u} \oplus \mathbf{v})(s) = \begin{cases} \mathbf{u}(s) \text{ if } s \in S, \\ \mathbf{v}(s) \text{ if } s \in T. \end{cases}$$

We suppose that $\mathbb{R}^{\emptyset} = \{\emptyset\}$ and $\mathbf{u} \oplus \emptyset = \mathbf{u}$.

Let $\rho\colon 2^S \to \mathbb{R} \cup \{\infty\}$, $\sigma\colon 2^S \to \mathbb{R} \cup \{-\infty\}$ such that $\rho(\emptyset) = \sigma(\emptyset) = 0$ and for every $X, Y \subseteq S$,

$$\begin{aligned} \rho(X) + \rho(Y) &\geq \rho(X \cup Y) + \rho(X \cap Y), \\ \sigma(X) + \sigma(Y) &\leq \sigma(X \cup Y) + \sigma(X \cap Y), \\ \rho(X) - \sigma(Y) &\geq \rho(X \setminus Y) - \sigma(Y \setminus X). \end{aligned}$$

(The first two rows state that ρ and σ are *submodular* and *supermodular*, respectively, and the last row states that ρ and σ are *compliant*.) Then the set

$$\mathbb{P} = \{\mathbf{u} \in \mathbb{R}^S; \sigma(X) \leq \textstyle\sum_{s \in X} \mathbf{u}(s) \leq \rho(X) \text{ for every } X \subseteq S\}$$

is called a *generalized polymatroid* (in abbreviation *g-polymatroid*) on the *ground set* S. Formally, we write $\mathbb{P} = (S, \rho, \sigma)$. If both ρ and σ are integer valued (i.e., are mappings to $\mathbb{Z} \cup \{\infty\}$ and $\mathbb{Z} \cup \{-\infty\}$, respectively), then $\mathbb{P}$ is called *integral.*

Let $\rho_\infty(\emptyset) = \sigma_\infty(\emptyset) = 0$ and $\rho_\infty(X) = \infty$, $\sigma_\infty(X) = -\infty$ for any $\emptyset \neq X \subseteq S$. Then $(S, \rho_\infty, \sigma_\infty) = \mathbb{R}^S$ is called the *free g-polymatroid* on S.

If $\sigma \equiv 0$, then ρ is monotone and nonnegative (i.e., $0 \leq \rho(X) \leq \rho(Y)$ if $X \subseteq Y \subseteq S$) and $\mathbb{P}$ is called a *polymatroid.* If $\mathbb{P}$ is polymatroid, then we formally write $\mathbb{P} = (S, \rho)$.

The following theorem brings together many results regarding constructions of (generalized) polymatroids.

Theorem 3.1. [15] *Let S, T be finite disjoint sets and $\mathbb{P}_1 = (S \cup T, \rho_1, \sigma_1)$, $\mathbb{P}_2 = (T, \rho_2, \sigma_2)$ be (integral) g-polymatroids. Suppose that $\rho_1(Y) \geq \sigma_2(Y)$, $\rho_2(Y) \geq \sigma_1(Y)$ for every $Y \subseteq T$. Then there exists an (integral) g-polymatroid $\mathbb{P} = (S, \rho, \sigma)$ such that for any $X \subseteq S$,*

$$\begin{aligned} \rho(X) &= \min_{Y \subseteq T} \left(\rho_1(X \cup Y) - \sigma_2(Y)\right), \\ \sigma(X) &= \max_{Y \subseteq T} \left(\sigma_1(X \cup Y) - \rho_2(Y)\right), \end{aligned}$$

and an (integral) vector $\mathbf{u} \in \mathbb{R}^S$ *is from* $\mathbb{P}$ *iff there exists an (integral) vector* $\mathbf{v} \in \mathbb{P}_2$ *so that* $\mathbf{u} \oplus \mathbf{v} \in \mathbb{P}_1$.

$\mathbb{P}$ is called the *partial intersection* of $\mathbb{P}_1$ and $\mathbb{P}_2$.

3.2. Abstract networks. For every $U \subseteq V(G)$, Δ_U denotes the set of arcs directed from $V(G) \setminus U$ to U (we write Δ_v if $U = \{v\}$). Note that $\Delta_U = (\omega_G^+(U))^{-1}$, where $\omega_G^+(W)$ was defined in Subsection 1.1. An *abstract g-polymatroidal flow network* $\mathcal{N}$ (in abbreviation an *abstract network*) is a loopless graph G where each vertex v of G is accompanied with a g-polymatroid $\mathbb{P}_v = (\Delta_v, \rho_v, \sigma_v)$. $\mathcal{N}$ is called *integral* if every $\mathbb{P}_v$ is integral. An $\mathbb{R}$-chain f in G is called a *flow* in $\mathcal{N}$ if for every vertex v of G, the restriction of f to Δ_v is from $\mathbb{P}_v$, (i.e., $\sigma_v(X) \le \sum_{s\in X} f(s) \le \rho_v(X)$ for every $X \subseteq \Delta_v$). If a flow in $\mathcal{N}$ is integer valued, then it is called *integral*. A *U-value* of a chain f is $\sum_{s\in\Delta_U} f(s)$ for every $U \subseteq V(G)$.

By a *U-cut* of $\mathcal{N}$ we mean a triple (U, A, B) so that $A = A' \setminus \Delta_U^{-1}$, $B = B' \setminus \Delta_U$, where the couple A', B' is a partition of $D(G)$ into two sets satisfying $(A')^{-1} = A'$, $(B')^{-1} = B'$. The *upper capacity* of the U-cut (U, A, B) is

$$c_{\mathrm{up}}(U, A, B) = \sum_{v\in U} \rho_v(\Delta_v \cap A) - \sum_{v\in V(G)\setminus U} \sigma_v(\Delta_v \cap B).$$

The *lower capacity* of the U-cut (U, A, B) is

$$c_{\mathrm{low}}(U, A, B) = \sum_{v\in U} \sigma_v(\Delta_v \cap A) - \sum_{v\in V(G)\setminus U} \rho_v(\Delta_v \cap B).$$

Clearly, $c_{\mathrm{up}}(U, A, B) = -c_{\mathrm{low}}(V(G) \setminus U, B, A)$. Note that we allow U, A or B to be empty.

