MCSHANE EQUI-INTEGRABILITY AND VITALI'S CONVERGENCE THEOREM

Jaroslav Kurzweil, Štefan Schwabik, Praha

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Abstract. The McShane integral of functions $f\colon I \to \mathbb{R}$ defined on an m-dimensional interval I is considered in the paper. This integral is known to be equivalent to the Lebesgue integral for which the Vitali convergence theorem holds.

For McShane integrable sequences of functions a convergence theorem based on the concept of equi-integrability is proved and it is shown that this theorem is equivalent to the Vitali convergence theorem.

Keywords: McShane integral

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We consider functions $f\colon I \to \mathbb{R}$ where $I \subset \mathbb{R}^m$ is a compact interval, $m \geqslant 1$.

A system (finite family) of point-interval pairs $\{(t_i, I_i),\ i = 1, \ldots, p\}$ is called an *M-system in I* if I_i are non-overlapping ($\operatorname{int} I_i \cap \operatorname{int} I_j = \emptyset$ for $i \neq j$, $\operatorname{int} I_i$ being the interior of I_i), t_i are arbitrary points in I.

Denote by μ the Lebesgue measure in $\mathbb{R}^m$.

An M-system in I is called an *M-partition of I* if $\bigcup_{i=1}^{p} I_i = I$.

Given $\Delta\colon I \to (0, +\infty)$, called a *gauge*, an M-system $\{(t_i, I_i),\ i = 1, \ldots, p\}$ in I is called *Δ-fine* if

$$I_i \subset B(t_i, \Delta(t_i)),\ \ i = 1, \ldots, p.$$

The set of Δ-fine partitions of I is nonempty for every gauge Δ (Cousin's lemma, see e.g. [1]).

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Definition 1. $f\colon I \to \mathbb{R}$ is McShane integrable and $J \in \mathbb{R}$ is its McShane integral if for every $\varepsilon > 0$ there exists a gauge $\Delta\colon I \to (0, +\infty)$ such that for every Δ-fine M-partition $\{(t_i, I_i),\ i = 1, \dots, p\}$ of I the inequality

$$\left| \sum_{i=1}^{p} f(t_i)\mu(I_i) - J \right| < \varepsilon$$

holds. We denote $J = \int_I f$.

N o t a t i o n. To simplify writing we will from now use the notation $\{(u_l, U_l)\}$ for M-systems instead of $\{(u_l, U_l);\ l = 1, \dots, r\}$ which specifies the number r of elements of the M-system. For a function $f\colon I \to \mathbb{R}$ and an M-system $\{(u_l, U_l)\}$ we write $\sum\limits_l f(u_l)\mu(U_l)$ instead of $\sum\limits_{l=1}^{r} f(u_l)\mu(U_l)$, etc.

Theorem 2. *$f\colon I \to \mathbb{R}$ is McShane integrable if and only if f is Lebesgue integrable.*

See [2] or [4].

Definition 3. A family $\mathcal{M}$ of functions $f\colon I \to \mathbb{R}$ is called equi-integrable if every $f \in \mathcal{M}$ is McShane integrable and for every $\varepsilon > 0$ there is a gauge Δ such that for any $f \in \mathcal{M}$ the inequality

$$\left| \sum_i f(t_i)\mu(I_i) - \int_I f \right| < \varepsilon$$

holds provided $\{(t_i, I_i)\}$ is a Δ-fine M-partition of I.

Theorem 4. *A family $\mathcal{M}$ of functions $f\colon I \to X$ is equi-integrable if and only if for every $\varepsilon > 0$ there exists a gauge $\Delta\colon I \to (0, +\infty)$ such that*

$$\left\| \sum_i f(t_i)\mu(I_i) - \sum_j f(s_j)\mu(K_j) \right\|_X < \varepsilon$$

for every Δ-fine M-partitions $\{(t_i, I_i)\}$ and $\{(s_j, K_j)\}$ of I and any $f \in \mathcal{M}$.

P r o o f. If $\mathcal{M}$ is equi-integrable then the condition clearly holds for the gauge δ which corresponds to $\frac{1}{2}\varepsilon > 0$ in the definition of equi-integrability.

If the condition of the theorem is fulfilled, then every individual function $f \in \mathcal{M}$ is McShane integrable (see e.g. [5]) with the same gauge δ for a given $\varepsilon > 0$ independently of the choice of $f \in \mathcal{M}$ and this proves the theorem. $\square$

Theorem 5. *Assume that $\mathcal{M} = \{f_k\colon I \to \mathbb{R};\ k \in \mathbb{N}\}$ is an equi-integrable sequence such that*

$$\lim_{k\to\infty} f_k(t) = f(t),\ t \in I.$$

Then the function $f\colon I \to \mathbb{R}$ is McShane integrable and

$$\lim_{k\to\infty} \int_I f_k = \int_I f$$

holds.

Proof. If Δ is the gauge from the definition of equi-integrability of the sequence f_k corresponding to the value $\varepsilon > 0$ then for any $k \in \mathbb{N}$

$$\left| \sum_i f_k(t_i)\mu(I_i) - \int_I f_k \right| < \varepsilon \tag{1}$$

for every Δ-fine M-partition $\{(t_i, I_i)\}$ of I.

