ON SOME SINGULAR SYSTEMS OF PARABOLIC FUNCTIONAL EQUATIONS

László Simon, Budapest

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Abstract. We will prove existence of weak solutions of a system, containing non-local terms u, w.

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1. Introduction

We will consider initial-boundary value problems for the system

$$D_t u - \sum_{i=1}^{n} D_i[a_i(t,x,u(t,x),Du(t,x)+g(w(t,x))Dw(t,x);u,w)] \\ + a_0(t,x,u(t,x),Du(t,x)+g(w(t,x))Dw(t,x);u,w) = G, \tag{1.1}$$

$$D_t w = F(t,x;u,w) \quad \text{in } Q_T = (0,T)\times\Omega \subset \mathbb{R}^{n+1},\ T\in(0,\infty) \tag{1.2}$$

where the functions

$$a_i\colon\ Q_T \times \mathbb{R}^{n+1} \times L^{p_1}(0,T;V_1) \times L^2(Q_T) \to \mathbb{R}$$

(with a closed linear subspace V_1 of the Sobolev space $W^{1,p_1}(\Omega)$, $2 \leqslant p_1 < \infty$) satisfy conditions which are generalizations of the usual conditions for quasilinear parabolic differential equations considered when using the theory of monotone type operators. Further,

$$F\colon\ Q_T \times L^{p_1}(0,T;V_1) \times L^2(Q_T) \to \mathbb{R}$$

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satisfies a Lipschitz condition. In the second part of the paper the case $g = 0$ and in the third part the general case will be considered.

Such problems with $g = 0$ arise, e.g., when considering diffusion and transport in porous media with variable porosity, see [4], [6]. In [6] a nonlinear system was numerically studied which consisted of a parabolic, an elliptic and an ordinary DE, describing the reaction-mineralogy-porosity changes in porous media. System (1.1), (1.2) is the case when the pressure is assumed to be constant. The case of general g was motivated by non-Fickian diffusion in viscoelastic polymers and by spread of morphogens (see [7], [8]). In [2], [5] similar degenerate systems of parabolic differential equations were considered without functional dependence and with more special differential equations, by using other methods.

2. Case $g = 0$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain having the uniform C^1 regularity property (see [1]) and let $p_1 \geqslant 2$ be a real number. Denote by $W^{1,p_1}(\Omega)$ the usual Sobolev space of real valued functions with the norm

$$\|u\| = \left[\int_\Omega (|Du|^{p_1} + |u|^{p_1})\right]^{1/p_1}.$$

Let $V_1 \subset W^{1,p_1}(\Omega)$ be a closed linear subspace containing $C_0^\infty(\Omega)$. Denote by $L^{p_1}(0,T;V_1)$ the Banach space of the set of measurable functions $u\colon (0,T) \to V_1$ such that $\|u\|_{V_1}^p$ is integrable, and define the norm by

$$\|u\|_{L^{p_1}(0,T;V_1)}^{p_1} = \int_0^T \|u(t)\|_{V_1}^{p_1}\,\mathrm{d}t.$$

For the sake of brevity we denote $L^{p_1}(0,T;V_1)$ by X_1^T. The dual space of X_1^T is $L^{q_1}(0,T;V_1^\star)$ where $1/p_1 + 1/q_1 = 1$ and $V_1^\star$ is the dual space of V_1 (see, e.g., [10], [11]). Further, let $X^T = X_1^T \times L^2(Q_T)$.

On functions a_i we assume:

(A_1) The functions $a_i\colon Q_T \times \mathbb{R}^{n+1} \times X^T \to \mathbb{R}$ satisfy the Carathéodory conditions for arbitrary fixed $(u,w) \in X_T$ $(i = 0,1,\dots,n)$.

(A_2) There exist bounded (nonlinear) operators $g_1\colon X^T \to \mathbb{R}^+$ and $k_1\colon X^T \to L^{q_1}(Q_T)$ such that

$$|a_i(t,x,\zeta_0,\zeta;u,w)| \leqslant g_1(u,w)[|\zeta_0|^{p_1-1} + |\zeta|^{p_1-1}] + [k_1(u,w)](t,x), \quad i = 0,1,\dots,n$$

for a.e. $(t,x) \in Q_T$, every $(\zeta_0,\zeta) \in \mathbb{R}^{n+1}$ and $(u,w) \in X^T$.