The next theorem gives a necessary and sufficient condition for an abstract network to admit a flow. Theorem 3.3 is the max-flow min-cut theorem for abstract networks.

Theorem 3.2. [15] *Let* $\mathcal{N}$ *be an (integral) abstract network on a loopless graph* G*. Then the following conditions are pairwise equivalent:*

1. $\mathcal{N}$ *admits an (integral) flow.*
2. *Every* $V(G)$*- and* $\emptyset$*-cut of* $\mathcal{N}$ *has nonnegative upper capacity.*
3. *Every* $V(G)$*- and* $\emptyset$*-cut of* $\mathcal{N}$ *has nonpositive lower capacity.*

Theorem 3.3. [15] *Let* $\mathcal{N}$ *be an abstract network on a loopless graph* G *admitting a flow and* $U \subseteq V(G)$*. Then the maximal (resp. minimal) U-value of a flow in* $\mathcal{N}$ *is equal to the minimal upper (resp. maximal lower) capacity of a U-cut in* $\mathcal{N}$*. Furthermore, if* $\mathcal{N}$ *is integral and the maximal (resp. minimal) U-value of* $\mathcal{N}$ *is finite, then there exists integral flow in* $\mathcal{N}$ *with the maximal (resp. minimal) U-value.*

Suppose that $\mathcal{N}$ is an abstract network on a loopless graph G, $U \subseteq V(G)$, and f is a flow in $\mathcal{N}$. Then the restriction of f to Δ_U is called a *U-transversal* of $\mathcal{N}$. The set of all U-transversals of $\mathcal{N}$ is called a *U-gammoid* of $\mathcal{N}$. (If $U = \{v\}$, then we speak about a *v-transversal* and a *v-gammoid* of $\mathcal{N}$.) The following theorem generalizes the known result of Edmonds and Fulkerson [EF], that transversals of a finite system of sets form a matroid.

Theorem 3.4. [15] *Let $\mathcal{N}$ be an (integral) abstract network on a loopless graph G with a collection of g-polymatroids $\mathbb{P}_v = (\Delta_v, \rho_v, \sigma_v)$ ($v \in V(G)$). Suppose that $\mathcal{N}$ admits a flow and has a vertex t such that $\mathbb{P}_t = \mathbb{R}^{\Delta_t}$. Then the t-gammoid of $\mathcal{N}$ is an (integral) g-polymatroid $\mathbb{P} = (\Delta_t, \rho, \sigma)$ such that*

$$\rho(X) = \min_{Z \subseteq E(G-t)} \sum_{v \in V(G-t)} -\sigma_v(\Delta_v \cap (X^{-1} \cup D(Z))),$$

$$\sigma(X) = \max_{Z \subseteq E(G-t)} \sum_{v \in V(G-t)} -\rho_v(\Delta_v \cap (X^{-1} \cup D(Z)))$$

for any $X \subseteq \Delta_t$. Furthermore, if $\mathcal{N}$ is integral, then every integral t-transversal of $\mathcal{N}$ can be extended to an integral flow in $\mathcal{N}$.

Theorems 3.1 and 3.4 are equivalent and both can serve as a general framework for various constructions of (generalized) polymatroids and matroids. We give a simple example. Let $\mathbb{P}_1 = (S_1, \rho_1, \sigma_1)$ and $\mathbb{P}_2 = (S_2, \rho_2, \sigma_2)$ be two g-polymatroids, $S_1 \cap S_2 = \emptyset$. Then $\mathbb{P}_1 \oplus \mathbb{P}_2 = \{\mathbf{u} \oplus \mathbf{v}; \mathbf{u} \in \mathbb{P}_1, \mathbf{v} \in \mathbb{P}_2\}$ is also a g-polymatroid called the *direct sum* of $\mathbb{P}_1$ and $\mathbb{P}_2$. It can be expressed in framework of Theorem 3.4 as follows. Take a graph G with vertices u, v, t so that u and v are not adjacent and u and t (resp. v and t) are joined by $|S_1|$ (resp. $|S_2|$) parallel edges. Take an Abstract network $\mathcal{N}'$ on G so that $\mathbb{P}_u$ and $\mathbb{P}_v$ are identified with $-\mathbb{P}_1 = (S_1, -\sigma_1, -\rho_1)$ and $-\mathbb{P}_2 = (S_2, -\sigma_2, -\rho_2)$, respectively, and $\mathbb{P}_t = \mathbb{R}^{\Delta_t}$. Then the t-gammoid of $\mathcal{N}'$ is in fact $\mathbb{P}_1 \oplus \mathbb{P}_2$.

The following theorem describes the behaviour of U-gammoids.

Theorem 3.5. [15] *Let $\mathcal{N}$ be an (integral) abstract network on a loopless graph G, $U \subseteq V(G)$ and $W = V(G) \setminus U$. Then the U-gammoid of $\mathcal{N}$ is the (integral) polyhedron arising as intersection of two generalized polymatroids. Furthermore, if $\mathcal{N}$ is integral, then any integral U-transversal can be extended to an integral flow from $\mathcal{N}$.*

Let $\mathcal{N}$ be an abstract g-polymatroidal flow network on a loopless graph G. Then restricting our attention only to an orientation of G, we get a flow model from [14]. This is called a *quasi-polymatroidal flow network on G*. This flow models is more similar to the models of Hassin [Has] and Lawler and Martel [LM1, LM3], but the formulation of Theorem 3.4 seems to be better for the model from [15].

