FRIEDRICHS EXTENSION OF OPERATORS DEFINED BY LINEAR HAMILTONIAN SYSTEMS ON UNBOUNDED INTERVAL

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(Received October 15, 2009)

Abstract. In this paper we consider a linear operator on an unbounded interval associated with a matrix linear Hamiltonian system. We characterize its Friedrichs extension in terms of the recessive system of solutions at infinity. This generalizes a similar result obtained by Marletta and Zettl for linear operators defined by even order Sturm-Liouville differential equations.

Keywords: linear Hamiltonian system, Friedrichs extension, self-adjoint operator, recessive solution, quadratic functional, positivity conjoined basis

MSC 2010: 47B25, 34L05, 34C10

1. Introduction

Friedrichs extension of linear differential operators is a topic frequently studied in literature. If $\mathcal{L}_{\min}$ is a minimal operator defined by a semibounded symmetric (for example, Sturm-Liouville) operator in a Hilbert space H, then Friedrichs proved in [7], [8] the existence of a self-adjoint extension of $\mathcal{L}_{\min}$ preserving the lower bound. This extension is known as the Friedrichs extension $\mathcal{L}_F$ of the minimal operator. One of the characterizations of the domain of $\mathcal{L}_F$, which will be used also in this work, is given in the classical result [6] by Freudenthal.

In this paper we consider the Hilbert space $L^2[a,\infty)$ of Lebesgue measurable complex-valued $2n$-vector functions satisfying $\int_a^\infty y^*(t)y(t)\,\mathrm{d}t < \infty$, where $y^* = \overline{y}^T$

[1] Corresponding author. Research supported by the Grant Agency of the Academy of Sciences of the Czech Republic under grant KJB100190701, by the Czech Science Foundation under grant 201/09/J009, and by the research project MSM 0021622409 of the Ministry of Education, Youth, and Sports of the Czech Republic.

[2] Research supported under grant 0964/2009 of Masaryk University.

is the conjugate transpose of y. The inner product is defined to be $\langle y, z\rangle := \int_a^\infty y^*(t)z(t)\,\mathrm{d}t$. Let $A, B, C\colon [a,\infty) \to \mathbb{C}^{n\times n}$ be locally integrable matrix functions such that $B(t)$ and $C(t)$ are Hermitian and $B(t)$ is positive semidefinite ($B(t) \geqslant 0$) for all $t \in [a,\infty)$. Put

$$y = \begin{pmatrix} x \\ u \end{pmatrix}, \quad \mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \mathcal{H}(t) := \begin{pmatrix} -C(t) & A^*(t) \\ A(t) & B(t) \end{pmatrix}$$

with vector and matrix blocks of dimension n. In this paper we study the Friedrichs extension of the symmetric operator defined by the differential expression

$$l[y](t) := -\mathcal{J}y' - \mathcal{H}(t)y = \begin{pmatrix} C(t)x - A^*(t)u - u' \\ x' - A(t)x - B(t)u \end{pmatrix}.$$

The equation $l[y](t) = 0$ defines a linear Hamiltonian system

$$\text{(H)} \qquad x' = A(t)x + B(t)u, \quad u' = C(t)x - A^*(t)u, \quad t \in [a,\infty).$$

It is well known that systems of the form (H) contain as special cases, for example, $2n$-th order scalar Sturm-Liouville differential equations or second order matrix Sturm-Liouville equations, see e.g. [12], [16].

An expository discussion about the boundary conditions which define the Friedrichs extension of second order Sturm-Liouville operators can be found in [21, Section 10.5]. More precisely, in [9], Friedrichs considered a second order Sturm-Liouville differential operator and proved that the "Friedrichs" extension on a finite interval can be determined by the Dirichlet boundary conditions. This result was later extended in various ways to an unbounded interval in [10], [17], [18] and to higher order operators in [13], [14]. In [2] (in a very special case) and in [20], the "Friedrichs" boundary conditions (i.e., the boundary conditions which determine the domain of the Friedrichs extension) were given for a class of higher order singular differential operators. More recently, in [12], the Friedrichs extension for singular differential operators of order $2n$ on finite or infinite interval was characterized by using the principal (recessive) solutions. The Friedrichs extension of a Hamiltonian operator in the limit point case is characterized in [23] by the Dirichlet boundary condition $x(a) = 0$, referring to the above notation $y = (x^*, u^*)^*$. The result of the present paper can be considered as a completion of this result for the Hamiltonian operator in the limit circle case. The Friedrichs extension constructed in this paper uses the recessive system of solutions of the Hamiltonian system (H). This result is of the same spirit as the corresponding result for higher order Sturm-Liouville equations in [12, Theorem 12], adopting the concept of $y = (x, u) \in L^2$ as in [23]. The key role in this result is played by the Lagrange bracket or its limit at infinity,