(A$_3$)
$$\sum_{i=1}^{n}[a_i(t,x,\zeta_0,\zeta;u,w)-a_i(t,x,\zeta_0,\zeta^\star;u,w)](\zeta_i-\zeta_i^\star) \geqslant [g_2(u)](t)|\zeta-\zeta^\star|^{p_1}, \quad t\in(0,T]$$

where

$$[g_2(u)](t)\geqslant \frac{c_2}{1+\|u\|^\sigma_{X_1^t}} \tag{2.1}$$

with some constants $c_2>0$, $0\leqslant\sigma<p_1-1$.

(A$_4$) There exists a (nonlinear) operator $k_2\colon X^T\to L^1(Q_T)$ such that

$$\sum_{i=0}^{n} a_i(t,x,\zeta_0,\zeta;u,w)\zeta_i \geqslant [g_2(u)](t)[|\zeta_0|^{p_1}+|\zeta|^{p_1}]-[k_2(u,w)](t,x)$$

for a.e. $(t,x)\in Q_T$, all $(\zeta_0,\zeta)\in\mathbb{R}^{n+1}$, $(u,w)\in X^T$ and

$$\|k_2(u,w)\|_{L^1(Q_T)}\leqslant c_3\left[\|u\|^\lambda+\|w\|^\mu+1\right] \tag{2.2}$$

with some nonnegative constants $\lambda<p_1-\sigma$, $\mu<2$.

(A$_5$) There exists $\delta\in(0,1]$ such that if $(u_k)\to u$ in $L^{p_1}(0,T;W^{1-\delta,p_1}(\Omega))$, a.e. in Q_T, $(\zeta_0^k)\to\zeta_0$, $(w_k)\to w$ weakly in $L^2(Q_T)$ then for $i=0,1,\dots,n$, a.e. $(t,x)\in Q_T$, and all $\zeta\in\mathbb{R}^n$ we have

$$a_i(t,x,\zeta_0^k,\zeta;u_k,w_k)-a_i(t,x,\zeta_0,\zeta;u_k,w)\to 0,\ k_1(u_k,w_k)\to k_1(u,w) \text{ in } L^1(Q_T).$$

(See (A$_1$).) Further, if conditions $(\zeta^k)\to\zeta$, $(w_k)\to w$ a.e. in Q_T are satisfied, too, then

$$a_i(t,x,\zeta_0^k,\zeta^k;u_k,w_k)\to a_i(t,x,\zeta_0,\zeta;u,w), \quad i=1,\dots,n$$

for a.e. $(t,x)\in Q_T$ and

$$a_0(t,x,\zeta_0^k,\zeta^k;u_k,w_k)\to a_0(t,x,\zeta_0,\zeta;u,w)$$

for a.e. $(t,x)\in Q_T$, in the last case assuming also that $(Du_k)\to Du$ a.e. in Q_T.

Assumptions on $F\colon Q_T\times\mathbb{R}\times X^T\to\mathbb{R}$:

(F$_1$) For each fixed $(u,w)\in X^T$, $F(\cdot,u;u,w)\in L^2(Q_T)$.

(F$_2$) F satisfies the following (global) Lipschitz condition: there exists a constant K such that for each $t\in(0,T]$, $(u,\tilde w),(u,\tilde w^\star)\in X^T$ we have

$$\int_{Q_t} \mathrm{e}^{-2c\tau}|F(\tau,x,\tilde w(\tau,x)\mathrm{e}^{c\tau};u,\tilde w\mathrm{e}^{ct})-F(\tau,x,\tilde w^\star(\tau,x)\mathrm{e}^{c\tau};u,\tilde w^\star\mathrm{e}^{ct})|^2\,\mathrm{d}\tau\,\mathrm{d}x \leqslant K\int_{Q_t}|\tilde w(\tau,x)-\tilde w^\star(\tau,x)|^2\,\mathrm{d}\tau\,\mathrm{d}x \tag{2.3}$$

for each positive number c. Further, there is a constant K_0 such that

$$\int_{Q_T} |F(t,x,0;u,0)|^2 \,\mathrm{d}t\,\mathrm{d}x \leqslant K_0(\|u\|^{\lambda}_{L^{p_1}(0,T;W^{1-\delta,p_1}(\Omega))} + 1).$$