3.3. Compatible systems of representatives. If $\mathbb{P} = (S, \rho)$ is an integral polymatroid such that $\rho(X) \leq |X|$ for every $X \subseteq S$, then $\mathbb{P}$ is called a *matroid* on S. It is usually identified with the system of sets $M = \{X \subseteq S; \rho(X) = |X|\}$ (see, e.g., Aigner [Ai], Oxley [O], Welsh [We]). We also write $M = (S, \rho)$. Subset X of S satisfying $\rho(X) = |X|$ is called *independent* in M. Maximal independent sets in M are called *bases* of M. It is known that all bases of a matroid have the same cardinality. If G is a graph, then the edge sets of all subforests of G form a matroid on $E(G)$, called the *cycle matroid* of G.

We say that a matroid M is *strongly base orderable* if for any two bases B_1, B_2 there exists a bijection $\pi : B_1 \to B_2$ such that for all subsets $A \subseteq B_1$ $(B_1 \setminus A) \cup \pi(A)$ is a base of M. It is known that the cycle matroid of a graph is strongly base orderable if and only if it does not contain a subgraph homeomorphic with K_4.

Let $\mathcal{A} = (A_t : t \in T)$ be a finite family of subsets of a finite set S. Then $\mathcal{X} = (X_t : t \in T)$ is a subsystem of $\mathcal{A}$ if $X_t \subseteq A_t$ for every $t \in T$. If $s \in S$ then denote by $X_s = \{t \in T; s \in X_t\}$ $(\subseteq T)$. The *length* of $\mathcal{X}$ is the value $\sum_{t \in T} |X_t|$. Two subsystems $\mathcal{X} = (X_t : t \in T)$ and $\mathcal{X}' = (X'_t : t \in T)$ are called *compatible* if $X_t \cap X'_t = \emptyset$ for every $t \in T$.

Let $\mathcal{M}_S = (M_s : s \in S)$ and $\mathcal{M}_T = (M_t : t \in T)$ be families of matroids, $M_s = (T, \rho_s)$ for every $s \in S$, and $M_t = (S, \rho_s)$ for every $t \in T$. Then a subsystem $\mathcal{X}$ is called an $(\mathcal{M}_S, \mathcal{M}_T)$*-system of representatives* of A if X_t and X_s are independent in M_t and M_s for every $t \in T$ and $s \in S$, respectively.

Theorem 3.6. [16] *$\mathcal{A}$ has k pairwise compatible $(\mathcal{M}_S, \mathcal{M}_T)$-systems of representatives of length d if and only if any two subsystems $\mathcal{X} = (X_t : t \in T)$ and $\mathcal{Y} = (Y_t : t \in T)$ of $\mathcal{A}$ satisfy*

$$k\left(\sum_{s \in S} \rho_s(X_s) + \sum_{t \in T} \rho_t(Y_t)\right) + \sum_{t \in T} |A_t \setminus (X_t \cup Y_t)| \geq kd.$$

Theorem 3.6 can be also expressed in language of an integral abstract network on a bipartite graph G where $\mathbb{P}_v$ is a strongly base orderable matroid for every $v \in V(G)$.

Let $U_{k,S}$ be the system of subsets of S with cardinality at most k. Then these form a strongly base orderable matroid on S. Suppose that M is a matroid on T and $M_s = M$ for every $s \in S$ and $M_t = U_{1,S}$ for every $t \in T$, then an $(\mathcal{M}_S, \mathcal{M}_T)$-system of representatives is called an *M-system of representatives.*

Let $M = (T, \rho)$ be a matroid. Then the covering theorem of Edmonds [Ed1] says that M has k independent sets whose union is T if and only if $k \cdot \rho(J) \geq |J|$ for every $J \subseteq T$. A stronger theorem holds for strongly base orderable matroids.

Theorem 3.7. [16] *Let S and T be finite sets and $M = (T, \rho)$ be a strongly base orderable matroid. Then the following conditions are pairwise equivalent.*

(1) *M has k independent sets whose union is T.*

(2) *$k \cdot \rho(J) \geq |J|$ for every $J \subseteq T$.*
(3) *Every family $\mathcal{C} = (C_t : t \in T)$ of subsets of S such that $|C_t| = k$ has an M-system of representatives.*
(4) *Every family $\mathcal{C} = (C_t : t \in T)$ of subsets of S such that $|C_t| = k$ has k pairwise compatible M-systems of representatives.*

By Edmonds [Ed1], items (1)–(3) from Theorem 3.7 are pairwise equivalent for arbitrary matroid M on T. We have conjectured in [16] that also Theorem 3.7 holds true for every matroid.

Note that a $U_{1,T}$-system of representatives of $\mathcal{A}$ is a *system of distinct representatives.* Thus Theorem 3.6 generalizes the Hall's theorem [Ha1] on distinct representatives.

3.4. Latin parallelepipeds and cubes. By a *latin $(n \times k)$-rectangle* we mean an $n \times k$ array $A = [a_{i,j}]$, $1 \leq i \leq n$, $1 \leq j \leq k$ such that $a_{i,j} \in \{1, \dots, n\}$ and if $a_{i,j} = a_{i',j'}$ then $i \neq i'$ and $j \neq j'$. In the case $n = k$, A is called a *latin square* of order n.

Let $A^{(1)} = [a^{(1)}_{i,j}]$, $A^{(2)} = [a^{(2)}_{i,j}]$, ... , $A^{(k)} = [a^{(k)}_{i,j}]$ be latin squares of order n. The k-tuple $\mathcal{A} = (A^{(1)}, A^{(2)}, \dots, A^{(k)})$ is called a *latin $n \times n \times k$-parallelepiped* if the elements $a^{(1)}_{i,j}$, $a^{(2)}_{i,j}$, ... , $a^{(k)}_{i,j}$ are pairwise distinct for every $1 \leq i, j \leq n$. In the case $k = n$, $\mathcal{A}$ is called a *latin cube* of order n.