$$[y, z](t) := y^*(t)\mathcal{J}z(t), \quad [y, \tilde{y}_j]_\infty := \lim_{t\to\infty} [y, \tilde{y}_j](t), \tag{1.1}$$

where $\tilde{y}_j$ for $j \in \{1, \dots, n\}$ are the recessive solutions of system (H), see Section 2. As we shall see, the Friedrichs extension is characterized by the zero values of $[y, \tilde{y}_j]_\infty$.

The paper is organized as follows. In the next section we briefly introduce the necessary notation and terminology from the theory of linear Hamiltonian systems, including the recessive system of solutions of (H) and the corresponding quadratic functional. In Section 3 we develop the tools from the spectral theory of differential operators and we state and prove our main result (Theorem 3.1).

2. Linear Hamiltonian systems

In this section we briefly discuss the properties of linear Hamiltonian systems and their solutions which will be needed in this paper. Vector solutions of (H) will be denoted by small letters, typically $y = (x, u)$, while matrix solutions of (H) will be denoted by capital letters, typically $Y = (X, U)$. Here x and u are n-vector valued (so that $y(t) \in \mathbb{C}^{2n}$), and X and U are $n \times n$-matrix valued (so that $Y(t) \in \mathbb{C}^{2n\times n}$). To be completely consistent with our notation we should write $y = (x^*, u^*)^*$ and $Y = (X^*, U^*)^*$. However, the above simplified notation is well adopted in the theory of linear Hamiltonian systems.

If $Y = (X, U)$ and $\widetilde{Y} = (\widetilde{X}, \widetilde{U})$ are any two solutions of (H), then their Wronskian matrix $Y^*(t)\mathcal{J}\widetilde{Y}(t) = \{X^*\widetilde{U} - U^*\widetilde{X}\}(t)$ is a constant matrix, which can be verified by differentiation. A solution $Y = (X, U)$ is a *conjoined basis* of (H) if $X^*(t)U(t)$ is Hermitian and $\operatorname{rank} Y(t) = n$ for some (and hence for any) point $t \in [a, \infty)$. Two conjoined bases $\widetilde{Y}$ and $\widehat{Y}$ are called *normalized* if their Wronskian matrix is the identity, i.e., $\widetilde{Y}^*(t)\mathcal{J}\widehat{Y}(t) \equiv I$. In this case we have (see e.g. [11, Proposition 1.1.5])

$$\begin{cases} \widetilde{X}^*\widetilde{U},\ \widehat{X}^*\widehat{U},\ \widetilde{X}\widehat{X}^*,\ \widetilde{U}\widehat{U}^* \ \text{ are Hermitian,} \\ \widetilde{X}^*\widehat{U} - \widetilde{U}^*\widehat{X} = I,\ \widehat{U}\widetilde{X}^* - \widetilde{U}\widehat{X}^* = I. \end{cases} \tag{2.1}$$

Following [16, pg. 316] and [1, pg. 172], a conjoined basis $\widetilde{Y} = (\widetilde{X}, \widetilde{U})$ of (H) is said to be a *recessive solution* (or *principal solution* at ∞) if $\widetilde{X}(t)$ is invertible for large t and for any other conjoined basis $Y = (X, U)$ for which the (constant) Wronskian matrix $Y^*\mathcal{J}\widetilde{Y}$ is nonsingular (such a solution is called *dominant*) we have

$$\lim_{t\to\infty} X^{-1}(t)\widetilde{X}(t) = 0. \tag{2.2}$$

The recessive solution is determined uniquely up to a right multiple by a nonsingular $n \times n$ matrix. An equivalent characterization of the recessive solution of (H) is

$$\lim_{t\to\infty} \left(\int_a^t \widetilde{X}^{-1}(\tau)B(\tau)\widetilde{X}^{*-1}(\tau)\,\mathrm{d}\tau \right)^{-1} = 0, \tag{2.3}$$

see [4, Theorem 3.1]. In the proof of our main result (Theorem 3.1) we will see another construction (the so-called Reid's construction) of the recessive solution which uses a pointwise limit of certain auxiliary solutions of (H).