(F_3) If $(u_k) \to u$ in $L^{p_1}(0,T;W^{1-\delta,p_1}(\Omega))$, a.e. in Q_T, $(\eta_k) \to \eta$ and $(w_k) \to w$ in $L^2(Q_T)$, a.e. in Q_T, then for a.e. $(t,x) \in Q_T$

$$F(t,x,\eta_k;u_k,w_k) \to F(t,x,\eta;u,w).$$

Remark. A sufficient condition for (2.3) to hold is the following inequality:

$$\begin{aligned}
&\int_{\Omega} |F(\tau,x,w(\tau,x);u,w) - F(\tau,x,w^\star(\tau,x);u,w^\star)|^2 \,\mathrm{d}x \\
&\leqslant K_1 \int_{Q_\tau} |w(s,x) - w^\star(s,x)|^2 \,\mathrm{d}s\,\mathrm{d}x \\
&\quad + K_2 \int_{\Omega} |w(\gamma(\tau),x) - w^\star(\gamma(\tau),x)|^2 \,\mathrm{d}x,\ \ \tau \in (0,T)
\end{aligned}$$

with some constants K_1, K_2 and a function $\gamma \in C^1$ satisfying $\gamma' > 0$, $0 \leqslant \gamma(\tau) \leqslant \tau$.

Definition. We define an operator $A = (A_1, A_2)\colon\ X^T \to (X^T)^\star$ by

$$\begin{aligned}
[A(u,w),(v,z)] &= [A_1(u,w),v] + [A_2(u,w),z],\\
[A_1(u,w),v] &= \int_{Q_T} \sum_{i=1}^{n} a_i(t,x,u(t,x),Du(t,x);u,w)D_i v \,\mathrm{d}t\,\mathrm{d}x\\
&\quad + \int_{Q_T} a_0(t,x,u(t,x),Du(t,x);u,w)v \,\mathrm{d}t\,\mathrm{d}x,\\
[A_2(u,w),z] &= \int_{Q_T} F(t,x,w(t,x);u,w)z \,\mathrm{d}t\,\mathrm{d}x,
\end{aligned}$$

$(u,w),(v,z) \in X^T$, where the brackets $[\cdot,\cdot]$ mean the dualities in spaces $(X^T)^\star, X^T$, $(X_1^T)^\star, X_1^T$, $[L^2(Q_T)]^\star$, $[L^2(Q_T)]$, respectively.

Theorem 2.1. *Assume* (A_1)–(A_5) *and* (F_1)–(F_3). *Then for any* $G \in (X_1^T)^\star$, $H \in L^2(Q_T)$ *there exists* $(u,w) \in X^T$ *such that* $D_t u \in (X_1^T)^\star$, $D_t w \in L^2(Q_T)$,

$$D_t u + A_1(u,w) = G, \quad u(0) = 0, \tag{2.4}$$

$$D_t w + A_2(u,w) = H, \quad w(0) = 0. \tag{2.5}$$

Sketch of the proof. Define a new unknown function $\tilde{w}$ (instead of w) by

$$\tilde{w}(t,x) = w(t,x)\mathrm{e}^{-ct},\ \text{ i.e. } w(t,x) = \tilde{w}(t,x)\mathrm{e}^{ct}$$

with constant $c > 0$. Further, define a function $\widetilde{F}$ and operators $\widetilde{A}_1, \widetilde{A}_2$ by

