A THIRD ORDER BOUNDARY VALUE PROBLEM SUBJECT TO NONLINEAR BOUNDARY CONDITIONS

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Abstract. Utilizing the theory of fixed point index for compact maps, we establish new results on the existence of positive solutions for a certain third order boundary value problem. The boundary conditions that we study are of nonlocal type, involve Stieltjes integrals and are allowed to be nonlinear.

Keywords: positive solution, nonlinear boundary conditions, third order problem, cone, fixed point index

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1. Introduction

In a very interesting paper [6], Graef and Webb studied the existence of multiple solutions for the nonlinear third order differential equation

$$u'''(t) = g(t)f(t, u(t)), \quad t \in (0, 1), \tag{1.1}$$

subject to the nonlocal boundary conditions (BCs)

$$u(0) = 0, \quad u'(p) = 0, \quad u''(1) = \lambda[u''], \tag{1.2}$$

where $p \in [1/2, 1]$ and $\lambda[\cdot]$ is a linear functional on the space $C[0, 1]$ given by a Stieltjes integral, namely

$$\lambda[v] = \int_0^1 v(s)\,\mathrm{d}\Lambda(s), \tag{1.3}$$

with $\mathrm{d}\Lambda$ a signed measure. The formulation (1.3) is quite general and includes, as special cases,

$$\lambda[v] = \sum_{i=1}^{m} \lambda_i v(\xi_i) \quad \text{and} \quad \lambda[v] = \int_0^1 \lambda(s)v(s)\,\mathrm{d}s,$$

that is, m-point and integral conditions.

Nonlocal boundary conditions, in the case of third order equations, have been studied recently by several authors, see for example the papers by Anderson and Davis [1], Clark and Henderson [4], Palamides and Palamides [24], Palamides and Smyrlis [25], Wang and Ge [26], Yang [31], Yao [33] and references therein.

One motivation given in [6] is that the BCs (1.2) can be seen as a generalization of the BCs that occur in a third order problem studied by Graef and Yang [7] and extended to the higher order case by Graef, Henderson and Yang [8].

The methodology in [6] is to rewrite the BVP (1.1)–(1.2) as a Hammerstein integral equation of the form

$$u(t) = \int_0^1 k_\lambda(t,s)g(s)f(s,u(s))\,\mathrm{d}s. \tag{1.4}$$

In order to establish existence and nonexistence results for the equation (1.4), Graef and Webb make use of a careful analysis of the Green function k_λ combined with an earlier theory developed by Webb and co-authors [29], [30].

Furthermore, in the paper [6], by making use of the results of [29] that deal with perturbed Hammerstein integral equations of the form

$$u(t) = \gamma(t)\tilde{\alpha}[u] + \delta(t)\tilde{\beta}[u] + \int_0^1 k(t,s)g(s)f(s,u(s))\,\mathrm{d}s, \tag{1.5}$$

the more general nonlocal BCs

$$u(0) = \tilde{\alpha}[u], \quad u'(p) = 0, \quad u''(1) + \tilde{\beta}[u] = \lambda[u''],$$

where $\tilde{\alpha}[\cdot]$ and $\tilde{\beta}[\cdot]$ are linear functionals on $C[0,1]$ given by Stieltjes integrals with signed measures, are studied.

In [14] Infante, motivated by earlier work of Guidotti and Merino [9], Infante and Webb [17], [18], Webb [27], [28], and Palamides, Infante and Pietramala [23], studied a thermostat model with nonlinear controllers. The approach used in [14] relied on an extension of the results of [29], valid for equations of the type (1.5), to the context of nonlinear perturbations of the form

$$u(t) = \gamma(t)H_1(\alpha[u]) + \delta(t)H_2(\beta[u]) + \int_0^1 k(t,s)g(s)f(s,u(s))\,\mathrm{d}s, \tag{1.6}$$

where H_1, H_2 are continuous functions such that there exist $h_{11}, h_{12}, h_{21}, h_{22} \in [0, \infty)$ with

$$(1.7) \qquad h_{11}v \leqslant H_1(v) \leqslant h_{12}v \quad \text{and} \quad h_{21}v \leqslant H_2(v) \leqslant h_{22}v$$

for every $v \geqslant 0$. Unlike the results of [29], due to some inequalities involved in the theory, the functionals $\alpha[\cdot]$ and $\beta[\cdot]$ are assumed to be given by *positive* measures.