One of the best known property of latin squares is that any latin $(n \times k)$-rectangle can be extended to a latin square of order n. This was proved by M. Hall [Ha2], using the theorem of distinct representatives of P. Hall [Ha1]. With respect to this fact it is natural to ask the following question: Given a latin $(n \times n \times k)$-parallelepiped, do there exist $n - k$ latin squares which may be added to the given parallelepiped to form a latin cube? This problem was posed in the Sixth Hungarian Colloqium on Combinatorics, 1981. In contrast with the classical case there are known constructions of latin $(n \times n \times (n-d))$-parallelepipeds that cannot be extended to a latin cube of order n. These constructions have been presented in Horák [Hor] (for $d = 2$ and $n = 2^k$, $k \geq 3$) and in Fu [Fu] (for $d = 2$ and $n = 6$ or $n \geq 12$). We have generalized these results.

Theorem 3.8. [17, 18] *For every $d \geq 2$ and $n \geq 2d + 1$ there exists a latin $(n \times n \times (n-d))$-parallelepiped that cannot be extended to a latin cube of order n.*

V. Conclusions

Results presented here belong to basic research in mathematics. The most interesting results are [1], [6] and [8]. In [1] is introduced a general framework for constructions of graphs without nowhere-zero flows. The main important technique is superposition, which is used in [6] for constructions of snarks with arbitrarily large girth. Similar methods can be used by constructing nonhamiltonian cubic graphs and other problems from graph theory and combinatorics.

Paper [8] contains a new approach for study of flow and coloring problems. This method is based on contraction-deletion principle and uses methods from linear algebra. We see potential for further development of these methods.

VI. List of The Papers Contained in The Thesis

[1] M. Kochol, *Superposition and constructions of graphs without nowhere-zero k-flows*, European Journal of Combinatorics **23** (2002), 281-306.

[2] M. Kochol, *Hypothetical complexity of the nowhere-zero* 5*-flow problem*, Journal of Graph Theory **28** (1998), 1-11.

[3] M. Kochol, *An equivalent version of the* 3*-flow conjecture*, Journal of Combinatorial Theory Series B **83** (2001), 258-261.

[4] M. Kochol, V. Lozin, B. Randerath, *The* 3*-colorability problem on graphs with maximum degree four*, SIAM Journal on Computing **32** (2003), 1128-1139.

[5] M. Kochol, *Cubic graphs without a Petersen minor have nowhere-zero* 5*-flows*, Acta Mathematica Universitatis Comenianae **68** (1999), 249-252.

[6] M. Kochol, *Snarks without small cycles*, Journal of Combinatorial Theory Series B **67** (1996), 34-47.

[7] M. Kochol, *A cyclically* 6*-edge-connected snark of order* 118, Discrete Mathematics **161** (1996), 297-300.

[8] M. Kochol, *Reduction of the* 5*-flow conjecture to cyclically* 6*-edge-connected snarks*, Journal of Combinatorial Theory Series B **90** (2004), 139-145.

[9] M. Kochol, *Stable dominating circuits in snarks*, Discrete Mathematics **233** (2001), 247-256.

[10] M. Kochol, *Equivalence of Fleischner's and Thomassen's conjectures*, Journal of Combinatorial Theory Series B **78** (2000), 277-279.

[11] A. Huck, M. Kochol, *Five cycle double covers of some cubic graphs*, Journal of Combinatorial Theory Series B **64** (1995), 119-125.

[12] H. Fleischner, M. Kochol, *A note about the dominating circuit conjecture*, Discrete Mathematics **259** (2002), 307-309.

[13] M. Kochol, *Equivalence between hamiltonicity and flow conjectures, and the sublinear defect property*, Discrete Mathematics **254** (2002), 221-230.

[14] M. Kochol, *Quasi polymatroidal flow networks*, Acta Mathematica Universitatis Comenianae **64** (1995), 83-97.

[15] M. Kochol, *Partial intersection theorem and flows in abstract networks*, SIAM Journal on Discrete Mathematics **11** (1998), 468-486.

[16] M. Kochol, *Compatible systems of representatives*, Discrete Mathematics **132** (1994), 115-126.

[17] M. Kochol, *Relatively narrow latin parallelepipeds that cannot be extended to a latin cube*, Ars Combinatoria **40** (1995), 247-260.

[18] M. Kochol, *Latin* $(n \times n \times (n-2))$*-parallelepipeds not completing to a latin cube*, Mathematica Slovaca **39** (1989), 121-125.

VII. References

[AVT] G. M. Adelson-Velskij, A. Titov, *On 4-chromatic cubic graphs*, Voprosy Kibernet. (Moscow) **1** (1974), 5-14. (in Russian)

[Ai] M. Aigner, *Combinatorial Theory*, Springer, Berlin, 1979.

[Ak] V. A. Aksionov, *The extension of a 3-coloring on planar graphs*, Diskret. Analiz **16** (1974), 3-19. (in Russian)

[AGZ] B. R. Alspach, L. Goddyn, C.-Q. Zhang, *Graphs with the circuit cover property*, Trans. Amer. Math. Soc. **344** (1994), 131-154.

[AH] K. Appel, W. Haken, *Every Planar Map Is Four Colorable*, Contemp. Math., Vol. 98, Amer. Math. Soc., Providence, 1989.

[AJ] P. Ash, B. Jackson, *Dominating cycles in bipartite graphs*, Progress in Graph Theory (J. A. Bondy, U. S. R. Murty, eds.), Academic Press, New York, 1984, pp. 81-87.

[Bi] G. D. Birkhoff, *The reducibility of maps*, Amer. J. Math. **35** (1913), 115-128.

[BM] J. A. Bondy, U. S. R. Murty, *Graph Theory with Applications*, Macmillan, London, 1976.

[Bo] O. V. Borodin, *A new proof of Grünbaum's 3 color theorem*, Discrete Math. **169** (1997), 177-183.

[Bro] R. L. Brooks, *On coloring the nodes of a network*, Proc. Cambrigde Phil. Soc. **37** (1941), 194-197.

[Bru] R. A. Brualdi, *Comments on bases in independence structures*, Bull. Australian Math. Soc. **2** (1969), 161-169.

[BS] R. A. Brualdi, E. B. Scrimger, *Exchange systems, matchings and transversals*, J. Combinatorial Theory **5** (1968), 244-257.