If the partition $\{(t_i, I_i)\}$ is fixed then the pointwise convergence $f_k \to f$ yields

$$\lim_{k\to\infty} \sum_i f_k(t_i)\mu(I_i) = \sum_i f(t_i)\mu(I_i).$$

Choose $k_0 \in \mathbb{N}$ such that for $k > k_0$ the inequality

$$\left| \sum_i f_k(t_i)\mu(I_i) - \sum_i f(t_i)\mu(I_i) \right| < \varepsilon$$

holds. Then we have

$$\begin{aligned}\left| \sum_i f(t_i)\mu(I_i) - \int_I f_k \right| &\leqslant \left| \sum_i [f(t_i)\mu(I_i) - f_k(t_i)\mu(I_i)] \right| \\ &\quad + \left| \sum_i f_k(t_i)\mu(I_i) - \int_I f_k \right| < 2\varepsilon\end{aligned}$$

for $k > k_0$.

This gives for $k, l > k_0$ the inequality

$$\left| \int_I f_k - \int_I f_l \right| < 4\varepsilon,$$

which shows that the sequence of real numbers $\int_I f_k$, $k \in \mathbb{N}$, is Cauchy and therefore

$$\lim_{k\to\infty} \int_I f_k = J \in \mathbb{R} \ \text{ exists.} \tag{2}$$

Let $\varepsilon > 0$. By hypothesis there is a gauge Δ such that (1) holds for all k whenever $\{(t_i, I_i)\}$ is a Δ-fine M-partition of I.

By (2) choose an $N \in \mathbb{N}$ such that $|\int_I f_k - J| < \varepsilon$ for all $k \geqslant N$. Suppose that $\{(t_i, I_i)\}$ is a Δ-fine M-partition of I. Since f_k converges to f pointwise there is a $k_1 \geqslant N$ such that

$$\left| \sum_i f_{k_1}(t_i)\mu(I_i) - \sum_i f(t_i)\mu(I_i) \right| < \varepsilon.$$

Therefore

$$\begin{aligned} \left| \sum_i f(t_i)\mu(I_i) - J \right| &\leqslant \left| \sum_i f(t_i)\mu(I_i) - \sum_i f_{k_1}(t_i)\mu(I_i) \right| \\ &\quad + \left| \sum_i f_{k_1}(t_i)\mu(I_i) - \int_I f_{k_1} \right| + \left| \int_I f_{k_1} - J \right| < 3\varepsilon \end{aligned}$$

and it follows that f is McShane integrable and $\lim\limits_{k\to\infty} \int_I f_k = J = \int_I f$. $\square$

R e m a r k 6. By a *figure* we mean a finite union of compact nondegenerate intervals in $\mathbb{R}^m$.

Let us mention the fact that if for the notion of an M-system $\{(t_i, I_i),\ i = 1, \dots, p\}$ the intervals I_i are replaced by figures, we can develop the same theory and M-systems and M-partitions of this kind can be used everywhere in our considerations.

Definition 7. Let $\mathcal{M}$ be a family of Lebesgue integrable functions $f\colon I \to \mathbb{R}$.

If for every $\varepsilon > 0$ there is a $\delta > 0$ such that for $E \subset I$ measurable with $\mu(E) < \delta$ we have $|\int_E f| < \varepsilon$ for every $f \in \mathcal{M}$ then the family $\mathcal{M}$ is called *uniformly absolutely continuous.*

Theorem 8. *Assume that a sequence of Lebesgue integrable functions $f_k\colon I \to \mathbb{R}$, $n \in \mathbb{N}$, is given such that f_k converge to f in measure.*

If the set $\{f_k;\ k \in \mathbb{N}\}$ is uniformly absolutely continuous then the function f is Lebesgue integrable and

$$\lim_{k\to\infty} \int_I f_k = \int_I f.$$

See [3, p. 168] or [1, p. 203, Theorem 13.3].

We will consider Theorem 8 in a less general form:

Theorem 9. *Assume that a sequence of Lebesgue integrable functions $f_k\colon I \to \mathbb{R}$, $k \in \mathbb{N}$, is given such that f_k converge to f pointwise in I.*

If the set $\{f_k;\ k \in \mathbb{N}\}$ is uniformly absolutely continuous then the function f is Lebesgue integrable and

$$\lim_{k\to\infty} \int_I f_k = \int_I f.$$

R e m a r k 10. It is possible to assume in Theorem 9 that f_k converge to f almost everywhere in I, but changing the values of f_k and f to 0 on a set N of zero Lebesgue measure ($\mu(N) = 0$) it can be seen easily that such a change has no effect on Lebesgue integrability and on the corresponding indefinite Lebesgue integrals.

Our goal is to show that the relaxed Vitali convergence Theorem 9 is a consequence of our convergence Theorem 4 for the McShane integral.

Lemma 11 (Saks-Henstock). *Assume that a family $\mathcal{M}$ of functions $f\colon I \to \mathbb{R}$ is equi-integrable. Given $\varepsilon > 0$ assume that the gauge Δ on I is such that*

$$\left| \sum_i f(t_i)\mu(I_i) - \int_I f \right| < \varepsilon$$

for every Δ-fine M-partition $\{(t_i, I_i)\}$ of I and $f \in \mathcal{M}$.

Then if $\{(r_j, K_j)\}$ is an arbitrary Δ-fine M-system we have

$$\left| \sum_j \left[f(r_j)\mu(K_j) - \int_{K_j} f \right] \right| \leqslant \varepsilon$$

for every $f \in \mathcal{M}$.

P r o o f. Since $\{(r_j, K_j)\}$ is a Δ-fine M-system the complement $I \setminus \operatorname{int}\bigl(\bigcup_j K_j\bigr)$ can be expressed as a finite system M_l, $l = 1, \dots, r$ of non-overlapping intervals in I. The functions $f \in \mathcal{M}$ are equi-integrable and therefore they are equi-integrable over each M_l and by definition for any $\eta > 0$ there is a gauge δ_l on M_l with $\delta_l(t) < \delta(t)$ for $t \in M_l$ such that for every $l = 1, \dots, r$ we have

$$\left| \sum_i f(s_i^l)\mu(J_i^l) - \int_{M_l} f \, \mathrm{d}\mu \right| < \frac{\eta}{r+1}$$

provided $\{(s_i^l, J_i^l)\}$ is a δ_l-fine M-partition of the interval M_l and $f \in \mathcal{M}$.

