INSTANTON-ANTI-INSTANTON SOLUTIONS OF DISCRETE YANG-MILLS EQUATIONS

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Abstract. We study a discrete model of the $SU(2)$ Yang-Mills equations on a combinatorial analog of $\mathbb{R}^4$. Self-dual and anti-self-dual solutions of discrete Yang-Mills equations are constructed. To obtain these solutions we use both the techniques of a double complex and the quaternionic approach.

Keywords: Yang-Mills equations, self-dual equations, anti-self-dual equations, instanton, anti-instanton, difference equations

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1. Introduction

We study an intrinsically defined discrete model of the $SU(2)$ Yang-Mills equations on a combinatorial analog of $\mathbb{R}^4$. It is known (see, for example, [5]) that a gauge potential can be defined as a certain $su(2)$-valued 1-form A (the connection 1-form). Then the gauge field F (the curvature 2-form) is given by

$$F = \mathrm{d}A + A \wedge A, \tag{1.1}$$

where $\wedge$ denotes the exterior multiplication. The Yang-Mills equations can be expressed in terms of the 2-forms F and $*F$ as

$$\mathrm{d}F + A \wedge F - F \wedge A = 0, \quad \mathrm{d}{*F} + A \wedge {*F} - {*F} \wedge A = 0, \tag{1.2}$$

where $*$ is the Hodge star operator.

We consider the self-dual and anti-self-dual equations

$$F = *F, \qquad F = - * F. \tag{1.3}$$

Equations (1.3) are nonlinear matrix first order partial differential equations. In the 4-dimensional Yang-Mills theories the self-dual (instanton) and anti-self-dual (anti-instanton) solutions of (1.3) are the absolute minima of the Yang-Mills action and satisfy the second-order Yang-Mills equations (1.2) (see [4]).

The purpose of this paper is to construct the self-dual and anti-self-dual solutions of discrete $SU(2)$ Yang-Mills equations which imitate the corresponding solutions of the continual theory. The ideas presented here are strongly influenced by the book of Dezin [2]. We develop discrete models of some objects in differential geometry, including the Hodge star operator, the differential and the exterior multiplication, in such a way that they preserve the geometric structure of their continual analogs. We continue the investigations which were originated in [3], [6]–[8]. The geometrical discretisation techniques used here extend those introduced in [2] and [6]. A combinatorial model of $\mathbb{R}^4$ based on the use of the double complex construction is taken from [8].

2. Quaternions and the $SU(2)$-connection

We begin with a brief review of some preliminaries about quaternions. The quaternions are formed from real numbers by adjoining three symbols $\mathbf{i}, \mathbf{j}, \mathbf{k}$, and an arbitrary quaternion x can be written as

$$x = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}, \tag{2.1}$$

where $x_1, x_2, x_3, x_4 \in \mathbb{R}$. The symbols $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy the identities

$$\begin{aligned} &\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \\ &\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}. \end{aligned} \tag{2.2}$$

It is clear that the space of quaternions is isomorphic to $\mathbb{R}^4$. By analogy with the complex numbers, x_1 is called the real part of x and $x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$ is called the imaginary part. In the sequel we will write

$$\operatorname{Im} x = x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}.$$

The conjugate quaternion of x is defined by

$$\bar{x} = x_1 - x_2\mathbf{i} - x_3\mathbf{j} - x_4\mathbf{k}.$$

Then the norm $|x|$ of a quaternion can be introduced as

$$|x|^2 = x\bar{x} = x_1^2 + x_2^2 + x_3^2 + x_4^2. \tag{2.3}$$

The algebra of quaternions can be represented as a sub-algebra of the 2×2 complex matrices $M(2, \mathbb{C})$. We identify the quaternion (2.1) with a matrix $f(x) \in M(2, \mathbb{C})$ by setting

$$f(x) = \begin{pmatrix} x_1 + x_2\mathrm{i} & x_3 + x_4\mathrm{i} \\ -x_3 + x_4\mathrm{i} & x_1 - x_2\mathrm{i} \end{pmatrix}. \tag{2.4}$$

Here i is the imaginary unit.