Here we focus on the boundary value problem (BVP)

$$u'''(t) = g(t)f(t, u(t)), \ t \in (0,1),$$
$$u(0) = H_1(\alpha[u]), \ u'(p) = H_2(\beta[u]), \ u''(1) = \lambda[u''], \ p \in [1/2, 1],$$

where the functions H_1, H_2 and the functionals $\alpha[\cdot]$ and $\beta[\cdot]$ are as above.

BVPs with nonlinear BCs have been studied recently by several authors, see for example the papers by Cabada, Minhós and Santos [3], Franco and O'Regan [5], Infante [11], [12], [14], Infante and Pietramala [16], Kong and Wang [19], Minhós [22], Yang [32] and references therein.

Here we utilize some of the results of [6] to show that our BVP fits exactly the framework of [14].

We prove, via the classical fixed point index theory, the existence of multiple positive solutions.

2. Some preliminary results on the integral equation

We first recall some results from [14]. The assumptions made on the terms that occur in the perturbed Hammerstein integral equation

$$u(t) = \gamma(t)H_1(\alpha[u]) + \delta(t)H_2(\beta[u]) + \int_0^1 k(t,s)g(s)f(s,u(s))\,\mathrm{d}s := Tu(t),$$

are as follows:

- $f\colon [0,1] \times [0,\infty) \to [0,\infty)$ is continuous.
- $k\colon [0,1] \times [0,1] \to [0,\infty)$ is continuous.
- There exist a subinterval $[a,b] \subseteq [0,1]$, a function $\Phi \in L^\infty[0,1]$, and a constant $c_1 \in (0,1]$ such that

$$k(t,s) \leqslant \Phi(s) \quad \text{for } t \in [0,1] \text{ and almost every } s \in [0,1],$$
$$k(t,s) \geqslant c_1\Phi(s) \quad \text{for } t \in [a,b] \text{ and almost every } s \in [0,1].$$

- $g\Phi \in L^1[0,1]$, $g \geqslant 0$ a.e., and $\int_a^b \Phi(s)g(s)\,\mathrm{d}s > 0$.

- A, B are functions of bounded variation. Here $\mathrm{d}A$ and $\mathrm{d}B$ are *positive* measures and we use the notation

$$\mathcal{K}_A(s) := \int_0^1 k(t,s)\,\mathrm{d}A(t) \text{ and } \mathcal{K}_B(s) := \int_0^1 k(t,s)\,\mathrm{d}B(t).$$

- $\gamma \in C[0,1]$, $\gamma(t) \geqslant 0$, $h_{12}\alpha[\gamma] < 1$. There exists $c_2 \in (0,1]$ such that

$$\gamma(t) \geqslant c_2\|\gamma\| \quad \text{for } t \in [a,b].$$

- $\delta \in C[0,1]$, $\delta(t) \geqslant 0$, $h_{22}\beta[\delta] < 1$. There exists $c_3 \in (0,1]$ such that

$$\delta(t) \geqslant c_3\|\delta\| \quad \text{for } t \in [a,b].$$

- $D_2 := (1 - h_{12}\alpha[\gamma])(1 - h_{22}\beta[\delta]) - h_{12}h_{22}\alpha[\delta]\beta[\gamma] > 0$.

Under the above hypotheses, the compact operator T leaves invariant the cone

$$K = \Big\{u \in C[0,1], u \geqslant 0\colon \min_{t\in[a,b]} u(t) \geqslant c\|u\|\Big\}, \tag{2.1}$$

where $c = \min\{c_1, c_2, c_3\}$. This type of cone was used first by Krasnosel'skiĭ, see e.g. [20], and D. Guo, see e.g. [10], and later by several authors.

We utilize the classical fixed point index theory for compact maps (see for example [2] or [10]) and we work with the following open bounded sets (relative to K):

$$K_\varrho = \{u \in K\colon \|u\| < \varrho\}, \quad V_\varrho = \Big\{u \in K\colon \min_{a\leqslant t\leqslant b} u(t) < \varrho\Big\}.$$

The set V_ϱ is equal to the set called $\Omega_{\varrho/c}$ in [21] (here c is from (2.1)). A key feature of these sets is that they can be nested, that is

$$K_\varrho \subset V_\varrho \subset K_{\varrho/c}.$$

We make use of the quantity

$$D_1 := (1 - h_{11}\alpha[\gamma])(1 - h_{21}\beta[\delta]) - h_{11}h_{21}\alpha[\delta]\beta[\gamma],$$

and observe that the condition $D_2 > 0$ implies $D_1 > 0$.