$$
\begin{aligned}
&\widetilde{F}(t,x,\eta;u,\tilde{w}) = \mathrm{e}^{-ct}F(t,x,\eta\mathrm{e}^{ct};u,\tilde{w}\mathrm{e}^{ct}) + c\eta,\\
&[\widetilde{A}_1(u,\tilde{w}),v] = [A_1(u,w),v] = \int_{Q_T}\sum_{i=1}^{n} a_i(t,x,u(t,x),Du(t,x);u,w)D_i v\,\mathrm{d}t\,\mathrm{d}x\\
&\qquad\qquad\qquad\qquad + \int_{Q_T} a_0(t,x,u(t,x),Du(t,x);u,w)v\,\mathrm{d}t\,\mathrm{d}x,\\
&[\widetilde{A}_2(u,\tilde{w}),z] = \int_{Q_T}\widetilde{F}(t,x,\tilde{w}(t,x);u,\tilde{w})z\,\mathrm{d}t\,\mathrm{d}x\\
&\qquad = \int_{Q_T}\mathrm{e}^{-ct}F(t,x,\tilde{w}(t,x)\mathrm{e}^{ct};u,\tilde{w}\mathrm{e}^{ct})z\,\mathrm{d}t\,\mathrm{d}x + c\int_{Q_T}\tilde{w}z\,\mathrm{d}t\,\mathrm{d}x.
\end{aligned}
$$

Clearly, (u,w) is a solution of (2.4), (2.5) if and only if $(u,\tilde{w})$ satisfies

$$D_t u + \widetilde{A}_1(u,\tilde{w}) = G, \quad u(0) = 0, \tag{2.6}$$

$$D_t\tilde{w} + \widetilde{A}_2(u,\tilde{w}) = \mathrm{e}^{-ct}H = \widetilde{H}, \quad \tilde{w}(0) = 0. \tag{2.7}$$

By (A_1)–(A_5), (F_1), (F_2) the operator $\widetilde{A}\colon X^T \to (X^T)^\star$ is bounded and demicontinuous (see [10], [11]).

By (F_2), $\widetilde{A}_2$ is monotone for sufficiently large $c > 0$), thus, by using (A_1)–(A_5), one can show that $\widetilde{A}$ is pseudomonotone with respect to the domain of $L = D_t$:

$$D(L) = \{(u,\tilde{w}) \in X^T\colon\ (D_t u, D_t\tilde{w}) \in (X^T)^\star, \quad u(0) = 0, \quad \tilde{w}(0) = 0\},$$

i.e.

$$
\begin{gathered}
(u_k,\tilde{w}_k) \to (u,\tilde{w}) \text{ weakly in } X^T,\\
(Lu_k, L\tilde{w}_k) \to (Lu, L\tilde{w}) \text{ weakly in } (X^T)^\star,
\end{gathered} \tag{2.8}
$$

and

$$\limsup_{k\to\infty}[\widetilde{A}(u_k,\tilde{w}_k),(u_k,\tilde{w}_k) - (u,\tilde{w})] \leqslant 0 \tag{2.9}$$

imply

$$\lim_{k\to\infty}[\widetilde{A}(u_k,\tilde{w}_k),(u_k,\tilde{w}_k) - (u,\tilde{w})] = 0 \tag{2.10}$$

and

$$\widetilde{A}(u_k,\tilde{w}_k) \to \widetilde{A}(u,\tilde{w}) \text{ weakly in } (X^T)^\star. \tag{2.11}$$

Because, by (2.8)

$$(u_k) \to u \quad \text{in } L^p(0,T;W^{1-\delta,p}(\Omega)) \text{ and a.e. in } Q_T \tag{2.12}$$

for a subsequence (again denoted by (u_k), for simplicity), see, e.g., [10]. We may choose the number $c > 0$ such that $c > K$ (see (F_2)). We have