It is well known that the unit quaternions, i.e., those that have the norm $|x| = 1$, form a group and this group is isomorphic to $SU(2)$. The 2×2 complex matrices

$$\mathbf{i} = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \tag{2.5}$$

realize a representation of the Lie algebra $su(2)$ of the group $SU(2)$. Note that multiplying by $-\mathrm{i}$ these tree matrices we obtain the standard Pauli matrices. Matrices (2.5) correspond to the units $\mathbf{i}, \mathbf{j}, \mathbf{k}$ given by (2.2). Thus the Lie algebra $su(2)$ can be viewed as the pure imaginary quaternions with the basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

Let the $SU(2)$-connection A be given by

$$A = \sum_{\mu} A_\mu(x)\, \mathrm{d}x^\mu, \tag{2.6}$$

where $A_\mu(x) \in su(2)$ and $x = (x_1, \dots, x_4)$ is a point of $\mathbb{R}^4$. On the other hand, A can be defined also as taking values in the space of pure imaginary quaternions. Let $f(x)$ be a function of the quaternion variable (2.1) with quaternion values. Then we can write A as

$$A = \mathrm{Im}(f(x)\, \mathrm{d}x), \tag{2.7}$$

where $f(x) = f_1(x) + f_2(x)\mathbf{i} + f_3(x)\mathbf{j} + f_4(x)\mathbf{k}$ and $\mathrm{d}x = \mathrm{d}x_1 + \mathrm{d}x_2\mathbf{i} + \mathrm{d}x_3\mathbf{j} + \mathrm{d}x_4\mathbf{k}$. Using the rules of multiplication (2.2) we have

$$A_1(x) = f_2(x)\mathbf{i} + f_3(x)\mathbf{j} + f_4(x)\mathbf{k}, \quad A_2(x) = f_1(x)\mathbf{i} + f_4(x)\mathbf{j} - f_3(x)\mathbf{k},$$
$$A_3(x) = -f_4(x)\mathbf{i} + f_1(x)\mathbf{j} + f_2(x)\mathbf{k}, \quad A_4(x) = f_3(x)\mathbf{i} - f_2(x)\mathbf{j} + f_1(x)\mathbf{k}.$$

Using (2.7) we can rewrite (1.1) as

$$F = \mathrm{Im}(\mathrm{d}f(x) \wedge \mathrm{d}x + f(x)\, \mathrm{d}x \wedge f(x)\, \mathrm{d}x). \tag{2.8}$$

In the quaternion notation the instanton and anti-instanton solutions can be found in Atiyah [1]. In Section 4 we will construct discrete analogs of these solutions.

3. Discrete model

We will use the double complex construction described in [8]. Let the tensor product $C(4) = C \otimes C \otimes C \otimes C$ of a 1-dimensional complex C be a combinatorial model of the Euclidean space $\mathbb{R}^4$ (for details see also [2]). The 1-dimensional complex C is defined in the following way. Let C^0 denote the real linear space of 0-dimensional chains generated by basis elements x_j (points), $j \in \mathbb{Z}$. It is convenient to introduce the shift operators τ, σ in the set of indices by

$$\tau j = j + 1, \qquad \sigma j = j - 1. \tag{3.1}$$

We denote the open interval $(x_j,\ x_{\tau j})$ by e_j. We will regard the set $\{e_j\}$ as a set of basis elements of the real linear space C^1 of 1-dimensional chains. Then the 1-dimensional complex (combinatorial real line) is the direct sum of the spaces introduced above: $C = C^0 \oplus C^1$. Together with the complex $C(4)$ we consider its double, namely, the complex $\widetilde{C}(4)$ of exactly the same structure (for details see [8]). We need the double to define a discrete analog of the Hodge star operator.