The following lemma gives a condition allowing the index to be 0 on the set V_ϱ.

Lemma 1 [14]. *Assume that there exists $\varrho > 0$ such that*

$$(2.2)\qquad f_{\varrho,\varrho/c}\bigg(\Big(\frac{c_2\|\gamma\|}{D_1}(1-h_{21}\beta[\delta])+\frac{c_3\|\delta\|}{D_1}h_{11}\beta[\gamma]\Big)\int_a^b \mathcal{K}_A(s)g(s)\,\mathrm{d}s + \Big(\frac{c_2\|\gamma\|}{D_1}h_{21}\alpha[\delta]+\frac{c_3\|\delta\|}{D_1}(1-h_{11}\alpha[\gamma])\Big)\int_a^b \mathcal{K}_B(s)g(s)\,\mathrm{d}s+\frac{1}{M}\bigg) > 1,$$

where

$$f_{\varrho,\varrho/c} = \inf\Big\{\frac{f(t,u)}{\varrho}\colon\ (t,u)\in[a,b]\times[\varrho,\varrho/c]\Big\} \quad \text{and} \quad \frac{1}{M} = \inf_{t\in[a,b]}\int_a^b k(t,s)g(s)\,\mathrm{d}s.$$

Then the fixed point index, $i_K(T,V_\varrho)$, is 0.

The next result gives a sufficient condition for the index to be 1 on the set K_ϱ.

Lemma 2 [14]. *Assume that there exists $\varrho > 0$ such that*

$$(2.3)\qquad f^{0,\varrho}\bigg(\Big(\frac{\|\gamma\|}{D_2}(1-h_{22}\beta[\delta])+\frac{\|\delta\|}{D_2}h_{12}\beta[\gamma]\Big)\int_0^1 \mathcal{K}_A(s)g(s)\,\mathrm{d}s + \Big(\frac{\|\gamma\|}{D_2}h_{22}\alpha[\delta]+\frac{\|\delta\|}{D_2}(1-h_{12}\alpha[\gamma])\Big)\int_0^1 \mathcal{K}_B(s)g(s)\,\mathrm{d}s+\frac{1}{m}\bigg) < 1,$$

where

$$f^{0,\varrho} = \sup\Big\{\frac{f(t,u)}{\varrho}\colon\ (t,u)\in[0,1]\times[0,\varrho]\Big\} \quad \text{and} \quad \frac{1}{m} = \sup_{t\in[0,1]}\int_0^1 k(t,s)g(s)\,\mathrm{d}s.$$

Then $i_K(T,K_\varrho)=1$.

3. The boundary value problem

Now we turn our attention to the BVP

$$(3.1)\qquad u'''(t) = g(t)f(t,u(t)),\quad t\in(0,1),$$

$$(3.2)\qquad u(0)=H_1(\alpha[u]),\quad u'(p)=H_2(\beta[u]),\quad u''(1)=\lambda[u''],\quad p\in[1/2,1].$$

In what follows we assume that $\lambda[1] < 1$ and by a solution of the BVP (3.1)–(3.2) we mean a solution of the corresponding perturbed integral equation

$$(3.3)\qquad u(t) = H_1(\alpha[u]) + tH_2(\beta[u]) + \int_0^1 k_\lambda(t,s)g(s)f(s,u(s))\,\mathrm{d}s,$$

where k_λ is the Green function associated to the BCs

$$u(0)=0, \quad u'(p)=0, \quad u''(1)=\lambda[u''],$$

that is,

$$k_\lambda(t,s):=\Big(tp-\frac{1}{2}t^2\Big)\Big(1+\frac{\Lambda(s)}{1-\lambda[1]}\Big)-t(p-s)\chi_{[0,p]}(s)+\frac{(t-s)^2}{2}\chi_{[0,t]}(s),$$

where

$$\Lambda(s):=\int_0^s \mathrm{d}\Lambda(t) \quad \text{and} \quad \chi_I(t):=\begin{cases}1, & t\in I,\\ 0, & t\notin I.\end{cases}$$

The function k_λ was investigated in Section 2 of [6] and a key property is given by the following theorem.