$$\begin{aligned}[\widetilde{A}_2(u_k,\tilde{w}_k),\tilde{w}_k-\tilde{w}] = [\widetilde{A}_2(u_k,\tilde{w}_k) - \widetilde{A}_2(u_k,\tilde{w}),\tilde{w}_k-\tilde{w}] \\ + [\widetilde{A}_2(u_k,\tilde{w}) - \widetilde{A}_2(u,\tilde{w}),\tilde{w}_k-\tilde{w}] + [\widetilde{A}_2(u,\tilde{w}),\tilde{w}_k-\tilde{w}]\end{aligned} \tag{2.13}$$

where by $c > K$, (F_2) the first term on the right hand side is nonnegative, the second term tends to 0 by (2.12), (F_2), (F_3), Vitali's theorem, and Cauchy-Schwarz inequality, while, finally, the third term converges to 0 by (2.8). Thus (2.9), (2.13) imply (for a subsequence)

$$\limsup_{k\to\infty}[\widetilde{A}_1(u_k,\tilde{w}_k),u_k-u] \leqslant 0. \tag{2.14}$$

By using (A_2), (A_3), (A_5), Vitali's theorem and Hölder's inequality, one obtains from (2.8), (2.14)

$$\lim_{k\to\infty}[\widetilde{A}_1(u_k,\tilde{w}_k),u_k-u)] = 0, \tag{2.15}$$

hence by (A_3) one obtains for a subsequence

$$\lim_{k\to\infty}\int_{Q_T}|Du_k-Du|^{p_1}\,\mathrm{d}t\,\mathrm{d}x = 0, \text{ thus } (Du_k)\to Du \text{ a.e. in } Q_T. \tag{2.16}$$

Further, by (2.15), (2.9), (2.13)

$$\lim_{k\to\infty}[\widetilde{A}_2(u_k,\tilde{w}_k),\tilde{w}_k-\tilde{w})] = 0, \tag{2.17}$$

thus by assumption (F_2) and due to $c > K$ (for a subsequence)

$$\lim_{k\to\infty}\int_{Q_T}|\tilde{w}_k-\tilde{w}|^2\,\mathrm{d}t\,\mathrm{d}x \text{ and so } (\tilde{w}_k)\to\tilde{w} \text{ a.e. in } Q_T. \tag{2.18}$$

Consequently, from (A_5), (2.12), (2.16) one obtains (by using Vitali's theorem)

$$\widetilde{A}_1(u_k,\tilde{w}_k)\to\widetilde{A}_1(u,\tilde{w}) \text{ weakly in } L^q(0,T;V_1^\star). \tag{2.19}$$

Similarly, (2.18), (F_2), (F_3) imply

$$\widetilde{A}_2(u_k, \tilde{w}_k) \to \widetilde{A}_2(u, \tilde{w}) \quad \text{weakly in } L^2(Q_T). \tag{2.20}$$

Thus (2.19), (2.20) imply (2.11) for a subsequence. Finally, (2.15), (2.17) imply (2.10) (for a subsequence). One can prove in the standard way that the last facts imply (2.10), (2.11) for the original sequence.

Finally, by (A_4), (F_2) (for sufficiently large $c > 0$), $\widetilde{A}$ is coercive:

$$\lim_{\|(u,\tilde{w})\|_{X^T} \to \infty} \frac{[\widetilde{A}(u, \tilde{w}), (u, \tilde{w})]}{\|u\| + \|\tilde{w}\|} = +\infty.$$

Since $\widetilde{A}\colon X^T \to (X^T)^\star$ is bounded, demicontinuous, pseudomonotone with respect to $D(L)$ and coercive, we obtain the existence of a solution $(u, \tilde{w})$ of (2.6), (2.7) and thus the existence of a solution (u, w) of (2.4), (2.5). (See, e.g. [3], [10].)

