

OSCILLATION OF NONLINEAR THREE-DIMENSIONAL  
DIFFERENCE SYSTEMS WITH DELAYS

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*Abstract.* In this paper the three-dimensional nonlinear difference system

$$\begin{aligned}\Delta x_n &= a_n f(y_{n-l}), \\ \Delta y_n &= b_n g(z_{n-m}), \\ \Delta z_n &= \delta c_n h(x_{n-k}),\end{aligned}$$

is investigated. Sufficient conditions under which the system is oscillatory or almost oscillatory are presented.

*Keywords:* difference equation, three-dimensional nonlinear system, oscillation

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## 1. INTRODUCTION

Consider a nonlinear three-dimensional difference system of the form

$$(1) \quad \begin{aligned}\Delta x_n &= a_n f(y_{n-l}), \\ \Delta y_n &= b_n g(z_{n-m}), \\ \Delta z_n &= \delta c_n h(x_{n-k}),\end{aligned}$$

where  $n_0 \in \mathbb{N} = \{1, 2, \dots\}$ ,  $l, m, k$  are given positive integers and  $\delta = \pm 1$ . Here  $a, b: N(n_0) \rightarrow \mathbb{R}_+ \cup \{0\}$ ,  $c: N(n_0) \rightarrow \mathbb{R}_+$ , where  $\mathbb{R}, \mathbb{R}_+$  denote the set of real numbers and the set of positive real numbers, respectively. Moreover,

$$(2) \quad \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n = \infty.$$

Assume that functions  $f, g, h: \mathbb{R} \rightarrow \mathbb{R}$  fulfil the following conditions: there exist positive constants  $M^*$ ,  $M^{**}$  and  $M^{***}$  such that

$$(3) \quad \frac{f(u)}{u^\alpha} \geq M^*, \quad \frac{g(u)}{u^\beta} \geq M^{**} \quad \text{and} \quad \frac{h(u)}{u^\gamma} \geq M^{***} \quad \text{for } u \neq 0$$

where  $\alpha$ ,  $\beta$  and  $\gamma$  are ratios of odd positive integers, and

$$(4) \quad \int_0^c \frac{du}{u^{\alpha\beta\gamma}} < \infty \quad \text{for any positive constant } c.$$

Set  $M = \min\{M^*, M^{**}, M^{***}\}$ .

We do not assume that functions  $f$ ,  $g$  and  $h$  are continuous or monotonic.

A solution  $(x, y, z)$  of system (1) is called oscillatory if all its components are oscillatory (that is, neither eventually positive nor eventually negative), and it is called nonoscillatory otherwise. The difference system (1) is called oscillatory if all its solutions are oscillatory. The difference system (1) is called almost oscillatory if all its solutions are oscillatory or

$$(5) \quad \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} z_n = 0.$$

A solution  $(x, y, z)$  of system (1) is called bounded if all its components are bounded. Otherwise it is called unbounded.

It is an interesting problem to extend the oscillation criteria for third order nonlinear difference equations to the case of nonlinear three-dimensional systems. The third order nonlinear difference equation was studied, among many others, by Andruch-Sobio and Drozdowicz [2], Andruch-Sobiło and Migda [3], [4], Migda, Schmeidel and Drozdowicz [7], and Schmeidel and Zbąszyniak [9].

The background for difference systems can be found in the well known monographs [1] by Agarwal, and Kocić and Ladas [6].

The oscillatory theory has been considered usually for two-dimensional difference systems (see, for example, [5], [10], [12] and [11] and the references therein).

Oscillatory results for three-dimensional systems are investigated by Thandapani and Ponnammal in [13].

## 2. NONOSCILLATORY RESULTS

We begin with some lemmas which will be useful in the sequel.