The sum

$$\sum_j f(r_j)\mu(K_j) + \sum_l \sum_i f(s_i^l)\mu(J_i^l)$$

represents an integral sum which corresponds to a certain δ-fine M-partition of I, namely $\{(r_j, K_j), (s_i^l, J_i^l)\}$, and consequently by the assumption we have

$$\left|\sum_j f(r_j)\mu(K_j) + \sum_l \sum_i f(s_i^l)\mu(J_i^l) - \int_I f \, \mathrm{d}\mu\right| < \varepsilon.$$

Hence

$$\begin{aligned}
&\left|\sum_j \left[f(r_j)\mu(K_j) - \int_{K_j} f \, \mathrm{d}\mu\right]\right| \\
&\qquad \leqslant \left|\sum_j f(r_j)\mu(K_j) + \sum_l \sum_i f(s_i^l)\mu(J_i^l) - \int_I f \, \mathrm{d}\mu\right| \\
&\qquad\quad + \sum_l \left|\sum_i f(s_i^l)\mu(J_i^l) - \int_{M_l} f \, \mathrm{d}\mu\right| < \varepsilon + r\frac{\eta}{r+1} < \varepsilon + \eta
\end{aligned}$$

Since this inequality holds for every $\eta > 0$ and $f \in \mathcal{M}$ we obtain immediately the statement of the lemma. $\square$

Looking at Lemma 11 we can see immediately that if the equi-integrable family $\mathcal{M}$ consists of a single McShane integrable function f, then the following standard Saks-Henstock Lemma holds.

Assume that $f\colon I \to \mathbb{R}$ is McShane integrable. Given $\varepsilon > 0$ assume that the gauge Δ on I is such that

$$\left|\sum_i f(t_i)\mu(I_i) - \int_I f\right| < \varepsilon$$

for every Δ-fine M-partition $\{(t_i, I_i)\}$ of I.

Then if $\{(r_j, K_j)\}$ is an arbitrary Δ-fine M-system we have

$$\left|\sum_j \left[f(r_j)\mu(K_j) - \int_{K_j} f\right]\right| \leqslant \varepsilon.$$

Proposition 12. *Assume that $f_k\colon I \to \mathbb{R}$, $k \in \mathbb{N}$, are McShane (=Lebesgue) integrable functions such that*

1. *$f_k(t) \to f(t)$ for $t \in I$,*
2. *the set $\{f_k;\ k \in \mathbb{N}\}$ is uniformly absolutely continuous.*

Then the set $\{f_k;\ k \in \mathbb{N}\}$ is equi-integrable.

P r o o f. Assuming 1 we will use Egoroff's Theorem (see [3] or [1, Th. 2.13, p. 22]) in the following form:

For every $j \in \mathbb{N}$ there is a measurable set $E_j \subset I$ such that $\mu(I \setminus E_j) < 1/j$, $E_j \subset E_{j+1}$ and $f_k(t) \to f(t)$ uniformly for $t \in E_j$, i.e. for every $\varepsilon > 0$ there is a $K_j \in \mathbb{N}$ such that for $k > K_j$ we have

$$|f_k(t) - f(t)| < \varepsilon \quad \text{for} \quad t \in E_j. \tag{3}$$

Let us mention that for $N = I \setminus \bigcup_{j=1}^{\infty} E_j$ we have $\mu(N) = 0$ because $\mu(N) \leqslant \mu(I \setminus E_j) < 1/j$ for every $j \in \mathbb{N}$.

By virtue of Remark 10 we may assume without any loss of generality that $f_k(t) = f(t) = 0$ for $k \in \mathbb{N}$ and $t \in N$.

Assume now that $\varepsilon > 0$ is given. By the assumption 2 there is a $j \in \mathbb{N}$ such that

$$\int_{I \setminus E_j} |f_k| < \varepsilon \quad \text{for all} \quad k \in \mathbb{N}. \tag{4}$$

Then (by (3) and (4))

$$\begin{aligned}
\int_I |f_k - f_l| &= \int_{E_j} |f_k - f_l| + \int_{I \setminus E_j} |f_k - f_l| \\
&\leqslant \int_{E_j} |f_k - f| + \int_{E_j} |f - f_l| + \int_{I \setminus E_j} |f_k| + \int_{I \setminus E_j} |f_l| \\
&< 2\varepsilon\mu(E_j) + 2\varepsilon \leqslant 2\varepsilon(\mu(I) + 1)
\end{aligned}$$

for all $k, l > K_j$. This shows that the sequence f_k, $k \in \mathbb{N}$, is Cauchy in the Banach space L of Lebesgue integrable functions on I and implies that the function $f\colon I \to \mathbb{R}$ also belongs to L and

$$\lim_{k \to \infty} \int_I |f_k - f| = 0, \tag{5}$$

i.e. there is a $K \in \mathbb{N}$ such that

$$\int_I |f_k - f| < \varepsilon \quad \text{for all} \quad k > K. \tag{6}$$

By Theorem 2 we know that all the functions f, f_k, $k \in \mathbb{N}$, are also McShane integrable and the values of their McShane and Lebesgue integrals are the same.

According to Definition 1 there exists a gauge $\Delta_1\colon I \to (0, +\infty)$ such that

$$\left| \sum_i f(t_i)\mu(I_i) - \int_I f \right| < \varepsilon \tag{7}$$

for every Δ_1-fine M-partition $\{(t_i, I_i)\}$ of I.