[CMRS] A. Cavicchioli, M. Meschiari, B. Ruini, F. Spaggiari, *A survey on snarks and new results: products, reducibility and a computer search*, J. Graph Theory **28** (1998), 57-86.

[Ce] U. A. Celmins, *On cubic graphs that do not have an edge-3-colouring*, Ph.D. Thesis, Dep. of Combinatorics and Optimization, University of Waterloo, Waterloo, Canada, 1984.

[CS] U. A. Celmins, E. R. Swart, *The constructions of snarks*, Research Report CORR 79-18, Dep. of Combinatorics and Optimization, University of Waterloo, Waterloo, Canada, 1979.

[CW] A. G. Chetwynd, R. J. Wilson, *Snarks and supersnarks*, The Theory and Applications of Graphs (Y. Alavi, G. Chartrand, D. L. Goldsmith, L. Lesniak-Foster, D. R. Lick, eds.), Wiley, New York, 1981, pp. 215-241.

[DM] J. Davies, C. McDiarmid, *Disjoint common transversals and exchange structures*, J. London Math. Soc. (2) **14** (1976), 55-62.

[Di] R. P. Dilworth, *A decomposition theorem for partially ordered sets*, Ann. Math. **51** (1950), 161-166.

[Ed1] J. Edmonds, *Lehman's switching game and a theorem of Tutte and Nash-Williams*, J. Res. Nat. Bur. Stand. Sect. B **69** (1965), 73-77.

[Ed2] J. Edmonds, *Submodular functions, matroids and certain polyhedra*, Combinatorial Structures and Their Applications (R. Guy, H. Hanani, N. Sauer, J. Schönheim, eds.), Gordon and Breach, New York, 1970, pp. 69-87.

[EF] J. Edmonds, D. R. Fulkerson, *Transversals and matroid partition*, J. Res. Nat. Bur. Stand. Sect. B **69** (1965), 147-153.

[FF] L. R. Ford, D. R. Fulkerson, *Flows in Networks*, Princeton Univ. Press, Princeton, 1962.

[Fl1] H. Fleischner, *Cycle decompositions,* 2*-coverings, removable cycles and the four-color disease*, Progress in Graph Theory (J. A. Bondy, U. S. R. Murty, eds.), Academic Press, New York, 1984, pp. 233-246.

[Fl2] H. Fleischner, *Eulerian Graphs and Related Topics, Part* 1*, Vol.* 2, Ann. Discrete Math., Vol. 50, North-Holland, Amsterdam, 1991.

[FJ] H. Fleischner, B. Jackson, *A note concerning some conjectures on cyclically* 4*-edge-connected* 3*-regular graphs*, Graph Theory in Memory of G. A. Dirac (L. D. Andersen, I. T. Jakobsen, C. Thomassen, B. Toft, P. D. Vestergaard, eds.), Ann. Discrete Math., Vol. 41, North-Holland, Amsterdam, 1989, pp. 171-177.

[Fr] A. Frank, *Generalized polymatroids*, Finite and Infinite sets (A. Hajnal, L. Lovász, eds.), North-Holland, Amsterdam, 1984, pp. 285–294.

[FT] A. Frank, É. Tardos, *Generalized polymatroids and submodular flows*, Math. Programming **42** (1988), 489–563.

[Fu] H.-L. Fu, *On latin* $(n \times n \times (n-2))$*-parallelepipeds*, Tamkang J. Math. **17** (1986), 107-111.

[Fj] S. Fujishige, *Submodular Functions and Optimization*, Ann. Discrete Math., Vol. 47, North-Holland, Amsterdam, 1991.

[GJS] M. R. Garey, D. S. Johnson, L. Stockmeyer, *Some simplified* NP*-complete graph problems*, Theor. Comput. Sci. **1** (1976), 237-267.

[Ga] M. Gardner, *Mathematical games: Snarks, Boojums and other conjectures related to the four-color-map theorem*, Sci. Am. **234** (April 1976), 126-130.

[Go1] L. Goddyn, *A girth requirement for the cycle double cover conjecture*, Cycles in Graphs (B. R. Alspach, C. D. Godsil, eds.), Ann. Discrete Math., Vol. 27, North-Holland, Amsterdam, 1985, pp. 13-26.

[Go2] L. Goddyn, *Cycle Covers of Graphs*, Ph. D. Thesis, University of Waterloo, Waterloo, Ontario, Canada, 1988.

[Go3] L. Goddyn, *Cycle double covers of graphs with Hamilton paths*, J. Combin. Theory Ser. B **46** (1989), 253-254.

[Gol] M. K. Goldberg, *Construction of class* 2 *graphs with maximum vertex degree* 3, J. Combin. Theory Ser. B **31** (1981), 282-291.

[Gr] H. Grötzsch, *Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel*, Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Rei-