Let $K(4)$ be a cochain complex with $gl(2,\mathbb{C})$-valued coefficients, where $gl(2,\mathbb{C})$ is the Lie algebra of the group $GL(2,\mathbb{C})$. Recall that $gl(2,\mathbb{C})$ consists of all complex 2×2 matrices $M(2,\mathbb{C})$ with bracket operation $[\cdot,\cdot]$. The complex $K(4)$ is a conjugate of $C(4)$ and we have $K(4) = K \otimes K \otimes K \otimes K$, where K is a conjugate of the 1-dimensional complex C. Basis elements of K can be written as $x^j,\ e^j$. Then an arbitrary p-dimensional basis element of $K(4)$ is given by $s^k_{(p)} = s^{k_1} \otimes s^{k_2} \otimes s^{k_3} \otimes s^{k_4}$, where s^{k_i} is either x^{k_i} or e^{k_i}, $k_i \in \mathbb{Z}$. Note that $s^k_{(p)}$ contains exactly p of 1-dimensional elements e^{k_i}. For a p-dimensional cochain $\varphi \in K(4)$ we have

$$\varphi = \sum_k \sum_p \varphi^{(p)}_k s^k_{(p)}, \tag{3.2}$$

where $\varphi^{(p)}_k \in gl(2,\mathbb{C})$. We will call cochains forms, emphasizing their relationship with the corresponding continual objects, differential forms. Denote by $\widetilde{K}(4)$ the complex of cochains over the double complex $\widetilde{C}(4)$. It is clear that $\widetilde{K}(4)$ has the same structure as $K(4)$. Let us introduce the operation $\tilde{\iota}\colon\ K(4) \to \widetilde{K}(4)$, $\tilde{\iota}\colon\ \widetilde{K}(4) \to K(4)$ by setting

$$\tilde{\iota} s^k_{(p)} = \tilde{s}^k_{(p)}, \qquad \tilde{\iota}\tilde{s}^k_{(p)} = s^k_{(p)}, \tag{3.3}$$

where $s^k_{(p)}$ and $\tilde{s}^k_{(p)}$ are basis elements of $K(4)$ and $\widetilde{K}(4)$. Hence for a p-form $\varphi \in K(4)$ we have $\tilde{\iota}\varphi = \tilde{\varphi}$.

For the definitions of d^c, $\cup$ and $*$ on $K(4)$, which are discrete analogs of the differential d, exterior multiplication $\wedge$ and the Hodge star operator respectively, we refer the reader to [8].

Let us consider a discrete 0-form with coefficients belonging to $M(2,\mathbb{C})$. We put

$$f=\sum_k f_k x^k, \tag{3.4}$$

where $x^k=x^{k_1}\otimes x^{k_2}\otimes x^{k_3}\otimes x^{k_4}$ is the 0-dimensional basis element of $K(4)$. Suppose that the matrices $f_k\in M(2,\mathbb{C})$ look like (2.4). Then f_k in quaternionic form can be expressed as

$$f_k=f_k^1+f_k^2\mathbf{i}+f_k^3\mathbf{j}+f_k^4\mathbf{k}. \tag{3.5}$$

Hence the form (3.4) can be viewed as a discrete form with quaternionic coefficients. We will call it simply the quaternionic form when no confusion can arise.

Let us denote by e the quaternionic 1-form

$$e=\sum_k e^k=\sum_k(e_1^k+e_2^k\mathbf{i}+e_3^k\mathbf{j}+e_4^k\mathbf{k}), \tag{3.6}$$

where e_i^k are the 1-dimensional basis elements of $K(4)$. Let $A\in K(4)$ be a discrete 1-form. We define the discrete $SU(2)$-connection A (discrete analog of (2.6)) to be

$$A=\sum_k\sum_{i=1}^4 A_k^i e_i^k, \tag{3.7}$$

where $A_k^i\in su(2)$. Using (3.4) and (3.6), we write (3.7) in the quaternionic form as

$$A=\mathrm{Im}(f\cup e)=\mathrm{Im}\left(\sum_k f_k e^k\right). \tag{3.8}$$

Then the A_k^i are given by

$$\begin{aligned} A_k^1&=f_k^2\mathbf{i}+f_k^3\mathbf{j}+f_k^4\mathbf{k}, & A_k^2&=f_k^1\mathbf{i}+f_k^4\mathbf{j}-f_k^3\mathbf{k},\\ A_k^3&=-f_k^4\mathbf{i}+f_k^1\mathbf{j}+f_k^2\mathbf{k}, & A_k^4&=f_k^3\mathbf{i}-f_k^2\mathbf{j}+f_k^1\mathbf{k}. \end{aligned} \tag{3.9}$$

