126 (2001) MATHEMATICA BOHEMICA No. 3, 631–637

PARTICULAR TRACE DECOMPOSITIONS AND APPLICATIONS OF TRACE DECOMPOSITION TO ALMOST PROJECTIVE INVARIANTS

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(Received July 30, 1999)

Abstract. First, by using the formulae of Krupka, the trace decomposition for some particular classes of tensors of types (1, 2) and (1, 3) is obtained. Second, it is proved that the traceless part of a tensor is an almost projective invariant of weight 1. We apply this result to Weyl curvature tensors.

Keywords: traceless tensor, trace decomposition, almost projective invariant

MSC 2000: 15A72, 53A55

Introduction

Let E be a real n-dimensional linear space, $n \geqslant 2$, $T^p_q E$ the linear space of tensors of type (p,q) on E. A tensor is said to be *traceless* if its traces are all zeros. By [2], [3, p. 303] *the trace decomposition problem* consists in finding a decomposition of a tensor in which the first term is traceless and the other terms are linear combinations of Kronecker's δ-tensors.

In [2], [3, p. 309] the following results are proved:

Theorem 0.1. *If $A = (A^i_k) \in T^1_1 E$, then there exists a unique traceless tensor $B = (B^i_k) \in T^1_1 E$ and a unique scalar C such that*

$$A^i_k = B^i_k + \delta^i_k C. \tag{0.1}$$

We have $C = \frac{1}{n} A^s_s$ and $B^i_k = A^i_k - \frac{1}{n}\delta^i_k A^s_s$.

Theorem 0.2. *If $A = (A^i_{kl}) \in T^1_2E$, then there exists a unique traceless tensor $B = (B^i_{kl}) \in T^1_2E$ and unique tensors $C = (C_k), D = (D_k) \in T^0_1E$ such that*

$$A^i_{kl} = B^i_{kl} + \delta^i_k C_l + \delta^i_l D_k. \tag{0.2}$$

These tensors are defined by

$$\begin{aligned} C_l &= \varrho(n,2)(nA^t_{tl} - A^t_{lt}) \\ D_k &= \varrho(n,2)(-A^t_{tk} + nA^t_{kt}) \\ B^i_{kl} &= A^i_{kl} - \varrho(n,2)[\delta^i_k(nA^t_{tl} - A^t_{lt}) + \delta^i_l(-A^t_{tk} + nA^t_{kt})] \end{aligned}$$

where $\varrho(n,2) = \frac{1}{n^2-1}$.

Theorem 0.3. *If $n \geqslant 3$ and $A = (A^i_{klm}) \in T^1_3E$, then there exists a unique traceless tensor $B = (B^i_{klm}) \in T^1_3E$ and unique tensors $C = (C_{lm}), D = (D_{km}), E = (E_{kl}) \in T^0_2E$ such that*

$$A^i_{klm} = B^i_{klm} + \delta^i_k C_{lm} + \delta^i_l D_{km} + \delta^i_m E_{kl}. \tag{0.3}$$

These tensors are defined by:

$$\begin{aligned} C_{kl} &= \varrho(n,3) \\ &\quad [n(n^2-3)A^t_{tkl} + (-n^2+2)A^t_{ktl} + nA^t_{klt} - 2A^t_{tlk} + nA^t_{ltk} + (-n^2+2)A^t_{lkt}], \\ D_{kl} &= \varrho(n,3) \\ &\quad [(-n^2+2)A^t_{tkl} + n(n^2-3)A^t_{ktl} + (-n^2+2)A^t_{klt} + nA^t_{tlk} - 2A^t_{ltk} + nA^t_{lkt}], \\ E_{kl} &= \varrho(n,3) \\ &\quad [nA^t_{tkl} + (-n^2+2)A^t_{ktl} + n(n^2-3)A^t_{klt} + (-n^2+2)A^t_{tlk} + nA^t_{ltk} - 2A^t_{lkt}], \\ B^i_{klm} &= A^i_{klm} - \delta^i_k C_{lm} - \delta^i_l D_{km} - \delta^i_m E_{kl}, \end{aligned}$$

where $\varrho(n,3) = \frac{1}{(n^2-1)(n^2-4)}$.