In this paper we essentially need the *existence* of recessive solution $\widetilde{Y} = (\widetilde{X}, \widetilde{U})$ with the property that

$$\widetilde{X}(t) \text{ is eventually nonsingular.} \tag{2.4}$$

Both properties, that is, the existence of $\widetilde{Y}$ and the eventual invertibility of $\widetilde{X}(t)$ are guaranteed for example by the requirement that the system (H) be nonoscillatory and eventually controllable. In this case the Sturmian separation theorem implies that every conjoined basis $Y = (X, U)$ of (H) has $X(t)$ eventually nonsingular. However, the two notions of nonoscillation and eventual controllability are not explicitly needed in this paper, so that we stay with the assumption on the existence of the recessive solution in our main result. In addition, in order to keep the values of a certain functional finite, we will need the assumption that the recessive solution $\widetilde{Y}$ satisfies

$$\lim_{t\to\infty} \widetilde{Y}(t) = 0. \tag{2.5}$$

R e m a r k 2.1. Assuming that the columns of $\widetilde{Y}$ belong to L^2, we conjecture that condition (2.5) is automatically satisfied, although we have not been able to find a reference for it or prove it.

If $\widetilde{Y}$ is the recessive solution of (H), then by definition its columns form the recessive system of (vector) solutions of (H), i.e., the functions $\tilde{y}_1 := \widetilde{Y}e_1, \ldots, \tilde{y}_n := \widetilde{Y}e_n$ form the recessive system of solutions, where e_j is the j-th unit vector.

With the Hamiltonian system (H) we consider the corresponding quadratic functional

$$\mathcal{F}(y) := \int_a^\infty \Omega(y, y)(t)\,\mathrm{d}t, \quad \Omega(y, \tilde{y})(t) := \{x^* C\tilde{x} + u^* B\tilde{u}\}(t), \tag{2.6}$$

where $y = (x, u)$ is *admissible*, i.e., x is locally absolutely continuous, u is locally integrable, $\operatorname{supp} y \subseteq (a, \infty)$, and y satisfies the first equation in system (H), the so-called equation of motion. The classical results [16, Chapter VII] of Reid characterize the positivity of the quadratic functional $\mathcal{F}$ for example by the nonoscillation of the Hamiltonian system (H) (although in [16] the results are formulated in terms of disconjugacy of system (H)).

Let $\mathrm{AC}_{\mathrm{loc}}$ be the set of all locally absolutely continuous functions $y\colon [a,\infty) \to \mathbb{C}^{2n}$. The motivation for the quadratic functional $\mathcal{F}$ is the following. For $y = (x,u) \in \mathrm{AC}_{\mathrm{loc}}$ (not necessarily admissible) we have

$$y^* l(y) = x^* C x + (x' - Ax)^* u - (x^* u)' + u^* (x' - Ax - Bu).$$

This formula is particularly simple when y is admissible. Therefore, the following result holds true.

Lemma 2.2. *For any admissible $y = (x,u) \in AC_{\mathrm{loc}} \cap L^2$ such that $l(y) \in L^2$ we have*

$$\mathcal{F}(y) = (x^* u)(t)\big|_a^\infty + \int_a^\infty \big\{x^*(Cx - A^* u - u')\big\}(t)\,\mathrm{d}t, \tag{2.7}$$

$$\mathcal{F}(y) = \langle y,\, l(y)\rangle + (x^* u)(t)\big|_a^\infty, \tag{2.8}$$

whenever (finite) $\lim\limits_{t\to\infty} (x^* u)(t)$ exists.