An arbitrary discrete 2-form $F\in K(4)$ can be written as

$$F=\sum_k\sum_{i<j}F_k^{ij}\varepsilon_{ij}^k, \tag{3.10}$$

where $F^{ij}_k \in gl(2,\mathbb{C})$, $1 \leqslant i,j \leqslant 4$, and ε^k_{ij} is the 2-dimensional basis element of $K(4)$. Let F be given by

$$F = \mathrm{d}^c A + A \cup A. \tag{3.11}$$

For convenience we also introduce the shift operator τ_i which acts in the set of indices as $\tau_i k = (k_1, \dots \tau k_i, \dots k_4)$, where τ is given by (3.1).

By the definitions of d^c and $\cup$, combining (3.7) and (3.11), we obtain

$$F^{ij}_k = \Delta_i A^j_k - \Delta_j A^i_k + A^i_k A^j_{\tau_i k} - A^j_k A^i_{\tau_j k}, \tag{3.12}$$

where $\Delta_i A^j_k = A^j_{\tau_i k} - A^j_k$.

It should be noted that in the continual case the curvature form F (1.1) takes values in the algebra $su(2)$ for any $su(2)$-valued connection form A. Unfortunately, this is not true in the discrete case because, generally speaking, the components $A^i_k A^j_{\tau_i k} - A^j_k A^i_{\tau_j k}$ of the form $A \cup A$ (see (3.12)) do not belong to $su(2)$.

To define an $su(2)$-valued discrete analog of the curvature 2-form we use the quaternionic form of A (3.8) and put it in (3.11). Then the discrete curvature form F is given by

$$F = \mathrm{Im}\{\mathrm{d}^c f \cup e + (f \cup e) \cup (f \cup e)\}. \tag{3.13}$$