Further, there exists a gauge $\Delta_2\colon I \to (0,+\infty)$ such that

$$\left|\sum_i f_k(t_i)\mu(I_i) - \int_I f_k\right| < \varepsilon \tag{8}$$

for every Δ_2-fine M-partition $\{(t_i, I_i)\}$ of I for all $k \leqslant K$, K given by (6). (A finite set of integrable functions is evidently equi-integrable.)

Similarly, for any $j \in \mathbb{N}$ we have a gauge $\delta_j\colon I \to (0,+\infty)$ such that

$$\left|\sum_i f_k(t_i)\mu(I_i) - \int_I f_k\right| < \frac{\varepsilon}{2^j} \tag{9}$$

for every δ_j-fine M-partition $\{(t_i, I_i)\}$ of I and all $k \leqslant K_j$.

Since $\mu(N) = 0$, for every $\delta > 0$ there is an open set $U \subset \mathbb{R}^m$ such that $N \subset U$ and $\mu(U) < \delta$. By virtue of the assumption 2 the value of δ can be chosen in such a way that

$$\left|\int_{U\cap I} f_k\right| < \varepsilon \quad \text{for all} \quad k \in \mathbb{N}, \tag{10}$$

cf. Definition 7.

For $t \in E_1 \setminus N$ define $\Delta_3(t) = \delta_1(t)$, for $t \in (E_2 \setminus E_1) \setminus N$ define $\Delta_3(t) = \delta_2(t), \ldots,$ for $t \in (E_j \setminus E_{j-1}) \setminus N$ define $\Delta_3(t) = \delta_j(t)$, etc.

If $t \in N$ then we define $\Delta_3(t) > 0$ such that for the ball $B(t, \Delta_3(t))$ (centered at t with the radius $\Delta_3(t)$) we have $B(t, \Delta_3(t)) \subset U$.

In this way the positive function Δ_3 defined on I represents a gauge.

Let us put $\Delta(t) = \min(\Delta_1(t), \Delta_2(t), \Delta_3(t))$ for $t \in I$. The function Δ is evidently a gauge on I.

Assume that $\{(t_i, I_i)\}$ is an arbitrary Δ-fine M-partition of I.

If $k \leqslant K$ then

$$\left|\sum_i f_k(t_i)\mu(I_i) - \int_I f_k\right| < \varepsilon$$

by (8).

If $k > K$ then

$$\begin{aligned}\left|\sum_i f_k(t_i)\mu(I_i) - \int_I f_k\right| &= \left|\sum_i \left[f_k(t_i)\mu(I_i) - \int_{I_i} f_k\right]\right| \\ &\leqslant \left|\sum_{j=1}^{\infty} \sum_{i\colon t_i \in (E_j \setminus E_{j-1})\setminus N} \left[f_k(t_i)\mu(I_i) - \int_{I_i} f_k\right]\right| \\ &\quad + \left|\sum_{i\colon t_i \in N} \left[f_k(t_i)\mu(I_i) - \int_{I_i} f_k\right]\right|.\end{aligned} \tag{11}$$

For the second term on the right hand side of (11) we know that if $t_i \in N$ then $f_k(t_i) = 0$ and $\bigcup_{i\colon t_i \in N} E_i \subset U$ and therefore by (10) we have

$$\left| \sum_{i\colon t_i \in N} \int_{I_i} f_k \right| \leqslant \left| \int_{\bigcup_i I_i\colon t_i \in N} f_k \right| \leqslant \left| \int_{U \cap I} f_k \right| < \varepsilon. \tag{12}$$

Concerning the first term on the right hand side of (11) we have

$$\begin{aligned} &\left| \sum_{j=1}^{\infty} \sum_{i\colon t_i \in (E_j \setminus E_{j-1}) \setminus N} \left[f_k(t_i)\mu(I_i) - \int_{I_i} f_k \right] \right| \\ &\qquad \leqslant \sum_{j=1}^{\infty} \left| \sum_{i\colon t_i \in (E_j \setminus E_{j-1}) \setminus N} \left[f_k(t_i)\mu(I_i) - \int_{I_i} f_k \right] \right|. \end{aligned} \tag{13}$$

If $k \leqslant K_j$ the the Saks-Henstock Lemma 11 yields by (9) the inequality

$$\left| \sum_{i\colon t_i \in (E_j \setminus E_{j-1}) \setminus N} \left[f_k(t_i)\mu(I_i) - \int_{I_i} f_k \right] \right| < \frac{\varepsilon}{2^j}. \tag{14}$$

If $k > K_j$ then (cf. (3))

$$\begin{aligned} &\left| \sum_{i\colon t_i \in (E_j \setminus E_{j-1}) \setminus N} \left[f_k(t_i)\mu(I_i) - \int_{I_i} f_k \right] \right| \\ &\leqslant \sum_{i\colon t_i \in (E_j \setminus E_{j-1}) \setminus N} \left| f_k(t_i)\mu(I_i) - \int_{I_i} f_k \right| \\ &\leqslant \sum_{i\colon t_i \in (E_j \setminus E_{j-1}) \setminus N} |f_k(t_i) - f(t_i)|\mu(I_i) + \sum_{i\colon t_i \in (E_j \setminus E_{j-1}) \setminus N} \left| f(t_i)\mu(I_i) - \int_{I_i} f \right| \\ &\quad + \sum_{i\colon t_i \in (E_j \setminus E_{j-1}) \setminus N} \int_{I_i} |f - f_k| \\ &< \varepsilon \sum_{i\colon t_i \in (E_j \setminus E_{j-1}) \setminus N} \mu(I_i) + \sum_{i\colon t_i \in (E_j \setminus E_{j-1}) \setminus N} \left| f(t_i)\mu(I_i) - \int_{I_i} f \right| \\ &\quad + \int_{\bigcup_i I_i\colon t_i \in (E_j \setminus E_{j-1}) \setminus N} |f - f_k|. \end{aligned}$$