3. Friedrichs extension

Denote

$$\mathcal{D}_0 := \{y \in \mathrm{AC}_{\mathrm{loc}},\ \operatorname{supp} y \subseteq (a,\infty)\},$$
$$\mathcal{D}_{\max} := \{y \in L^2 \cap \mathrm{AC}_{\mathrm{loc}},\ l(y) \in L^2\}.$$

i.e., $\mathcal{D}_0$ is the set of absolutely continuous $\mathbb{C}^{2n}$-valued functions with compact support in (a,∞) and $\mathcal{D}_{\max}$ is the maximal set of functions allowed in $l(y)$ so that $l(y) \in L^2$. Then it is well known that $\mathcal{D}_0$ is dense in L^2 and that the expression $l(y)$ defines a symmetric operator on $\mathcal{D}_0$ (as a consequence of the Lagrange identity (3.3) below). The *maximal operator* $\mathcal{L}_{\max}$ generated by $l(y)$ is defined as $\mathcal{L}_{\max}\colon \mathcal{D}_{\max} \to L^2$, $\mathcal{L}_{\max}(y) := l(y)$, and then the *minimal operator* $\mathcal{L}_{\min}$ is the closure of the restriction of the maximal operator to the set $\mathcal{D}_0$. It follows that $y(a) = 0$ for any $y \in \mathcal{D}_{\min}$ and that $\mathcal{L}_{\min} = \mathcal{L}_{\max}^*$ (the adjoint operator in L^2), i.e.,

$$\langle l(y),\, z\rangle = \langle y,\, l(z)\rangle \quad \text{for all } y \in \mathcal{D}_{\min},\ z \in \mathcal{D}_{\max}. \tag{3.1}$$

Since the matrix $\mathcal{H}(t)$ is Hermitian and since $\mathcal{J}^* = -\mathcal{J} = \mathcal{J}^{-1}$, for any $y, z \in \mathcal{D}_{\max}$ we have

$$\begin{aligned} l^*(y) z - y^* l(z) &= (-\mathcal{J} y' - \mathcal{H} y)^* z - y^*(-\mathcal{J} z' - \mathcal{H} z) \\ &= (y^* \mathcal{J} z)' + y^*(\mathcal{H} - \mathcal{H}^*) z = (y^* \mathcal{J} z)', \end{aligned} \tag{3.2}$$

which is known as Green's formula. And since $y^*\mathcal{J}z = [y, z]$, integrating equation (3.2) over $[a, \infty)$ yields the Lagrange indentity

$$\langle l(y),\, z\rangle - \langle y,\, l(z)\rangle = [y, z](t)\big|_a^{\infty}. \tag{3.3}$$

Hence, for any $y, z \in \mathcal{D}_{\max}$ both the inner products on the left-hand side of (3.3) are finite, so that the limit

$$[y, z]_\infty = \big\langle l(y),\, z\big\rangle - \big\langle y,\, l(z)\big\rangle + [y, z](a) \tag{3.4}$$

exists and is finite. And since $[y, z](a) = 0$ for any $y \in \mathcal{D}_{\min}$ and $z \in \mathcal{D}_{\max}$ (use $y(a) = 0$ for $y \in \mathcal{D}_{\min}$), equations (3.4) and (3.1) imply that the domain of the minimal operator has the form

$$\mathcal{D}_{\min} = \{y \in \mathcal{D}_{\max},\ y(a) = 0,\ [y, z]_\infty = 0 \text{ for every } z \in \mathcal{D}_{\max}\},$$

see also [19, Lemma 7]. The idea of our main result (Theorem 3.1 below) is to enlarge the domain $\mathcal{D}_{\min}$ by a suitable selection of finitely many functions z from $\mathcal{D}_{\max}$ satisfying $[y, z]_\infty = 0$. These functions z turn out to be the recessive solutions of (H).