Putting (3.9) in (3.12) we find that

$$\begin{aligned}
F^{12}_k = {} & (\Delta_1 f^1_k - \Delta_2 f^2_k - f^3_k f^3_{\tau_1 k} - f^4_k f^4_{\tau_1 k} - f^3_k f^3_{\tau_2 k} - f^4_k f^4_{\tau_2 k})\mathbf{i} \\
& + (\Delta_1 f^4_k - \Delta_2 f^3_k + f^2_k f^3_{\tau_1 k} + f^4_k f^1_{\tau_1 k} + f^1_k f^4_{\tau_2 k} + f^3_k f^2_{\tau_2 k})\mathbf{j} \\
& + (-\Delta_1 f^3_k - \Delta_2 f^4_k + f^2_k f^4_{\tau_1 k} - f^3_k f^1_{\tau_1 k} - f^1_k f^3_{\tau_2 k} + f^4_k f^2_{\tau_2 k})\mathbf{k} \\
& - f^2_k f^1_{\tau_1 k} - f^3_k f^4_{\tau_1 k} + f^4_k f^3_{\tau_1 k} + f^1_k f^2_{\tau_2 k} + f^4_k f^3_{\tau_2 k} - f^3_k f^4_{\tau_2 k}, \\
F^{13}_k = {} & (-\Delta_1 f^4_k - \Delta_3 f^2_k + f^3_k f^2_{\tau_1 k} - f^4_k f^1_{\tau_1 k} - f^1_k f^4_{\tau_3 k} + f^2_k f^3_{\tau_3 k})\mathbf{i} \\
& + (\Delta_1 f^1_k - \Delta_3 f^3_k - f^2_k f^2_{\tau_1 k} - f^4_k f^4_{\tau_1 k} - f^4_k f^4_{\tau_3 k} - f^2_k f^2_{\tau_3 k})\mathbf{j} \\
& + (\Delta_1 f^2_k - \Delta_3 f^4_k + f^2_k f^1_{\tau_1 k} + f^3_k f^4_{\tau_1 k} + f^4_k f^3_{\tau_3 k} + f^1_k f^2_{\tau_3 k})\mathbf{k} \\
& + f^2_k f^4_{\tau_1 k} - f^3_k f^1_{\tau_1 k} - f^4_k f^2_{\tau_1 k} - f^4_k f^2_{\tau_3 k} + f^1_k f^3_{\tau_3 k} + f^2_k f^4_{\tau_3 k}, \\
F^{14}_k = {} & (\Delta_1 f^3_k - \Delta_4 f^2_k + f^3_k f^1_{\tau_1 k} + f^4_k f^2_{\tau_1 k} + f^2_k f^4_{\tau_4 k} + f^1_k f^3_{\tau_4 k})\mathbf{i} \\
& + (-\Delta_1 f^2_k - \Delta_4 f^3_k - f^2_k f^1_{\tau_1 k} + f^4_k f^3_{\tau_1 k} + f^3_k f^4_{\tau_4 k} - f^1_k f^2_{\tau_4 k})\mathbf{j} \\
& + (\Delta_1 f^1_k - \Delta_4 f^4_k - f^2_k f^2_{\tau_1 k} - f^3_k f^3_{\tau_1 k} - f^3_k f^3_{\tau_4 k} - f^2_k f^2_{\tau_4 k})\mathbf{k} \\
& - f^2_k f^3_{\tau_1 k} + f^3_k f^2_{\tau_1 k} - f^4_k f^1_{\tau_1 k} + f^3_k f^2_{\tau_4 k} - f^2_k f^3_{\tau_4 k} + f^1_k f^4_{\tau_4 k},
\end{aligned}$$

$$
\begin{aligned}
F_k^{23} = {} & (-\Delta_2 f_k^4 - \Delta_3 f_k^1 + f_k^4 f_{\tau_2 k}^2 + f_k^3 f_{\tau_2 k}^1 + f_k^1 f_{\tau_3 k}^3 + f_k^2 f_{\tau_3 k}^4)\mathbf{i} \\
& + (\Delta_2 f_k^1 - \Delta_3 f_k^4 - f_k^1 f_{\tau_2 k}^2 + f_k^3 f_{\tau_2 k}^4 + f_k^4 f_{\tau_3 k}^3 - f_k^2 f_{\tau_3 k}^1)\mathbf{j} \\
& + (\Delta_2 f_k^2 + \Delta_3 f_k^3 + f_k^1 f_{\tau_2 k}^1 + f_k^4 f_{\tau_2 k}^4 + f_k^4 f_{\tau_3 k}^4 + f_k^1 f_{\tau_3 k}^1)\mathbf{k} \\
& + f_k^1 f_{\tau_2 k}^4 - f_k^4 f_{\tau_2 k}^1 + f_k^3 f_{\tau_2 k}^2 - f_k^4 f_{\tau_3 k}^1 + f_k^1 f_{\tau_3 k}^4 - f_k^2 f_{\tau_3 k}^3, \\
F_k^{24} = {} & (\Delta_2 f_k^3 - \Delta_4 f_k^1 + f_k^4 f_{\tau_2 k}^1 - f_k^3 f_{\tau_2 k}^2 - f_k^2 f_{\tau_4 k}^3 + f_k^1 f_{\tau_4 k}^4)\mathbf{i} \\
& + (-\Delta_2 f_k^2 - \Delta_4 f_k^4 - f_k^1 f_{\tau_2 k}^1 - f_k^3 f_{\tau_2 k}^3 - f_k^3 f_{\tau_4 k}^3 - f_k^1 f_{\tau_4 k}^1)\mathbf{j} \\
& + (\Delta_2 f_k^1 + \Delta_4 f_k^3 - f_k^1 f_{\tau_2 k}^2 - f_k^4 f_{\tau_2 k}^3 - f_k^3 f_{\tau_4 k}^4 - f_k^2 f_{\tau_4 k}^1)\mathbf{k} \\
& - f_k^1 f_{\tau_2 k}^3 + f_k^4 f_{\tau_2 k}^2 + f_k^3 f_{\tau_2 k}^1 + f_k^3 f_{\tau_4 k}^1 - f_k^2 f_{\tau_4 k}^4 - f_k^1 f_{\tau_4 k}^3, \\
F_k^{34} = {} & (\Delta_3 f_k^3 + \Delta_4 f_k^4 + f_k^1 f_{\tau_3 k}^1 + f_k^2 f_{\tau_3 k}^2 + f_k^2 f_{\tau_4 k}^2 + f_k^1 f_{\tau_4 k}^1)\mathbf{i} \\
& + (-\Delta_3 f_k^2 - \Delta_4 f_k^1 + f_k^4 f_{\tau_3 k}^1 + f_k^2 f_{\tau_3 k}^3 + f_k^3 f_{\tau_4 k}^2 + f_k^1 f_{\tau_4 k}^4)\mathbf{j} \\
& + (\Delta_3 f_k^1 - \Delta_4 f_k^2 + f_k^4 f_{\tau_3 k}^2 - f_k^1 f_{\tau_3 k}^3 - f_k^3 f_{\tau_4 k}^1 + f_k^2 f_{\tau_4 k}^4)\mathbf{k} \\
& + f_k^4 f_{\tau_3 k}^3 + f_k^1 f_{\tau_3 k}^2 - f_k^2 f_{\tau_3 k}^1 - f_k^3 f_{\tau_4 k}^4 - f_k^2 f_{\tau_4 k}^1 + f_k^1 f_{\tau_4 k}^2.
\end{aligned}
$$