This together with (14) gives for $k \in \mathbb{N}$ the estimate

$$\begin{aligned} \left| \sum_{i\colon t_i \in (E_j \setminus E_{j-1}) \setminus N} \left[f_k(t_i)\mu(I_i) - \int_{I_i} f_k \right] \right| &< \frac{\varepsilon}{2^j} + \varepsilon \sum_{i\colon t_i \in (E_j \setminus E_{j-1}) \setminus N} \mu(I_i) \\ &+ \sum_{i\colon t_i \in (E_j \setminus E_{j-1}) \setminus N} \left| f(t_i)\mu(I_i) - \int_{I_i} f \right| + \int_{\bigcup_i I_i\colon t_i \in (E_j \setminus E_{j-1}) \setminus N} |f - f_k|. \end{aligned}$$

Summing over j and using (7) and (6) together with the Saks-Henstock Lemma 11 we obtain

$$\sum_{j}^{\infty}\left|\sum_{i:\, t_i\in(E_j\setminus E_{j-1})\setminus N}\left[f_k(t_i)\mu(I_i)-\int_{I_i} f_k\right]\right|<\varepsilon+\varepsilon\mu(I)+\varepsilon+\varepsilon$$

and taking into account (11) and (12) we conclude

$$\left|\sum_i f_k(t_i)\mu(I_i)-\int_I f_k\right|<(4+\mu(I))\varepsilon \quad \text{for all} \quad k\in\mathbb{N}.$$

This inequality proves that the sequence f_k, $k\in\mathbb{N}$, is equi-integrable. $\square$

Lemma 13. *Assume that $f_k\colon\ I\to\mathbb{R}$, $k\in\mathbb{N}$, are McShane (Lebesgue) integrable functions such that*

1. *$f_k(t)\to f(t)$ for $t\in I$,*
2. *the set $\{f_k;\ k\in\mathbb{N}\}$ is equi-integrable.*

Then for every $\varepsilon>0$ there is an $\eta>0$ such that for any finite family $\{J_j\colon\ j=1,\dots,p\}$ of non-overlapping intervals in I with $\sum\limits_j\mu(J_j)<\eta$ we have

$$\left|\sum_j\int_{J_j} f_k\right|<\varepsilon,\quad k\in\mathbb{N}.$$

P r o o f. Let $\varepsilon>0$ be given. Since f_k are equi-integrable on I, there exists a gauge δ on I such that $|\sum\limits_i f_k(t_i)\mu(I_i)-\int_I f_k|<\varepsilon$ for $k\in\mathbb{N}$ whenever $\{(t_i,I_i)\}$ is a δ-fine M-partition of I. Fixing a δ-fine M-partition $\{(t_i,I_i)\}$ of I let $k_0\in\mathbb{N}$ be such that

$$|f_k(t_i)-f(t_i)|<\varepsilon \text{ for } \ k>k_0,$$

put $C=\max\{|f(t_i)|,|f_k(t_i)|;\ i,k\leqslant k_0\}$ and set $\eta=\varepsilon(C+1)^{-1}$.

Suppose that $\{J_j\colon\ j=1,\dots,p\}$ is a finite family of non-overlapping intervals in I such that $\sum\limits_j\mu(J_j)<\eta$. By subdividing these intervals if necessary, we may assume that for each j, $J_j\subseteq I_i$ for some i. For each i let $M_i=\{j;\ J_j\subseteq I_i\}$ and let

$$D=\{(t_i,J_j)\colon\ j\in M_i,i\}.$$

Note that D is a δ-fine M-system in I.

Using the Saks-Henstock Lemma 11 we get

$$\begin{aligned}\Big|\sum_j \int_{J_j} f_k\Big| &\leqslant \Big|\sum_j \Big[\int_{J_j} f_k - f_k(t_i)\mu(J_j)\Big]\Big| + \sum_j |f_k(t_i)|\mu(J_j)\\ &\leqslant \varepsilon + (C+\varepsilon)\sum_j \mu(J_j) < \varepsilon + (C+\varepsilon)\eta < \varepsilon\Big(2 + \frac{\varepsilon}{C+1}\Big)\end{aligned}$$

and this proves the lemma. □

Lemma 14. *Assume that $f_k\colon I \to \mathbb{R}$, $k \in \mathbb{N}$, are McShane (Lebesgue) integrable functions such that*

1. *$f_k(t) \to f(t)$ for $t \in I$,*
2. *the set $\{f_k;\ k \in \mathbb{N}\}$ forms an equi-integrable sequence.*

Then for every $\varepsilon > 0$ there exists an $\eta > 0$ such that

(a) *if F is closed, G open, $F \subset G \subset I$, $\mu(G \setminus F) < \eta$ then there is a gauge $\xi\colon I \to (0,\infty)$ such that*

$$\begin{aligned} B(t,\xi(t)) \subset G \quad &\text{for } t \in G,\\ B(t,\xi(t)) \cap I \subset I \setminus F \quad &\text{for } t \in I \setminus F\end{aligned}$$

and

(b) *for ξ-fine M-systems $\{(u_l,U_l)\}$, $\{(v_m,V_m)\}$ satisfying*

$$u_l, v_m \in G,\ F \subset \operatorname{int} \bigcup_{u_l \in F} U_l,\ F \subset \operatorname{int} \bigcup_{v_m \in F} V_m$$

we have

$$\Big|\sum_l f_k(u_l)\mu(U_l) - \sum_m f_k(v_m)\mu(V_m)\Big| \leqslant \varepsilon \tag{15}$$

for every $k \in \mathbb{N}$.