he **8** (1959), 109-120.
[Ha1] P. Hall, *On representatives of subsets*, J. London Math. Soc. **10** (1935), 26-30.
[Ha2] M. Hall Jr., *An existence theorem for latin squares*, Bull. Amer. Math. Soc. **51** (1945), 387-388.
[Has] R. Hassin, *Minimum cost flow with set-constraints*, Networks **12** (19-82), 1–21.
[He] P. J. Heawood, *On the four-color map theorem*, Quart. J. Pure Appl. Math. **29** (1898), 270-285.
[Hol] J. Holyer, *The NP-completeness of edge-coloring*, SIAM J. Comput. **10** (1981), 718-720.
[Hof] A. J. Hoffman, *Some recent applications of the theory of linear inequalities to extremal combinatorial analysis*, Combinatorial Analysis (R. Bellman, M. Hall Jr., eds.), Proc. Sympos. Appl. Math., Vol. 10, Amer. Math. Soc., Providence, RI, 1960, pp. 113-127.
[Hor] P. Horák, *Latin parallelepipeds and cubes*, J. Combin. Theory Ser. A **33** (1982), 213-214.
[I] R. Isaacs, *Infinite families of nontrivial trivalent graphs which are not Tait colorable*, Amer. Math. Monthly **82** (1975), 221-239.
[Jac] B. Jackson, *On circuit covers, circuit decompositions and Euler tours in graphs*, Surveys in Combinatorics (K. Walker, ed.), London Math. Soc. Lecture Notes Ser. 187, Cambridge Univ. Press, Cambridge, 1993, pp. 191-210.
[J1] F. Jaeger, *Flows and generalized coloring theorems in graphs*, J. Combin. Theory Ser. B **26** (1979), 205-216.
[J2] F. Jaeger, *On circular flows in graphs*, Finite and Infinite Sets (A. Hajnal, L. Lovász, V. T. Sós, eds.), Coll. Math. Soc. János Bolyai, Vol. 37, North-Holland, Amsterdam, 1984, pp. 391-402.
[J3] F. Jaeger, *A survey of the cycle double cover conjecture*, Cycles in Graphs (B. R. Alspach, C. D. Godsil, eds.), Annals Disc. Math., Vol. 27, North-Holland, Amsterdam, 1985, pp. 1-12.
[J4] F. Jaeger, *Nowhere-zero flow problems*, Selected topics in Graph Theory 3 (L. W. Beineke, R. J. Wilson, eds.), Academic Press, New York, 1988, pp. 71-95.
[JLPT] F. Jaeger, N. Linial, C. Payan, M. Tarsi, *Group connectivity of graphs – a nonhomogeneous analogue of nowhere-zero flow properties*, J. Combin. Theory Ser. B **56** (1992), 165-182.
[JS] F. Jaeger, T. Swart, *Conjecture* 1, Combinatorics 79 (M. Deza, I. G. Rosenberg, eds.), Ann. Discrete Math., Vol 9, North-Holland, Amsterdam, 1980, p. 305.
[JT] T. R. Jensen, B. Toft, *Graph Coloring Problems*, Wiley, New York, 1995.
[Ke] A. B. Kempe, *A memoir of the theory of mathematical form*, Phil. Trans. Roy. Soc. London **177** (1886), 1-70.

[Ki] P. A. Kilpatrick, *Tutte's first colour-cycle conjecture*, Ph.D. Thesis, Cape Town, South Africa, 1975.

[Ko] M. M. Kovalev, *Maximization of convex functions on supermatroids*, Dokl. Akad. Nauk BSSR **27** (1983), 584–587. (In Russian)

[Ku] J. P. S. Kung, *Bimatroids and invariants*, Adv. Math. **30** (1978), 238–249.

[LM1] E. L. Lawler, C. U. Martel, *Computing maximal "polymatroidal" network flows*, Math. Oper. Res. **7** (1982), 334–347.

[LM2] E. L. Lawler, C. U. Martel, *Flow network formulation of polymatroid optimization problem*, Bonn Workshop on Combinatorial Optimization (A. Bachem, M. Grötschel, B. Korte, eds.), Ann. Discrete Math., Vol. 16, North-Holland, Amsterdam, 1982, pp. 189–200.

[LM3] E. L. Lawler, C. U. Martel, *Polymatroidal flows with lower bounds*, Discrete Appl. Math. **15** (1986), 291–313.

[Lo] L. Lovász, *Flats in matroids and geometric graphs*, Combinatorial Surveys (P. Cameron, ed.), Academic Press, New York, 1977, pp. 45–86.

[LP] L. Lovász, M. D. Plummer, *Matching Theory*, Ann. Discrete Math., Vol. 29, North-Holland, Amsterdam, 1986.

[Ma] J. H. Mason, *A Characterization of Transversal Independence Spaces*, Ph.D. Thesis, Univ of Wisconsin, Madison, WI, 1970.

[MS] M. M. Matthews, D. P. Sumner, *Hamiltonian results in $K_{1,3}$-free graphs*, J. Graph Theory **8** (1984), 139-146.

[Me] K. Menger, *Zur algemeinen Kurventheorie*, Fund. Math. **10** (1927), 96-115.

[Mi] G. J. Minty, *A theorem on three-coloring the edges of a trivalent graph*, J. Combin. Theory **2** (1967), 164-167.

[NW] G. L. Nemhauser, L. A. Wolsey, *Integer and Combinatorial Optimization*, Wiley, New York, 1988.

[O] J. Oxley, *Matroid Theory*, Oxford University Press, Oxford, 1992.

[Pe] J. Petersen, *Sur le théorème de Tait*, Interméd. Math. **15** (1898), 225-227.

[Pr] M. Preissmann, *Sur les Colorations des Arêtes des Graphes Cubiques*, Thèse de Doctorat de 3$^{\text{eme}}$, Grenoble, France, 1981.

[Ri1] R. Rizzi, *Indecomposable r-graphs and some other counterexamples*, J. Graph Theory **32** (1999), 1-15.

[Ri2] R. Rizzi, *On 4-connected graphs without even cycle decompositions*, Disctrete Math. (to appear).

[RSST] N. Robertson, D. Sanders, P. Seymour, R. Thomas, *The four-color theorem*, J. Combin. Theory Ser. B **70** (1997), 2-44.

[RW] R. W. Robinson, N. C. Wormald, *Almost all cubic graphs are hamiltonian*, Random Structures Algorithms **3** (1982), 117-125.

[Ry] Z. Ryjáček, *On a closure concept in claw-free graphs*, J. Combin. Theory Ser. B **70** (1997), 217-224.

[Sa] G. Sabidussi, *Conjecture* 2.4, Cycles in Graphs (B. R. Alspach, C. D. Godsil, eds.), Ann. Discrete Math., Vol. 27, North-Holland, Amsterdam, 1985, p. 462.

[Se1] P. D. Seymour, *Sums of circuits*, Graph Theorey and Related Topics (J. A. Bondy, U. S. R. Murty, eds.), Academic Press, New York, 1979, pp. 341-355.

[Se2] P. D. Seymour, *Nowhere-zero* 6*-flows*, J. Combin. Theory Ser. B **30** (1981), 130-135.