To obtain (3.13) we must take the imaginary part of these equations.

Theorem 3.1. *The discrete curvature F in* (3.11) *is $su(2)$-valued if and only if*

$$
\begin{aligned}
-f_k^2 f_{\tau_1 k}^1 - f_k^3 f_{\tau_1 k}^4 + f_k^4 f_{\tau_1 k}^3 + f_k^1 f_{\tau_2 k}^2 + f_k^4 f_{\tau_2 k}^3 - f_k^3 f_{\tau_2 k}^4 &= 0, \\
f_k^2 f_{\tau_1 k}^4 - f_k^3 f_{\tau_1 k}^1 - f_k^4 f_{\tau_1 k}^2 - f_k^4 f_{\tau_3 k}^2 + f_k^1 f_{\tau_3 k}^3 + f_k^2 f_{\tau_3 k}^4 &= 0, \\
-f_k^2 f_{\tau_1 k}^3 + f_k^3 f_{\tau_1 k}^2 - f_k^4 f_{\tau_1 k}^1 + f_k^3 f_{\tau_4 k}^2 - f_k^2 f_{\tau_4 k}^3 + f_k^1 f_{\tau_4 k}^4 &= 0, \\
f_k^1 f_{\tau_2 k}^4 - f_k^4 f_{\tau_2 k}^1 + f_k^3 f_{\tau_2 k}^2 - f_k^4 f_{\tau_3 k}^1 + f_k^1 f_{\tau_3 k}^4 - f_k^2 f_{\tau_3 k}^3 &= 0, \\
-f_k^1 f_{\tau_2 k}^3 + f_k^4 f_{\tau_2 k}^2 + f_k^3 f_{\tau_2 k}^1 + f_k^3 f_{\tau_4 k}^1 - f_k^2 f_{\tau_4 k}^4 - f_k^1 f_{\tau_4 k}^3 &= 0, \\
f_k^4 f_{\tau_3 k}^3 + f_k^1 f_{\tau_3 k}^2 - f_k^2 f_{\tau_3 k}^1 - f_k^3 f_{\tau_4 k}^4 - f_k^2 f_{\tau_4 k}^1 + f_k^1 f_{\tau_4 k}^2 &= 0.
\end{aligned}
$$