P r o o f. Denote $\Phi_k(J) = \int_J f_k$ for an interval $J \subset I$ (the indefinite integral or primitive of f_k) and put $\hat{\varepsilon} = \varepsilon/10$.

Since f_k are equi-integrable we obtain by the Saks-Henstock Lemma 11 that there is a gauge Δ on I such that

$$\Big|\sum_j [f_k(r_j)\mu(K_j) - \Phi_k(K_j)]\Big| \leqslant \hat{\varepsilon} \tag{16}$$

for every Δ-fine M-system $\{(r_j,K_j)\}$ and $k \in \mathbb{N}$.

Assume that

$$\{(w_p, W_p)\} \text{ is a fixed } \Delta\text{-fine } M\text{-partition of } I. \tag{17}$$

Let $k_0 \in \mathbb{N}$ be such that

$$|f_k(w_p) - f(w_p)| < 1$$

for $k > k_0$ and all p. Put $\kappa = \max\limits_{p,k \leqslant k_0} \{1 + |f(w_p)|, |f_k(w_p)|\}$. Then

$$|f_k(w_p)| < \kappa \quad \text{for all } \ k \in \mathbb{N} \ \text{ and } \ p. \tag{18}$$

Assume that $\eta > 0$ satisfies

$$\eta \cdot \kappa \leqslant \hat{\varepsilon} \tag{19}$$

and take

$$0 < \xi(t) \leqslant \Delta(t), \ t \in I. \tag{20}$$

Since the sets G and $I \backslash F$ are open, the gauge ξ can be chosen such that $B(t, \xi(t)) \subset G$ for $t \in G$ and $B(t, \xi(t)) \cap I \subset I \setminus F$ for $t \in I \setminus F$.

This shows part (a) of the lemma.

Since $\{(w_p, W_p)\}$ is a partition of I, we have $\bigcup\limits_p W_p = I$ and therefore

$$\begin{aligned} \sum_l f_k(u_l)\mu(U_l) &= \sum_p \sum_{l\colon u_l \in F} \sum_{m\colon v_m \in F} f_k(u_l)\mu(W_p \cap U_l \cap V_m) \\ &+ \sum_p \sum_{l\colon u_l \in F} f_k(u_l)\mu\Big(W_p \cap U_l \Big\backslash \bigcup_{m\colon v_m \in F} V_m\Big) \\ &+ \sum_p \sum_{l\colon u_l \in I \backslash F} f_k(u_l)\mu(W_p \cap U_l) \end{aligned} \tag{21}$$

and similarly

$$\begin{aligned} \sum_m f_k(v_m)\mu(V_m) &= \sum_p \sum_{l\colon u_l \in F} \sum_{m\colon v_m \in F} f_k(v_m)\mu(W_p \cap U_l \cap V_m) \\ &+ \sum_p \sum_{m\colon u_m \in F} f_k(v_m)\mu\Big(W_p \cap V_m \Big\backslash \bigcup_{l\colon u_l \in F} U_l\Big) \\ &+ \sum_p \sum_{m\colon v_m \in I \backslash F} f_k(v_m)\mu(W_p \cap V_m). \end{aligned} \tag{22}$$

The M-systems

$$\{(u_l, W_p \cap U_l \cap V_m);\ p, u_l \in F, v_m \in F\},$$
$$\{(w_p, W_p \cap U_l \cap V_m);\ p, u_l \in F, v_m \in F\}$$

are Δ-fine and therefore, by (16), we have the inequalities

$$\Bigl|\sum_p \sum_{l\colon u_l\in F} \sum_{m\colon v_m\in F} f_k(u_l)\mu(W_p \cap U_l \cap V_m) - \Phi_k(W_p \cap U_l \cap V_m)\Bigr| \leqslant \hat{\varepsilon},$$
$$\Bigl|\sum_p \sum_{l\colon u_l\in F} \sum_{m\colon v_m\in F} f_k(w_p)\mu(W_p \cap U_l \cap V_m) - \Phi_k(W_p \cap U_l \cap V_m)\Bigr| \leqslant \hat{\varepsilon}.$$

Hence

$$\Bigl|\sum_p \sum_{l\colon u_l\in F} \sum_{m\colon v_m\in F} f_k(u_l)\mu(W_p \cap U_l \cap V_m) - \sum_p \sum_{l\colon u_l\in F} \sum_{m\colon v_m\in F} f_k(w_p)\mu(W_p \cap U_l \cap V_m)\Bigr| \leqslant 2\hat{\varepsilon}$$

and similarly also

$$\Bigl|\sum_p \sum_{l\colon u_l\in F} \sum_{m\colon v_m\in F} f_k(v_m)\mu(W_p \cap U_l \cap V_m) - \sum_p \sum_{l\colon u_l\in F} \sum_{m\colon v_m\in F} f_k(w_p)\mu(W_p \cap U_l \cap V_m)\Bigr| \leqslant 2\hat{\varepsilon}.$$

Therefore

$$\Bigl|\sum_p \sum_{l\colon u_l\in F} \sum_{m\colon v_m\in F} f_k(u_l)\mu(W_p \cap U_l \cap V_m) - \sum_p \sum_{l\colon u_l\in F} \sum_{m\colon v_m\in F} f_k(v_m)\mu(W_p \cap U_l \cap V_m)\Bigr| \leqslant 4\hat{\varepsilon}. \tag{23}$$