Remark 0.4. (i) For $n = 2$ and $A \in T^1_3E$ the trace decomposition of A is not unique (theorem 2 (a) of [2], [3, p. 309]).

(ii) In [3, p. 305] it is proved that the traced decomposition problem has a solution for every $A \in T^p_q(M)$ and for every p, q with $p \leqslant q$. Moreover, the traceless part is unique.

(iii) For the generalization of the trace decomposition problem to spaces with complex structure see [4] and for spaces with quaternionic structure see [5].

Acknowledgement. The author wishes to express his gratitude to the referee for his valuable remarks.

1. The trace decomposition of some particular tensors of types (1, 2) and (1, 3)

In this section we restrict our attention to tensors of types $(1,2)$ and $(1,3)$ because the most important tensors given by a linear connection, namely the torsion tensor and the curvature tensor, are of these types. After a straightforward computation we obtain

Proposition 1.1. *If $A = (A^i_{kl}) \in T^1_2 E$ has the form*

$$A^i_{kl} = E^i_{\{k}F_{l\}} = E^i_k F_l + E^i_l F_k \tag{1.1}$$

then the trace decomposition of A is

$$A^i_{kl} = B^i_{kl} + \delta^i_k C_l + \delta^i_l C_k \tag{1.2}$$

with $C_l = \frac{1}{n+1} A^t_{tl}$ and $B^i_{kl} = A^i_{kl} - \frac{1}{n+1}(\delta^i_k A^t_{tl} + \delta^i_l A^t_{tk})$.

Corollary 1.2. *If $A \in T^1_2 E$ has the form (1.1) with $F = \delta$, i.e.*

$$A^i_{kl} = \delta^i_k G_l + \delta^i_l G_k, \tag{1.3}$$

then in the trace decomposition (1.2) of A we have $C_k = G_k, B^i_{kl} = 0$.

Proposition 1.3. *If $A = (A^i_{kl}) \in T^1_2 E$ has the form*

$$A^i_{kl} = E^i_{[k}F_{l]} = E^i_k F_l - E^i_l F_k \tag{1.4}$$

then the trace decomposition of A is

$$A^i_{kl} = B^i_{kl} + \delta^i_k C_l - \delta^i_l C_k \tag{1.5}$$

with $C_l = \frac{1}{n-1} A^t_{tl}$ and $B^i_{kl} = A^i_{kl} - \frac{1}{n-1}(\delta^i_k A^t_{tl} + \delta^i_l A^t_{kt})$.

Corollary 1.4. *If $A \in T^1_2 E$ has the form (1.4) with $F = \delta$, i.e.*

$$A^i_{kl} = \delta^i_k G_l - \delta^i_l G_k, \tag{1.6}$$

then in the trace decomposition (1.5) of A we have $C_l = G_l, B^i_{kl} = 0$.

E x a m p l e 1.5.

(i) Let $\nabla, \widetilde{\nabla}$ be two linear connections on a smooth n-dimensional manifold M. Then $A = \widetilde{\nabla} - \nabla \in T^1_2(M)$. H. Weyl proved that ∇ and $\widetilde{\nabla}$ have the same autoparallel curves if and only if $\exists \psi \in T^0_1(M)$ such that

$$A = \delta \otimes \psi + \psi \otimes \delta, \tag{1.7}$$

i.e. (1.3), and he called the transformation $\nabla \to \widetilde{\nabla}$ given by (1.7) a *projective transformation* ([6, p. 178]) or sometimes a *geodesic mapping.* In particular, if $\nabla, \widetilde{\nabla}$ are the Levi-Civita connections for the Riemannian metrics $g, \tilde{g}$ on M and A is given by (1.7) then the Riemannian spaces (M, g), $(M, \tilde{g})$ are said to be *in geodesic representation* (or *correspondence*) ([6, p. 322]).