[Se3] P. D. Seymour, *Nowhere-zero flows*, Handbook of Combinatorics (R. L. Graham, M. Grötschel, L. Lovász, eds.), North-Holland, Amsterdam, 1995, pp. 289-299.

[Sc1] A. Schrijver, *Matroids and linking systems*, J. Combin. Theory Ser. B **26** (1979), 349–369.

[Sc2] A. Schrijver, *Total dual integrality from directed graphs, crossing families, and sub- and supermodular functions*, Progress in Combinatorial Optimization (W. R. Pulleyblank, ed.), Academic Press, Toronto, Ontario, 1984, pp. 315–362.

[Sc3] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, New York, 1986.

[Sz] G. Szekeres, *Polyhedral decomposition of cubic graphs*, J. Austral. Math. Soc. **8** (1973), 367-387.

[Tai] P. G. Tait, *Remarks on the colouring of maps*, Proc. Roy. Soc. Edinburgh **10** (1880), 729.

[Tar] M. Tarsi, *Semi-duality and the cycle double cover conjecture*, J. Combin. Theory Ser. B **41** (1986), 332-340.

[Th1] C. Thomassen, *Reflections on graph theory*, J. Graph Theory **10** (1986), 309-324.

[Th2] C. Thomassen, *Grötzsch's* 3*-color theorem and its counterparts for the torus and the projective plane*, J. Combin. Theory Ser. B **62** (1994), 268-279.

[Tu1] W. T. Tutte, *A contribution to the theory of chromatic polynomials*, Canad. J. Math. **6** (1954), 80-91.

[Tu2] W. T. Tutte, *A class of Abelian groups*, Canad. J. Math. **8** (1956), 13-28.

[Tu3] W. T. Tutte, *On the algebraic theory of graph colorings*, J. Combin. Theory **1** (1966), 15-50.

[Wa] J. J. Watkins, *Snarks*, Ann. New York Acad. Sci. **576** (1989), 606-622.

[WW] J. J. Watkins, R. J. Wilson, *A survey of snarks*, Graph Theory, Combinatorics, and Applications (Y. Alavi, G. Chartrand, O. R. Oellermann, A. J. Schwenk, eds.), Wiley, New York, 1991, pp. 1129-1144.

[We] D. J. A. Welsh, *Matroid Theory*, Academic Press,, London, 1976.

[Y] D. H. Younger, *Integer flows*, J. Graph Theory **7** (1983), 349-357.

[Z] C.-Q. Zhang, *Integral Flows and Cycle Covers of Graphs*, Dekker, New York, 1997.

VIII. List of Author's Publications

[k1] M. Kochol, *Construction of crossing-critical graphs*, Discrete Mathematics **66** (1987), 311-313.

[k2] M. Kochol, *Generalized Pascal triangles with maximal left periods*, Computers and Artificial Intelligence **6** (1987), 59-70.

[k3] M. Kochol, *Efficient monotone circuit for threshold functions*, Information Processing Letters **32** (1989), 121-122.

[k4] M. Kochol, *Latin* $(n \times n \times (n-2))$*-parallelepipeds not completing to a latin cube*, Mathematica Slovaca **39** (1989), 121-125.

[k5] M. Kochol, *Varieties of modular p-algebras not containing* $M_{3,3}$, Algebra Universalis **28** (1991), 559-588.

[k6] M. Kochol, *Latin parallelepipeds not completing to a cube*, Mathematica Slovaca **41** (1991), 3-9.

[k7] M. Kochol, *Linear jump of crossing number for non Kuratowski edge of a graph*, Radovi Matematički **7** (1991), 177-184.

[k8] M. Kochol, *The notion and basic properties of M-transversals*, Discrete Mathematics **104** (1992), 191-196.

[k9] M. Kochol, *A note on the arboricity of graphs*, Graphs and Combinatorics **8** (1992), 313-315.

[k10] M. Kochol, *Constructive approximation of a ball by polytopes*, Mathematica Slovaca **44** (1994), 99-105.

[k11] M. Kochol, *About a generalization of transversals*, Mathematica Bohemica **119** (1994), 143-149.

[k12] M. Kochol, *Compatible systems of representatives*, Discrete Mathematics **132** (1994), 115-126.

[k13] A. Huck and M. Kochol, *Five cycle double covers of some cubic graphs*, Journal of Combinatorial Theory Series B **64** (1995), 119-125.

[k14] M. Kochol, *Quasi polymatroidal flow networks*, Acta Mathematica Universitatis Comenianae **64** (1995), 83-97.

[k15] M. Kochol, *Relatively narrow latin parallelepipeds that cannot be extended to a latin cube*, Ars Combinatoria **40** (1995), 247-260.

[k16] M. Kochol, *Symmetrized and continuous generalization of transversals*, Mathematica Bohemica **121** (1996), 95-106.

[k17] M. Kochol, *Snarks without small cycles*, Journal of Combinatorial Theory Series B **67** (1996), 34-47.

[k18] M. Kochol, *A cyclically* 6*-edge-connected snark of order* 118, Discrete Mathematics **161** (1996), 297-300.

[k19] M. Kochol, *Hypothetical complexity of the nowhere-zero* 5*-flow problem*, Journal of Graph Theory **28** (1998), 1-11.

[k20] M. Kochol, *Partial intersection theorem and flows in abstract networks*, SIAM Journal on Discrete Mathematics **11** (1998), 468-486.