Proof. From the above, the assertion follows immediately. □

Theorem 3.2. *Let e be given by* (3.6) *and let $\bar{e}$ be the conjugate quaternion of e. Then the 2-form $e \cup \bar{e}$ is self-dual, i.e.,*

$$e \cup \bar{e} = *\tilde{\iota}(e \cup \bar{e}), \tag{3.14}$$

and $\bar{e} \cup e$ is anti-self-dual, i.e.,

$$\bar{e} \cup e = - * \tilde{\iota}(\bar{e} \cup e). \tag{3.15}$$

Proof. Denote

$$e_i = \sum_k e_i^k, \qquad \varepsilon_{ij} = \sum_k \varepsilon_{ij}^k.$$

This implies $e_i \cup e_j = \varepsilon_{ij}$ and $e_j \cup e_i = -\varepsilon_{ij}$ for all $i < j$. Then we have

$$\begin{aligned} e \cup \bar{e} &= (e_1 + e_2\mathbf{i} + e_3\mathbf{j} + e_4\mathbf{k}) \cup (e_1 - e_2\mathbf{i} - e_3\mathbf{j} - e_4\mathbf{k}) \\ &= -2\{(e_1 \cup e_2 + e_3 \cup e_4)\mathbf{i} + (e_1 \cup e_3 - e_2 \cup e_4)\mathbf{j} + (e_1 \cup e_4 + e_2 \cup e_3)\mathbf{k}\} \\ &= -2\{(\varepsilon_{12} + \varepsilon_{34})\mathbf{i} + (\varepsilon_{13} - \varepsilon_{24})\mathbf{j} + (\varepsilon_{14} + \varepsilon_{23})\mathbf{k}\}. \end{aligned}$$

By the definition of $*$ and using (3.3), we get

$$*\tilde{\iota}(e \cup \bar{e}) = -2\tilde{\iota}\{(\tilde{\varepsilon}_{34} + \tilde{\varepsilon}_{12})\mathbf{i} + (-\tilde{\varepsilon}_{24} + \tilde{\varepsilon}_{13})\mathbf{j} + (\tilde{\varepsilon}_{23} + \tilde{\varepsilon}_{14})\mathbf{k}\} = e \cup \bar{e}.$$

In the same way we obtain (3.15). □

Corollary 3.3. *For any quaternionic* 0*-form* f, *the form* $f \cup e \cup \bar{e}$ *is self-dual and* $f \cup \bar{e} \cup e$ *is anti-self-dual.*

Discrete self-dual and anti-self-dual equations (discrete analogs of equations (1.3)) are defined by

$$F = \tilde{\iota} * F, \qquad F = -\tilde{\iota} * F. \tag{3.16}$$

Using (3.10), by the definitions of $\tilde{\iota}$ and $*$, the first equation (self-dual) of (3.16) can be rewritten as

$$F_k^{12} = F_k^{34}, \qquad F_k^{13} = -F_k^{24}, \qquad F_k^{14} = F_k^{23}. \tag{3.17}$$

By analogy with the continual case the solutions of (3.16) are called instantons and anti-instantons respectively.

4. Discrete instanton and anti-instanton

Again in analogy with the continual case consider (3.8), where the components of f are given by

$$f_k = \frac{\overline{k}}{1 + |k|^2}. \tag{4.1}$$

Here $k = k_1 + k_2\mathbf{i} + k_3\mathbf{j} + k_4\mathbf{k}$, $k_i \in \mathbb{Z}$, and the norm $|k|$ is defined by (2.3). Putting this in (3.9) we obtain

$$\begin{aligned} A_k^1 &= \frac{-k_2\mathbf{i} - k_3\mathbf{j} - k_4\mathbf{k}}{1 + |k|^2}, \qquad A_k^2 = \frac{k_1\mathbf{i} - k_4\mathbf{j} + k_3\mathbf{k}}{1 + |k|^2}, \\ A_k^3 &= \frac{k_4\mathbf{i} + k_1\mathbf{j} - k_2\mathbf{k}}{1 + |k|^2}, \qquad A_k^4 = \frac{-k_3\mathbf{i} + k_2\mathbf{j} + k_1\mathbf{k}}{1 + |k|^2}. \end{aligned} \tag{4.2}$$