A generalization of (1.7) is

$$A = \delta \otimes \psi_1 + \psi_2 \otimes \delta \tag{1.8}$$

with $\psi_1, \psi_2 \in T^0_1(M)$, which is called an *almost projective transformation.* In the next section we define almost projective transformations for tensors of type (p, q), $p \leqslant q$.

(ii) If ∇ is a linear connection on a smooth n-dimensional manifold M then the torsion T of ∇ is in $T^1_2(M)$. ∇ is called *semisymmetric* if $\exists\, t \in T^0_1(M)$ such that

$$T = \delta \otimes t - t \otimes \delta \tag{1.8$'$}$$

i.e. (1.6) holds ([7, p. 194]).

Proposition 1.6. *If $n \geqslant 3$ and $A = (A^i_{klm}) \in T^1_3 E$ has the form*

$$A^i_{klm} = F^i_{\{k} G_{lm\}} = F^i_k G_{lm} + F^i_l G_{mk} + F^i_m G_{kl} \tag{1.9}$$

then in the trace decomposition of A (see Th. 0.3) we have

$$C_{kl} = E_{kl} = \frac{1}{n^2 - 4}(nA^t_{tkl} - 2A^t_{ktl}),$$
$$D_{kl} = \frac{1}{n^2 - 4}(-2A^t_{tkl} + nA^t_{ktl}).$$

Corollary 1.7. *If $n \geqslant 3$ and $A \in T^1_3 E$ has the form (1.9) with $F = \delta$, i.e.*

$$A^i_{klm} = \delta^i_k G_{lm} + \delta^i_l G_{mk} + \delta^i_m G_{kl}, \tag{1.10}$$

then in the trace decomposition of A we have $C_{kl} = E_{kl} = G_{kl}, D_{kl} = G_{lk}, B^i_{klm} = 0$.

Proposition 1.8. *If $n \geqslant 3$ and $A = (A^i_{klm}) \in T^1_3 E$ has the form:*

$$A^i_{klm} = F_{[k} G_{lm]} = F^i_k G_{lm} - F^i_l G_{mk} + F^i_m G_{kl} \tag{1.11}$$

then in the trace decomposition of A (see Th. 0.3) we have

$$\begin{aligned}
\frac{C_{kl}}{\varrho(n,3)} &= (n^3-3n)F^t_t G_{kl} + (5n-n^3)F^t_k G_{lt} \\
&\quad + (n^3-3n)F^t_l G_{tk} - 2F^t_k G_{tl} - 2F^t_t G_{lk} + (2-2n^2)F^t_l G_{kt}, \\
\frac{D_{kl}}{\varrho(n,3)} &= (2-2n^2)F^t_t G_{kl} - 2F^t_k G_{lt} - 2F^t_l G_{tk} + (n^3-3n)F^t_k G_{tl} \\
&\quad + (5n-n^3)F^t_t G_{lk} + (n^3-3n)F^t_l G_{kt}, \\
\frac{E_{kl}}{\varrho(n,3)} &= (n^3-3n)F^t_t G_{kl} + (n^3-3n)F^t_k G_{lt} \\
&\quad + (5n-n^3)F^t_l G_{tk} + (6-2n^2)F^t_k G_{tl} - 2F^t_t G_{lk} - 2F^t_l G_{kt}.
\end{aligned}$$

Corollary 1.9. *If $n \geqslant 3$ and $A \in T^1_3 E$ has the form (1.11) with $F = \delta$, i.e.*

$$A^i_{klm} = \delta^i_k G_{lm} - \delta^i_l G_{mk} + \delta^i_m G_{kl}$$

then in the trace decomposition of A we have $C_{kl} = E_{kl} = G_{kl}$, $D_{kl} = -G_{lk}$, $B^i_{klm} = 0$.

2. Applications to almost projective invariants

Let M be a smooth n-dimensional manifold, $C^\infty(M)$ the ring of real-valued functions on M, $T^p_q(M)$ the vector space of tensors of type (p,q) on M, $\mathcal{T}(M)$ the tensorial algebra of M.