Theorem 3.1 [6]. *Suppose that $\Lambda(s)\geqslant 0$ for $s\leqslant p$ and $\Lambda(s)/(1-\lambda[1])\geqslant -(s-p)/(1-p)$ for $s>p$, and let*

$$\Phi(s):=\begin{cases}\dfrac{p^2}{2}+\dfrac{p^2}{2}\dfrac{\Lambda(s)}{1-\lambda[1]}, & s\geqslant p,\\[2ex] \dfrac{s^2}{2}+\dfrac{p^2}{2}\dfrac{\Lambda(s)}{1-\lambda[1]}, & s<p.\end{cases}$$

Then, for $t\in[0,1]$ and $s\in[0,1]$, we have

$$c(t)\Phi(s)\leqslant k_\lambda(t,s)\leqslant \Phi(s),$$

where $c(t):=(2tp-t^2)/p^2$.

In order to satisfy the conditions of Section 2, we need

$$h_{12}\alpha[1]<1, \quad h_{22}\beta[t]<1, \quad (1-h_{12}\alpha[1])(1-h_{22}\beta[t]),-h_{12}h_{22}\alpha[t]\beta[1]>0,$$

and, by fixing $[a,b]\subset(0,1)$, we obtain

$$c:=\min\{a,a(2p-a)/p^2,b(2p-b)/p^2\}. \tag{3.4}$$

By means of the fixed point index results of Section 2, we can state a result on the existence of one or of two positive solutions. Note that, provided the nonlinearity f possesses a suitable oscillatory behavior, it is possible to state, with arguments similar to those in [21], a theorem on the existence of three or more positive solutions.

Theorem 3.2. *Let $[a,b] \subset (0,1)$ and let c be as in (3.4). Then equation (3.3) has a positive solution in K if one of the following conditions holds.*

(S_1) *There exist $\varrho_1, \varrho_2 \in (0,\infty)$ with $\varrho_1 < \varrho_2$ such that (2.3) is satisfied for ϱ_1 and (2.2) is satisfied for ϱ_2.*

(S_2) *There exist $\varrho_1, \varrho_2 \in (0,\infty)$ with $\varrho_1 < c\varrho_2$ such that (2.2) is satisfied for ϱ_1 and (2.3) is satisfied for ϱ_2.*

Equation (3.3) has at least two positive solutions in K if one of the following conditions holds.

(D_1) *There exist $\varrho_1, \varrho_2, \varrho_3 \in (0,\infty)$ with $\varrho_1 < \varrho_2 < c\varrho_3$ such that (2.3) is satisfied for ϱ_1, (2.2) is satisfied for ϱ_2 and (2.3) is satisfied for ϱ_3.*

(D_2) *There exist $\varrho_1, \varrho_2, \varrho_3 \in (0,\infty)$ with $\varrho_1 < c\varrho_2$ and $\varrho_2 < \varrho_3$ such that (2.2) is satisfied for ϱ_1, (2.3) is satisfied for ϱ_2 and (2.2) is satisfied for ϱ_3.*

The next example illustrates the applicability of our result.

Example 1. Consider the BVP

$$u'''(t) = f(u(t)), \quad t \in (0,1),$$
$$u(0) = H_1(u(1/4)), \quad u'(2/3) = H_2(u(1/2)), \quad u'(3/4) = u'(1),$$

where the functions H_1, H_1 are chosen in a way similar to that used in [15], that is

$$H_1(w) = \begin{cases} \frac{2}{3}w, & 0 \leqslant w \leqslant 1, \\ \frac{1}{3}w + \frac{1}{3}, & w \geqslant 1, \end{cases} \qquad H_2(w) = \begin{cases} \frac{9}{10}w, & 0 \leqslant w \leqslant 1, \\ \frac{9}{20}w + \frac{9}{20}, & w \geqslant 1. \end{cases}$$

In this case we have

$$h_{11} = 1/3, \quad h_{21} = 9/20, \quad h_{12} = 2/3, \quad h_{22} = 9/10.$$

We fix $[a,b] = [1/8, 7/8]$ and, by direct calculation, we obtain

$$D_1 = 23/48, \quad D_2 = 1/30, \quad m = 324/31, \quad M(1/8, 7/8) = 36864/1325.$$

This value for m corrects the typo ($m = 567/55$) present in [6].

Therefore all terms appearing in (2.2) and (2.3) can be computed and the growth assumptions for the nonlinearity f are

$$f^{0,\varrho} < 0.24820 \quad \text{and} \quad f_{\varrho,\varrho/c} > 5.7245.$$

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