Example. Conditions (A_1)–(A_5) are satisfied if e.g.

$$\begin{aligned} a_i(t, x, \zeta_i, \zeta; u, w) &= b(H(u))\zeta_i|\zeta|^{p_1-2}, \quad i = 1, 2, \ldots, n, \\ a_0(t, x, \zeta_i, \zeta; u, w) &= b(H(u))\zeta_0|\zeta_0|^{p_1-2} + b_0(F_0(u)) + b_1(F_1(w)) \end{aligned}$$

where b, b_0, b_1 are continuous functions satisfying with some positive constants c_3, c_4, c_5 the inequalities

$$\begin{aligned} b(\theta) &\geqslant \frac{c_3}{1 + |\theta|^\sigma} \quad (0 \leqslant \sigma < p_1 - 1), \\ |b_0(\theta)| &\leqslant c_4(|\theta|^{\lambda-1} + 1) \text{ where } 1 \leqslant \lambda < p_1 - \sigma, \\ |b_1(\theta)| &\leqslant c_5(|\theta|^{\mu_1-1} + 1) \text{ where } 0 \leqslant \mu_1 < 2 - 2/p_1 \end{aligned}$$

and

$$\begin{aligned} &H\colon L^{p_1}(0, T; W^{1-\delta, p_1}(\Omega)) \to C(\overline{Q_T}), \\ &F_0\colon L^{p_1}(0, T; W^{1-\delta, p_1}(\Omega)) \to L^{p_1}(Q_T), \quad F_1\colon L^2(Q_T) \to \mathbb{R} \end{aligned}$$

are linear continuous operators. If b is between two positive constants, H may be the same as F_0.

Conditions (F_1)–(F_3) are satisfied if e.g.

$$F(t, x, \eta; u, w) = \beta(\eta)\gamma_1(H_1(u)) + \gamma_2(H_2(u))\delta(G(w)) + \gamma_3(H_3(u))$$

where β, δ are globally Lipschitz functions, γ_1, γ_2 are continuous and bounded, γ_3 is continuous and satisfies

$$|\gamma_3(\theta)| \leqslant c_5|\theta|^{\lambda/2} + c_6$$

with some constants c_5, c_6, and

$$G\colon\ L^2(Q_T) \to L^2(Q_T), \quad H_1, H_2\colon\ L^{p_1}(0,T; W^{1-\delta,p_1}(\Omega)) \to L^{p_1}(Q_T),$$
$$H_3\colon\ L^{p_1}(0,T; W^{1-\delta,p_1}(\Omega)) \to L^2(Q_T)$$

are continuous linear operators such that G satisfies for all $w \in L^2(Q_T)$

$$\int_\Omega |[G(w)](\tau,x)|^2 \,\mathrm{d}x \leqslant K_1 \int_{Q_\tau} |w(s,x)|^2 \,\mathrm{d}s\,\mathrm{d}x + K_2 \int_\Omega |w(\gamma(s),x)|^2 \,\mathrm{d}x$$

where $\gamma \in C^1$, $\gamma' > 0$, $\gamma(s) \leqslant s$.

3. Case $g \neq 0$

Now we shall consider equations (1.1), (1.2) with a bounded, continuous function g. This problem will be transformed to the case $g = 0$, considered in Theorem 2.1. Let $f = \int g$, $f(0) = 0$, $p_1 = p > 2$.

Define

$$\widetilde{X}^T = L^p(0,T; W^{1,p}(\Omega)) \times L^p(0,T; W^{1,p}(\Omega))$$

and an operator $A_1\colon\ \widetilde{X}_T \to \left(X_1^T\right)^\star$ for $(u,w) \in \widetilde{X}^T$, $v \in X_1^T$ by

$$[A_1(u,w),v] = \int_{Q_T} \left\{ \sum_{i=1}^n a_i(t,x,u,Du + g(w)Dw; u,w) D_i v \right\} \mathrm{d}t\,\mathrm{d}x$$
$$+ \int_{Q_T} a_0(t,x,u(t,x), Du(t,x) + g(w(t,x))Dw(t,x); u,w) v \,\mathrm{d}t\,\mathrm{d}x.$$