**Lemma 1.** *Assume that condition (3) holds. Let  $(x, y, z)$  be a solution of system (1) and let the sequence  $x$  be nonoscillatory. Then  $(x, y, z)$  is nonoscillatory and sequences  $x, y, z$  are monotonic for sufficiently large  $n$ .*

*Proof.* We note that condition (3) implies the usual signed condition

$$(6) \quad uf(u) > 0, \quad ug(u) > 0, \quad uh(u) > 0 \quad \text{for } u \neq 0.$$

The proof follows directly from condition (6) and from system (1). □

**Lemma 2.** *Assume that conditions (2) and (3) hold. Let  $(x, y, z)$  be a nonoscillatory solution of system (1). If*

$$(7) \quad \lim_{n \rightarrow \infty} x_n \text{ is finite}$$

then

$$\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} z_n = 0.$$

**Lemma 3.** *Assume that conditions (2) and (3) hold and  $(x, y, z)$  is a nonoscillatory solution of system (1). Then one of the following three cases holds*

- (I)  $\operatorname{sgn} x_n = \operatorname{sgn} y_n = \operatorname{sgn} z_n,$
- (II)  $\operatorname{sgn} x_n = \operatorname{sgn} z_n \neq \operatorname{sgn} y_n,$
- (III)  $\operatorname{sgn} x_n = \operatorname{sgn} y_n \neq \operatorname{sgn} z_n$

for large  $n$ . Moreover, if  $\delta = -1$  in system (1) then every nonoscillatory solution of (1) fulfils condition (I) or (II), if  $\delta = 1$  then every nonoscillatory solution of (1) fulfils condition (I) or (III).

**Lemma 4.** *Assume that conditions (2) and (3) hold. Then every solution  $(x, y, z)$  of system (1) fulfilling condition (II) is bounded.*

The proofs of Lemmas 2, 3 and 4 are analogous to the proofs of lemmas which are presented in [8], and hence are omitted.

### 3. OSCILLATION THEOREMS

In this section we establish sufficient conditions under which system (1) is oscillatory or almost oscillatory.

**Theorem 1.** *Let  $\delta = -1$  in system (1). Assume that conditions (2) and (3) hold,*

$$(8) \quad \sum_{i=1}^{\infty} c_i \left( \sum_{j=1}^{i-k-1} a_j \left( \sum_{s=1}^{j-l-1} b_s \right)^{\alpha} \right)^{\gamma} = \infty$$

and

$$(9) \quad \sum_{i=m}^{\infty} b_i \left( \sum_{j=i-m}^{\infty} c_j \right)^{\beta} = \infty.$$

Then system (1) is almost oscillatory.

*Proof.* Without loss of generality we assume that  $x_n > 0$ . By Lemma 1, this implies that the sequences  $y$  and  $z$  are nonoscillatory sequences. Hence  $(x, y, z)$  is a nonoscillatory solution of system (1). (If not, Theorem 1 holds.) By Lemma 3, such a solution fulfils condition (I) or (II).

Suppose that condition (I) holds for large  $n$ , say  $n \geq n_1 \geq n_0$ . Hence the sequence  $z$  is decreasing for  $n \geq n_1$ . Set  $n_2 = n_1 + k + l + m$ . Summing the second equation of system (1) from  $n_2$  to  $n - 1$  we have  $y_n - y_{n_2} = \sum_{i=n_2}^{n-1} b_i g(z_{i-m})$  for  $n \geq n_2$ . Since  $y_{n_2} > 0$ , we get  $y_n > \sum_{i=n_2}^{n-1} b_i g(z_{i-m})$ . From (6), we have  $g(z_{n-m}) > 0$ . By (3), we get  $g(z_{n-m}) \geq M(z_{n-m})^{\beta} > 0$ . Therefore, using the fact that  $z$  is decreasing we infer that  $y_n > M(z_{n-m-1})^{\beta} \sum_{i=n_2}^{n-1} b_i$ . Hence

$$(10) \quad (y_{n-l})^{\alpha} > M^{\alpha} (z_{n-l-m-1})^{\alpha\beta} \left( \sum_{i=n_2}^{n-l-1} b_i \right)^{\alpha}.$$