For $p \leqslant q$ consider the subspace $\delta(T^p_q(M))$ of $T^p_q(M)$ generated by the tensors fields $X = (X^{i_1\ldots i_p}_{j_1\ldots j_q})$ of the form ([3, p. 306])

$$\begin{aligned}
X^{i_1\ldots i_p}_{j_1\ldots j_q} &= \delta^{i_1}_{j_1} X^{(1)i_2\ldots i_p}_{(1)j_2\ldots j_q} + \delta^{i_1}_{j_2} X^{(1)i_2\ldots i_p}_{(2)j_1 j_3\ldots j_q} + \ldots + \delta^{i_1}_{j_q} X^{(1)i_2\ldots i_p}_{(q)j_1\ldots j_{q-1}} \\
&\quad + \delta^{i_2}_{j_1} X^{(2)i_1 i_3\ldots i_p}_{(1)j_2\ldots j_q} + \delta^{i_2}_{j_2} X^{(2)i_1 i_3\ldots i_p}_{(2)j_1 j_3\ldots j_q} + \ldots + \delta^{i_2}_{j_q} X^{(2)i_1 i_3\ldots i_p}_{(q)j_1\ldots j_{q-1}} \\
&\quad \ldots \\
&\quad + \delta^{i_p}_{j_1} X^{(p)i_1\ldots i_{p-1}}_{(1)j_2\ldots j_q} + \delta^{i_p}_{j_2} X^{(p)i_1\ldots i_{p-1}}_{(2)j_1 j_3\ldots j_q} + \ldots + \delta^{i_p}_{j_q} X^{(p)i_1\ldots i_{p-1}}_{(q)j_1\ldots j_{q-1}}.
\end{aligned}$$

Definition 2.1. (i) If $A \in T^p_q(M)$, a transformation

$$\widetilde{A} = \varrho A + X \tag{2.1}$$

with $\varrho \in C^\infty(M)$ and $X \in \delta(T^p_q(M))$ is called an *almost projective transformation.*

(ii) With respect to (2.1) a tensor field $N \in \mathcal{T}(M)$ derived from A by any means is called an *almost projective invariant of weight* k (a natural number) if

$$\widetilde{N} = \varrho^k N \tag{2.2}$$

where $\widetilde{N}$ is derived from $\widetilde{A}$ in the same manner as N is derived from A.

Proposition 2.2. *If $A \in T^p_q(M)$ then the traceless part of A is an almost projective invariant of weight* 1.

Proof. Let $g = (g_{ij})$ be a Riemannian metric on M. Then g admits a lift, denoted by $\langle , \rangle$, to $T^p_q(M)$. It is a simple computation that $B \in T^p_q(M)$ is traceless if and only if B is $\langle , \rangle$-orthogonal to $\delta(T^p_q t(M))$. Then A has the decomposition $A = B + X(A)$ where B is the traceless part of A (which is unique!) and $X(A) \in \delta(T^p_q(M))$.

Returning to (2.1) we get

$$\widetilde{A} = \varrho A + X = \varrho(B + X(A)) + X = \varrho B + (\varrho X(A) + X).$$

Obviously $\varrho X(A) + X \in \delta(T^p_q(M))$ and the uniqueness of the traceless part yields that the traceless part of $\widetilde{A}$ is

$$\widetilde{B} = \varrho B.$$

$\square$

Remark 2.3. For the case $p = 1$, $q = 2$ this result appears in [1].

3. Applications to Weyl curvature tensors

Let $g = (g_{ij})$ be a semi-Riemannian metric on M and $R = (R^i_{jkl}) \in T^1_3(M)$ the curvature tensor of g. In [3, p. 314] it is proved that the traceless part of R is exactly *the Weyl projective curvature tensor,* and if we consider the tensor $R^{ij}_{kl} = g^{js} R^i_{skl} \in T^2_2(M)$ then the traceless part of (R^{ij}_{kl}) is exactly *the Weyl conformal curvature tensor*. Applying the result of the previous section we get

Proposition 3.1. *For a semi-Riemannian metric the Weyl projective curvature tensor and the Weyl conformal curvature tensor are almost projective invariants of weight* 1.

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