Further, assume

(F_4) F has the form $F(t,x;u,w) = F_1(t,x,[h(u)](t,x), w(t,x))$ where F_1 is continuously differentiable with respect to the last three variables, the partial derivatives are bounded and either $h(u) = u$ or $h\colon\ L^p(Q_T) \to L^p(0,T; W^{1,p}(\Omega))$ is a continuous linear operator such that $h(u) \in L^p(0,T; C^1(\overline{\Omega}))$ for all $u \in L^p(Q_T)$ and the following estimate holds for any $\tau \in [0,T]$ with a suitable constant:

$$\int_\Omega |[h(u)](\tau,x)|^2 \,\mathrm{d}x \leqslant \text{const} \int_{Q_\tau} |u(s,x)|^2 \,\mathrm{d}s\,\mathrm{d}x.$$

Further, there exists a constant $c_0 > 0$ such that

$$F_1(t,x,\zeta_0,\eta)\eta < 0 \quad \text{if } |\eta| \geqslant c_0.$$

Theorem 3.1. *Assume that* (A_1)–(A_5) *and* (F_1)–(F_4) *are satisfied with* $p_1 = p > 2$, $\delta = 1$, $\sigma < p - 2$ *such that for the operators* g_1, k_1, g_2, k_2 *in* (A_2)–(A_4) *we have*

$$g_1(u,w)^q \leqslant \mathit{const}\; g_2(u,w), \quad k_1(u,w)^q \leqslant \mathit{const}\; k_2(u,w) \text{ if } w(t,x) \leqslant c_0 \text{ a.e.}$$

Further, let g *be a bounded, continuous function. Then for any* $G \in (X_1^T)^\star$ *there exists* $(u,w) \in \widetilde{X}^T$ *such that* $u + f(w) \in L^p(0,T;V_1)$,

$$D_t u + D_t[f(w)] \in (X_1^T)^\star, \quad D_t w \in L^2(Q_T),$$

$$D_t u + A_1(u,w) = G, \quad u(0) = 0, \tag{3.1}$$

$$D_t w = F(t,x;u,w) \quad \text{for a.e. } (t,x) \in Q_T,\; w(0) = w. \tag{3.2}$$

S k e t c h o f t h e p r o o f. Instead of u introduce a new unknown function $\tilde{u}$ by

$$\tilde{u}(t,x) = u(t,x) + f(w(t,x)) \quad (\text{where } f = \textstyle\int g,\; f(0) = 0). \tag{3.3}$$

By using the formulas

$$D_t \tilde{u} = D_t u + f'(w) D_t w, \quad D\tilde{u} = Du + f'(w) Dw \tag{3.4}$$

we obtain that $(u,w) \in \widetilde{X}^T$ is a solution of (3.1), (3.2) if and only if $(\tilde{u},w) \in \widetilde{X}^T$ satisfies

$$D_t \tilde{u} + \widetilde{A}_1(\tilde{u},w) = G, \quad \tilde{u}(0) = 0, \tag{3.5}$$

$$D_t w = F(t,x;\tilde{u} - f(w), w), \quad w(0) = 0 \tag{3.6}$$

where

$$\begin{aligned}
&[\widetilde{A}_1(\tilde{u},w), v] \\
&= \int_{Q_T} \left\{ \sum_{i=1}^n a_i(t,x,\tilde{u}(t,x) - f(w(t,x)), D\tilde{u}(t,x); \tilde{u} - f(w), w) D_i v \right\} \mathrm{d}t\,\mathrm{d}x \\
&\quad + \int_{Q_T} \{a_0(t,x,\tilde{u} - f(w), D\tilde{u}; \tilde{u} - f(w), w) - f'(w) F(t,x;\tilde{u} - f(w), w)\}\, v \,\mathrm{d}t\,\mathrm{d}x.
\end{aligned}$$

One can show that by Theorem 2.1 there is a solution $(\tilde{u},w) \in X^T$ of (3.5), (3.6) (such that $D_t w \in L^2(Q_T)$). Then one proves that $w \in L^p(0,T;W^{1,p}(\Omega))$, hence $(\tilde{u},w) \in \widetilde{X}^T$ and thus with $u = \tilde{u} - f(w)$, (u,w) satisfies (3.1), (3.2).

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Author's address: *László Simon*, L. Eötvös University, Budapest, Hungary, 1111 Pázmány P. sétány 1/C, e-mail: `simonl@ludens.elte.hu`.