Since $\{(u_l, U_l)\}$ is a ξ-fine M-system with $u_l \in G$, we obtain by the properties of the gauge ξ given in (a) and from the assumption $F \subset \operatorname{int} \bigcup_{u_l\in F} U_l$, $F \subset \operatorname{int} \bigcup_{v_m\in F} V_m$ that

$$\Bigl(\bigcup_{p,u_l\in F} W_p \cap U_l \setminus \bigcup_{v_m\in F} V_m\Bigr) \cup \bigcup_{p,u_l\in I\setminus F} W_p \cap U_l \subset G \setminus F. \tag{24}$$

Further, the M-systems

$$\left\{\left(u_l, W_p \cap U_l \setminus \bigcup_{v_m \in F} V_m\right);\ p, u_l \in F\right\} \cup \{(u_l, W_p \cap U_l);\ p, u_l \in I \setminus F\},$$
$$\left\{\left(w_p, W_p \cap U_l \setminus \bigcup_{v_m \in F} V_m\right);\ p, u_l \in F\right\} \cup \{(w_p, W_p \cap U_l);\ p, u_l \in I \setminus F\}$$

are Δ-fine (note that $W_p \cap U_l \setminus \bigcup_{v_m \in F} V_m$ and $W_p \cap U_l$ are figures in general). Therefore by (16) we have

$$\Big|\sum_{p,u_l \in F}\Big[f_k(u_l)\mu\Big(W_p \cap U_l \setminus \bigcup_{v_m \in F} V_m\Big) - \Phi_k\Big(W_p \cap U_l \setminus \bigcup_{v_m \in F} V_m\Big)\Big]$$
$$+ \sum_{p,u_l \in I\setminus F}[f_k(u_l)\mu(W_p \cap U_l) - \Phi_k(W_p \cap U_l)]\Big| \leqslant \hat{\varepsilon},$$
$$\Big|\sum_{p,u_l \in F}\Big[f_k(w_p)\mu\Big(W_p \cap U_l \setminus \bigcup_{v_m \in F} V_m\Big) - \Phi_k\Big(W_p \cap U_l \setminus \bigcup_{v_m \in F} V_m\Big)\Big]$$
$$+ \sum_{p,u_l \in I\setminus F}[f_k(w_p)\mu(W_p \cap U_l) - \Phi_k(W_p \cap U_l)]\Big| \leqslant \hat{\varepsilon}.$$

This yields

$$\Big|\sum_{p,u_l \in F} f_k(u_l)\mu\Big(W_p \cap U_l \setminus \bigcup_{v_m \in F} V_m\Big) + \sum_{p,u_l \in I\setminus F} f_k(u_l)\mu(W_p \cap U_l)$$
$$- \sum_{p,u_l \in F} f_k(w_p)\mu\Big(W_p \cap U_l \setminus \bigcup_{v_m \in F} V_m\Big) - \sum_{p,u_l \in I\setminus F} f_k(w_p)\mu(W_p \cap U_l)\Big| \leqslant 2\hat{\varepsilon}.$$

By virtue of (24), (18), the assumption $\mu(G \setminus F) < \eta$ and (19) we have

$$\Big|\sum_{p,u_l \in F} f_k(w_p)\mu\Big(W_p \cap U_l \setminus \bigcup_{v_m \in F} V_m\Big) + \sum_{p,u_l \in I\setminus F} f_k(w_p)\mu(W_p \cap U_l)\Big| \leqslant \kappa \cdot \eta \leqslant \hat{\varepsilon}$$

and therefore

$$\Big|\sum_{p,u_l \in F} f_k(u_l)\mu\Big(W_p \cap U_l \setminus \bigcup_{v_m \in F} V_m\Big) + \sum_{p,u_l \in I\setminus F} f_k(u_l)\mu(W_p \cap U_l)\Big| \leqslant 3\hat{\varepsilon} \tag{25}$$

and similarly also

$$\Big|\sum_{p,v_m \in F} f_k(v_m)\mu\Big(W_p \cap V_m \setminus \bigcup_{u_l \in F} U_l\Big) + \sum_{p,v_m \in I\setminus F} f_k(w_m)\mu(W_p \cap V_m)\Big| \leqslant 3\hat{\varepsilon}. \tag{26}$$

From (21), (22), (23), (25) and (26) we get

$$\Big|\sum_l f_k(u_l)\mu(U_l) - \sum_m f_k(v_m)\mu(V_m)\Big| \leqslant 10\hat{\varepsilon} \leqslant \varepsilon$$

and (15) is satisfied. This proves part (b) of the lemma. □

Theorem 15. *Assume that $f_k\colon I \to \mathbb{R}$, $k \in \mathbb{N}$, are McShane integrable functions such that*

1. *$f_k(t) \to f(t)$ for $t \in I$,*
2. *the set $\{f_k;\ k \in \mathbb{N}\}$ is equi-integrable.*

Then $f_k \cdot \chi_E$, $k \in \mathbb{N}$, is an equi-integrable sequence for every measurable set $E \subset I$.

Proof. Let $\varepsilon > 0$ be given and let $\eta > 0$ corresponds to ε by Lemma 14. Assume that $E \subset I$ is measurable. Then there exist $F \subset I$ closed and $G \subset I$ open such that $F \subset E \subset G$ where $\mu(G \setminus F) < \eta$. Assume that the gauge $\xi\colon I \to (0, \infty)$ is given as in the Lemma 14 and that $\{(u_l, U_l)\}$, $\{(v_m, V_m)\}$ are ξ-fine M-partitions of I.