It is convenient to denote

$$M_k^i = \frac{1}{(1+|k|^2)(1+|\tau_i k|^2)}, \qquad i = 1,2,3,4. \tag{4.3}$$

Substituting (4.2) in (3.12) and using (4.3) we find the components F_k^{ij}, for example,

$$\begin{aligned} F_k^{12} = {} & \{M_k^1(1+k_2^2-k_1^2-k_1) + M_k^2(1+k_1^2-k_2^2-k_2)\}\mathbf{i} \\ & + \{M_k^1(k_4k_1+k_2k_3) - M_k^2(k_3k_2+k_4k_1)\}\mathbf{j} \\ & + \{M_k^1(k_2k_4-k_1k_3) + M_k^2(k_1k_3-k_2k_4)\}\mathbf{k} \\ & + M_k^1(k_1k_2+k_2) - M_k^2(k_1k_2+k_1). \end{aligned}$$

Note that the last term in F_k^{ij} has the form $M_k^i(k_ik_j+k_j) - M_k^j(k_ik_j+k_i)$. Hence, by Theorem 3.1, the curvature F defined by (4.2) is $su(2)$-valued if and only if

$$M_k^i(k_ik_j+k_j) - M_k^j(k_ik_j+k_i) = 0 \tag{4.4}$$

for any $k_i \in \mathbb{Z}$, $i, j = 1,2,3,4$ and $i < j$. An easy computation shows that equation (4.4) has only the solutions

$$\mu = k_1 = k_2 = k_3 = k_4, \quad k_i \in \mathbb{Z}. \tag{4.5}$$

Thus, the $su(2)$-valued discrete curvature 2-form F can be written in quaternionic form as

$$F = \sum_{k, k_i = \mu} M_\mu(2-2\mu)\{(\varepsilon_{12}^k - \varepsilon_{34}^k)\mathbf{i} + (\varepsilon_{13}^k + \varepsilon_{24}^k)\mathbf{j} + (\varepsilon_{14}^k - \varepsilon_{23}^k)\mathbf{k}\}, \tag{4.6}$$

where $M_\mu = M_k^1 = M_k^2 = M_k^3 = M_k^4$. From (4.3) we have $M_\mu = \frac{1}{2(1+4\mu^2)(1+\mu+2\mu^2)}$. Since $k_i = \mu$, in (4.6) we can write ε_{ij}^μ instead of ε_{ij}^k. If we consider the 0-form

$$\omega = \sum_\mu M_\mu(1-\mu)x^\mu, \qquad \mu \in \mathbb{Z}, \tag{4.7}$$

and use the relation (see the proof of Theorem 3.2)

$$\bar{e} \cup e = 2\{(\varepsilon_{12} - \varepsilon_{34})\mathbf{i} + (\varepsilon_{13} + \varepsilon_{24})\mathbf{j} + (\varepsilon_{14} - \varepsilon_{23})\mathbf{k}\},$$

then F can be written as

$$F = \omega \cup \bar{e} \cup e.$$

In view of Corollary 3.3, F is anti-self-dual, i.e., $F = -\tilde{\iota} * F$. Thus under the condition (4.5), A with components (4.1) describes an anti-instanton.

In the same manner we can see that the quaternionic 1-form

$$A = \mathrm{Im}(f \cup \bar{e}),$$

where f has the components

$$f_k = \frac{k}{1 + |k|^2},$$

leads to an instanton solution of (3.17). Indeed, in this case the discrete curvature (3.13) has the form $F = \omega \cup e \cup \bar{e}$. Consequently, F is self-dual.

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