Summing the first equation of system (1) from  $n_2$  to  $n - 1$  and using (3), we have  $x_n \geq M \sum_{i=n_2}^{n-1} a_i (y_{i-l})^{\alpha}$ . Using (10) in the above inequality, we obtain

$$(11) \quad x_n \geq M^{1+\alpha} \sum_{i=n_2}^{n-1} a_i (z_{i-l-m-1})^{\alpha\beta} \left( \sum_{j=n_2}^{i-l-1} b_j \right)^{\alpha}.$$

As the sequence  $z$  is decreasing we have

$$(12) \quad x_n \geq M^{1+\alpha} (z_{n+k})^{\alpha\beta} \sum_{i=n_2}^{n-1} a_i \left( \sum_{j=n_2}^{i-l-1} b_j \right)^\alpha.$$

From the third equation of system (1), we get  $-\Delta z_{n+k} = c_{n+k} h(x_n)$ . By (3), we have  $-\Delta z_{n+k} \geq c_{n+k} M(x_n)^\gamma$ . Using (12) in the above equality, we obtain

$$-\Delta z_{n+k} \geq c_{n+k} M^{2+\alpha} (z_{n+k})^{\alpha\beta\gamma} \left( \sum_{i=n_2}^{n-1} a_i \left( \sum_{j=n_2}^{i-l-1} b_j \right)^\alpha \right)^\gamma.$$

Hence, we obtain

$$-\frac{\Delta z_n}{(z_n)^{\alpha\beta\gamma}} \geq c_n M^{2+\alpha} \left( \sum_{i=n_2}^{n-k-1} a_i \left( \sum_{j=n_2}^{i-l-1} b_j \right)^\alpha \right)^\gamma \text{ for } n \geq n_2.$$

Summing the above inequality from  $n_2$  to  $n-1$  we obtain

$$-\sum_{i=n_2}^{n-1} \frac{\Delta z_n}{(z_n)^{\alpha\beta\gamma}} \geq M^{2+\alpha} \sum_{i=n_2}^{n-1} c_i \left( \sum_{j=n_2}^{i-k-1} a_j \left( \sum_{s=n_2}^{j-l-1} b_s \right)^\alpha \right)^\gamma.$$

For  $z_{n+1} < u < z_n$  we have

$$\int_{z_{n+1}}^{z_n} \frac{du}{(u)^{\alpha\beta\gamma}} \geq \frac{-\Delta z_n}{(z_n)^{\alpha\beta\gamma}} \text{ for } n \geq n_2.$$

Hence

$$\int_0^{z_{n_2}} \frac{du}{(u)^{\alpha\beta\gamma}} \geq M^{2+\alpha} \sum_{i=n_2}^{\infty} c_i \left( \sum_{j=n_2}^{i-k-1} a_j \left( \sum_{s=n_2}^{j-l-1} b_s \right)^\alpha \right)^\gamma$$

which, by (4) and (8), is a contradiction. Therefore condition (I) cannot hold.

Suppose that condition (II) from Lemma 3 holds for large  $n$ , say  $n \geq n_3$ . Then  $y_n < 0$ . From the first equation of system (1) we get that  $x$  is a nonincreasing sequence. Therefore a nonnegative limit of the sequence  $x$  exists and

$$\lim_{n \rightarrow \infty} x_n = L^* < \infty.$$

By Lemma 2 we have

$$(13) \quad \lim_{n \rightarrow \infty} y_n = 0.$$

We will prove that  $L^* = 0$ . Suppose on the contrary that  $L^* > 0$ . Then  $x_n \geq L^*$  for  $n \geq n_3$ . Summing the third equation of system (1) from  $n$  to  $\infty$ , we get  $z_n = \sum_{i=n}^{\infty} c_i h(x_{i-k})$ . Then, by (3), we have

$$z_n \geq M(x_{i-k})^\gamma \sum_{i=n}^{\infty} c_i \geq M(L^*)^\gamma \sum_{i=n}^{\infty} c_i \quad \text{for } n \geq n_4 = n_3 + k + m.$$