By virtue of (a) in Lemma 14 we have

$$\text{if } \ u_l \in E \ \text{ then } U_l \subset G, F \subset \operatorname{int} \bigcup_{u_l \in F} U_l$$

and

$$\text{if } \ v_m \in E \ \text{ then } V_m \subset G, F \subset \operatorname{int} \bigcup_{v_m \in F} V_m.$$

Hence by (b) from Lemma 14 we have

$$\Big|\sum_{l, u_l \in E} f_k(u_l)\mu(U_l) - \sum_{m, v_m \in E} f_k(v_m)\mu(V_m)\Big| \leqslant \varepsilon$$

and therefore also

$$\Big|\sum_l f_k(u_l)\chi_E(u_l)\mu(U_l) - \sum_m f_k(v_m)\chi_E(v_m)\mu(V_m)\Big| \leqslant \varepsilon.$$

This is the Bolzano-Cauchy condition from Theorem 4 for equi-integrability of the sequence $f_k \cdot \chi_E$, $k \in \mathbb{N}$, and the proof is complete. □

Proposition 16. *Assume that $f_k\colon I \to \mathbb{R}$, $k \in \mathbb{N}$, are McShane integrable functions such that*

1. *$f_k(t) \to f(t)$ for $t \in I$,*
2. *the set $\{f_k;\ k \in \mathbb{N}\}$ is equi-integrable.*

Then for every $\varepsilon > 0$ there is an $\eta > 0$ such that if $E \subset I$ is measurable with $\mu(E) < \eta$ then

$$\left|\int_I f_k \cdot \chi_E\right| = \left|\int_E f_k\right| \leqslant 2\varepsilon$$

for every $k \in \mathbb{N}$.

Proof. Let $\varepsilon > 0$ be given and let $\eta > 0$ correspond to ε by Lemma 13 and assume that $\mu(E) < \eta$. Then there is an open set $G \subset I$ such that $E \subset G$ and $\mu(G) < \eta$.

The equi-integrability of f_k implies the existence of a gauge $\Delta\colon I \to (0, +\infty)$ such that for every Δ-fine M-partition $\{(t_i, I_i)\}$ of I the inequality

$$\left|\sum_i f_k(t_i)\mu(I_i) - \int_I f_k\right| < \varepsilon$$

holds.

By Theorem 15 the integrals $\int_I f_k \cdot \chi_E$, $k \in \mathbb{N}$, exist and for every $\theta > 0$ there is a gauge $\delta\colon I \to (0, +\infty)$ which satisfies $B(t, \delta(t)) \subset G$ if $t \in G$, $\delta(t) \leqslant \Delta(t)$ for $t \in I$ and

$$\left|\sum_m f_k(v_m) \cdot \chi_E(v_m)\mu(V_m) - \int_I f_k \cdot \chi_E\right| \leqslant \theta$$

holds for any δ-fine M-partition $\{(v_m, V_m)\}$ of I.

If $v_m \in E \subset G$ then $V_m \subset G$ and $\sum\limits_{m, v_m \in E} \mu(V_m) \leqslant \eta$.

Since $\{(v_m, V_m);\ v_m \in E\}$ is a Δ-fine M-system, we have by the Saks-Henstock Lemma 11 the inequality

$$\left|\sum_{m, v_m \in E} \left[f_k(v_m)\mu(V_m) - \int_{V_m} f_k\right]\right| \leqslant \varepsilon$$

and by Lemma 13 we get

$$\left|\sum_{m, v_m \in E} \int_{V_m} f_k\right| \leqslant \varepsilon.$$

Hence

$$\left|\int_E f\right| \leqslant \theta + \left|\sum_{m, v_m \in E} f_k(v_m)\mu(V_m)\right| \leqslant \theta + \left|\sum_{m, v_m \in E} \left[f_k(v_m)\mu(V_m) - \int_{V_m} f_k\right]\right| + \left|\sum_{m, v_m \in E} \int_{V_m} f_k\right| \leqslant \theta + 2\varepsilon.$$

This proves the statement because $\theta > 0$ can be chosen arbitrarily small. $\square$

Using Proposition 12 and 16 and the concept of uniform absolute continuity of a sequence of functions given in Definition 7 we obtain the following.

Theorem 17. *Assume that $f_k\colon I \to \mathbb{R}$, $k \in \mathbb{N}$, are McShane integrable functions such that $f_k(t) \to f(t)$ for $t \in I$.*

Then the set $\{f_k;\ k \in \mathbb{N}\}$ forms an equi-integrable sequence if and only if $\{f_k;\ k \in \mathbb{N}\}$ is uniformly absolutely continuous.

Concluding remarks 18. Theorem 17 shows that the relaxed Vitali convergence Theorem 9 is equivalent to our convergence Theorem 4 which uses the concept of equi-integrability.

Therefore Theorem 4 is in the sense of Gordon [1] also a sort of primary theorem because the Lebesgue dominated convergence theorem and the Levi monotone convergence theorem follow from Theorem 4 (see [1, p. 203]).

Note also that if we are looking at the Vitali convergence Theorem 8 where the sequence f_k, $k \in \mathbb{N}$, is assumed to converge to f in measure then by the Riesz theorem [3] there is a subsequence f_{k_l} which converges to f for all $t \in I \setminus N$ where $\mu(N) = 0$. If we set $f_{k_l}(t) = f(t)$ for $t \in N$ then Theorem 17 yields that the assumption of the Vitali convergence Theorem implies that the original sequence f_k, $k \in \mathbb{N}$, contains a subsequence which is equi-integrable.

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Author's address: *Jaroslav Kurzweil*, *Štefan Schwabik*, Matematický ústav AV ČR, Žitná 25, 115 67 Praha 1, Czech Republic, e-mail: `kurzweil@math.cas.cz`, `schwabik@math.cas.cz`.