[k21] M. Kochol, *An elementary proof of the fundamental theorem of algebra*, International Journal of Mathematical Education in Science and Technology **30** (1999), 614-615.
[k22] M. Kochol, *Cubic graphs without a Petersen minor have nowhere-zero 5-flows*, Acta Mathematica Universitatis Comenianae **68** (1999), 249-252.
[k23] M. Kochol, *Equivalence of Fleischner's and Thomassen's conjectures*, Journal of Combinatorial Theory Series B **78** (2000), 277-279.
[k24] M. Kochol, *Stable dominating circuits in snarks*, Discrete Mathematics **233** (2001), 247-256.
[k25] M. Kochol, *An equivalent version of the 3-flow conjecture*, Journal of Combinatorial Theory Series B **83** (2001), 258-261.
[k26] M. Kochol, *Polynomials associated with nowhere-zero flows*, Journal of Combinatorial Theory Series B **84** (2002), 260-269.
[k27] M. Kochol, *Superposition and constructions of graphs without nowhre-zero k-flows*, European Journal of Combinatorics **23** (2002), 281-306.
[k28] M. Kochol, *Tension polynomials of graphs*, Journal of Graph Theory **40** (2002), 137-146.
[k29] M. Kochol, *Equivalences between hamiltonicity and flow conjectures, and the sublinear defect property*, Discrete Mathematics **254** (2002), 221-230.
[k30] H. Fleischner and M. Kochol, *A note about the dominating circuit conjecture*, Discrete Mathematics **259** (2002), 307-309.
[k31] M. Kochol, *Linear algorithm for 3-coloring of locally connected graphs*, Experimental and Efficient Algorithms (K. Jansen, M. Margraf, M. Mastrolli, and J. D. P. Rolim, eds.), Lecture Notes in Computer Science, Vol. 2647,, Springer-Verlag, Berlin, 2003, pp. 191-194.
[k32] M. Kochol, V. Lozin and B. Randerath, *The 3-colorability problem on graphs with maximum degree four*, SIAM Journal on Computing **32** (2003), 1128-1139.
[k33] M. Kochol, *On bases of the cycle and cut spaces in digraphs*, Ars Combinatoria **68** (2003), 231-234.
[k34] M. Kochol, *A theorem about elementary cuts and flow polynomials*, Graphs and Combinatorics **19** (2003), 389-392.
[k35] M. Kochol, *Tension-flow polynomials on graphs*, Discrete Mathematics **274** (2004), 173-185.
[k36] M. Kochol, *Reduction of the 5-flow conjecture to cyclically 6-edge-connected snarks*, Journal of Combinatorial Theory Series B **90** (2004), 139-145.
[k37] M. Kochol, *Snarks and flow-snarks constructed from coloring-snarks*, Discrete Mathematics **278** (2004), 165-174.
[k38] M. Kochol, *Constructions of graphs without nowhere-zero flows from Boolean formulas*, Ars Combinatoria **70** (2004), 257-264.

[k39] M. Kochol, *A note on approximation of a ball by polytopes*, Discrete Optimization **1** (2004), 229-231.

[k40] M. Kochol, 3-*coloring and* 3-*clique-ordering of locally connected graphs*, Journal of Algorithms (to appear).

[k41] M. N. Ellingham, H. Fleischner, M. Kochol and E. Wenger, *Colorability of planar graphs with isolated nontriangular faces*, Graphs and Combinatorics (to appear).

[k42] M. Kochol, *Decomposition formulas for the flow polynomial*, European Journal of Combinatorics (to appear).

[k43] M. Kochol, *Restrictions on smallest counterexamples to the* 5-*flow conjecture*, Combinatorica (to appear).

Summary

This thesis consists from author's 18 papers published in mathematical journals. The papers deal with several aspects of flows in graphs. The results are divided into three parts. In the first part we deal with snarks (nontrivial cubic graphs without 3-edge-coloring) and nowhere-zero group- and integer-valued flows in graphs. The second part contains results about path, circuits, cycle double coverings of graphs and related problems. The third part is devoted to generalization of classical flows in networks and some relative areas from transversal theory and latin squares.

We build general methods on constructing graphs without nowhere-zero k-flows. We call such graphs k-snarks. We study flows in multi-terminal networks and generalize some classical results which have been known for flows in graphs. This enables us to get new results about k-flows, k-snarks and snarks.

We prove that the 3-Flow Conjecture suffices to verify for 5-edge-connected graphs and that the smallest counterexample to the 5-Flow Conjecture must be cyclically 6-edge-connected snark. We show that if there exists a bridgeless 5-snark (resp. k-edge-connected 3-snark), then the problem to determine whether a bridgeless (resp. k-edge-connected) graph has a nowhere-zero 5-flow (resp. 3-flow) is NP-complete. We prove that cubic graphs not containing a subgraph homeomorphic to the Petersen graph have nowhere-zero 5-flows.

We construct snarks with arbitrary large girth, thereby obtaining a counterexample to the conjecture of Jaeger and Swart. We introduce constructions of a new infinite families of cyclically 6-edge-connected snarks. We also construct families of snarks with high degree of "uncolorability" or having other special properties.

We study 3-coloring problem for some classes of graphs with maximal degree four. We show that this problem is NP-complete for some classes but can be solved in linear time for others.

We also deal with conjectures about circuits in graphs. We show that every cyclically 4-edge-connected cubic graph has a dominating circuit (i.e., circuit whose vertices are incident with all edges of the graph) if and only if any two edges of a cyclically 4-edge-connected cubic graph are contained in a dominating circuit. We prove that every 4-connected line graph is hamiltonian if and only if every cyclically 4-edge-connected cubic graph has either an edge-3-coloring or a dominating cycle. We construct snarks with a stable dominating circuit (that means there is no other circuit covering all its vertices). We show that if a graph has either oddness 2 or a Hamilton path, then it has a cycle double covering consisting from 5 cycles. We prove that many conjectures and theorems about graphs could be proved if one could show that they are true apart from some "errors", provided that the number of these errors grows asymptotically slower than the order of these graphs.

We introduce a flow model from combinatorial optimization using quasi polymatroids as constraints for the flow conservation. Quasi polymatroids

are polyhedra arising from generalized polymatroids after reflexions of some coordinates. We also introduce abstract network flow model which is only formally different from the first one, and has many common features with the flow models used by studying k-snarks. Using abstract networks we introduce a general framework for various results regarding constructions of matroids and (generalized) polymatroids – for instance, the basic operations on (generalized) polymatroids and constructions of transversal matroids and gammoids.

We also study certain generalizations of the Hall's theorem of distinct representatives and construct latin parallelepipeds that cannot be extended to a latin cube.