Summing the second equation of system (1) from  $n_4$  to  $n - 1$ , we obtain  $y_n = y_{n_4} + \sum_{i=n_4}^{n-1} b_i g(z_{i-m})$  for  $n \geq n_4$ . Thus, by (3), we get

$$y_n \geq y_{n_4} + M \sum_{i=n_4}^{n-1} b_i (z_{i-m})^\beta \geq y_{n_4} + M^{1+\beta} (L^*)^{\gamma\beta} \sum_{i=n_4}^{n-1} b_i \left( \sum_{j=i-m}^{\infty} c_j \right)^\beta.$$

Hence, by (9), we have  $\lim_{n \rightarrow \infty} y_n = \infty$ . This contradicts (13), so  $\lim_{n \rightarrow \infty} x_n = 0$ . Hence, by Lemma 2, we have  $\lim_{n \rightarrow \infty} z_n = 0$ .

This completes the proof.  $\square$

**Theorem 2.** *Let  $\delta = 1$  in (1). Assume that conditions (2) and (3) hold, and*

$$(14) \quad \sum_{n=1}^{\infty} c_n = \infty.$$

*Then every bounded solution  $(x, y, z)$  of system (1) is oscillatory.*

**Proof.** Without loss of generality we assume that  $x_n > 0$ . By Lemma 1 the sequence  $(x, y, z)$  is a nonoscillatory solution of system (1). (If not, Theorem 1 holds.) By Lemma 3, such a solution fulfils condition (I) or (III).

Let  $(x, y, z)$  be a nonoscillatory solution of system (1) for which condition (I) holds. Then we get  $x_n > 0$ ,  $y_n > 0$  and  $z_n > 0$  for large  $n$ , say  $n \geq n_5$ . Hence the sequence  $y$  is eventually nondecreasing. Summing the first equation of system (1) from  $n_6 = n_5 + l$  to  $n - 1$  we have

$$x_n = x_{n_6} + \sum_{i=n_6}^{n-1} a_i f(y_{i-l}) \quad \text{for } n \geq n_6.$$

Therefore, by positivity of sequences  $x$  and  $y$  and by (3), we get

$$x_n \geq M \sum_{i=n_6}^{n-1} a_i (y_{i-l})^\alpha.$$

Since  $y$  is nondecreasing we have  $x_n \geq M(y_{n_6-l})^\alpha \sum_{i=n_6}^{n-1} a_i$ . Thus, using (2), we obtain that  $\lim_{n \rightarrow \infty} x_n = \infty$ . Hence there is no nonoscillatory bounded solution of system (1) which fulfils condition (I).

Let  $(x, y, z)$  be a nonoscillatory solution of system (1) for which condition (III) holds. Without loss of generality  $x_n > 0$  for large  $n$ , say  $n \geq n_7$ . Hence  $x$  is a nondecreasing sequence. Then there exists a positive limit of the sequence  $x$ . Set  $\lim_{n \rightarrow \infty} x_n = L^{**}$ . Assume that  $L^{**} < \infty$ . Hence, by Lemma 2, we have  $\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} z_n = 0$ .

Summing the third equation of system (1) from  $n_8 = n_7 + k$  to  $n - 1$  we have

$$z_n = z_{n_8} + \sum_{i=n_8}^{n-1} c_i h(x_{i-k}) \text{ for } n \geq n_8.$$

Therefore, by positivity of the sequence  $x$  and by (3), we get  $z_n \geq z_{n_8} + M \sum_{i=n_8}^{n-1} c_i (x_{i-k})^\gamma$ . Since  $x$  is a nondecreasing sequence we have

$$z_n \geq z_{n_8} + M(x_{n_8-k})^\gamma \sum_{i=n_8}^{n-1} c_i.$$

The left hand side of the above inequality tends to zero whereas the right hand side, by (14), tends to infinity. This contradiction excludes that  $L^{**} < \infty$ . Hence  $\lim_{n \rightarrow \infty} x_n = \infty$ . So, there is no nonoscillatory bounded solution of system (1) which fulfils condition (III).

Hence the thesis of Theorem 2 holds. □

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