We assume that the minimal operator is bounded below (positive), i.e., there exists $\varepsilon > 0$ such that

$$\langle l(y),\, y\rangle \geqslant \varepsilon\langle y,\, y\rangle \quad \text{for all } y \in \mathcal{D}_{\min}, \tag{3.5}$$

where $\mathcal{D}_{\min}$ is the domain of the minimal operator. This assumption is not really restrictive, since the nonoscillation and eventual controllability of (H) imply that the operator $\mathcal{L}_{\min}$ is bounded below, say by a constant $\gamma \in \mathbb{R}$, that is, $\langle l(y),\, y\rangle \geqslant \gamma\langle y,\, y\rangle$ for every $y \in \mathcal{D}_{\min}$. This follows from the fact that the corresponding eigenvalue problem has a smallest (although possibly negative) eigenvalue. The proof of this fact can be found in [11, Theorem 7.6.2] or in [12, pg. 414]. The construction of the Friedrichs extension then applies to the operator $\mathcal{L}_{\min} - \gamma\mathcal{I}$, where $\mathcal{I}$ is the identity operator.

Recalling the definition of the Lagrange bracket in (1.1), we next present the main result of this paper. We refer to [12, Theorem 12] for the case of higher order Sturm-Liouville operators, and to a part of [15, Theorem 4.4] or [21, Theorem 10.5.3] for the second order operators.

Theorem 3.1. *Assume that* (3.5) *holds and that the Hamiltonian system* (H) *possesses the recessive system of solutions* $\widetilde{Y} = (\tilde{y}_1, \ldots, \tilde{y}_n)$ *satisfying conditions* (2.4) *and* (2.5). *Then the domain of the Friedrichs extension* $\mathcal{L}_F$ *of the minimal operator* $\mathcal{L}_{\min}$ *is given by*

$$\mathcal{D}_F = \bigl\{y = (x, u) \in \mathcal{D}_{\max},\ x(a) = 0,\ [y, \tilde{y}_j]_\infty = 0 \textit{ for all } j = 1, \ldots, n\bigr\}.$$

We will construct the domain $\mathcal{D}_F$ of the Friedrichs extension by using the result of [6], see also [21, Definition 10.5.1] and [13, Section 3]. In particular, for our setting we conclude that

$$\begin{aligned}(3.6)\qquad \mathcal{D}_F = \bigl\{&y \in \mathcal{D}_{\max},\ \exists y_s \in \mathcal{D}_{\min} \text{ with } y_s \to y \text{ in } L^2 \text{ as } s \to \infty \\ &\text{and } \langle l(y_s - y_r),\, y_s - y_r\rangle \to 0 \text{ as } s, r \to \infty\bigr\}.\end{aligned}$$

Then we have the inclusions $\mathcal{D}_0 \subseteq \mathcal{D}_{\min} \subseteq \mathcal{D}_F \subseteq \mathcal{D}_{\max}$.

Remark 3.2. Let $q = q_\pm := \dim \operatorname{Ker}(\mathcal{L}_{\min} - \mathrm{i}\mathcal{I})$ be the deficiency indices of the operator $\mathcal{L}_{\min}$. If $q = 0$, then the operator $\mathcal{L}_{\min}$ is self-adjoint and $\mathcal{D}_{\min} = \mathcal{D}_F$. If the operator $\mathcal{L}_{\min}$ is not self-adjoint, i.e., if $1 \leqslant q \leqslant n$, then a self-adjoint extension of $\mathcal{L}_{\min}$ is given as a restriction of the operator $\mathcal{L}_{\max}$ to the domain

$$\overline{\mathcal{D}} := \bigl\{y = (x, u) \in \mathcal{D}_{\max},\ x(a) = 0,\ [y, y_j]_\infty = 0 \text{ for all } j = 1, \ldots, q\bigr\}.$$

Here $y_1, \ldots, y_q \in \mathcal{D}_{\max}$ are such that $[y_i, y_j]_\infty = 0$ for all $i, j \in \{1, \ldots, q\}$ and such that $y_1, \ldots, y_q$ are linearly independent modulo $\mathcal{D}_{\min}$ (i.e., no nontrivial linear combination of $y_1, \ldots, y_q$ belongs to $\mathcal{D}_{\min}$). The set $\overline{\mathcal{D}}$ is called a *Lagrangian subspace* in [22].

We are now ready to prove the main result of this paper.

Proof of Theorem 3.1. Let $\widetilde{Y} = (\widetilde{X}, \widetilde{U})$ be the recessive solution of (H) satisfying conditions (2.4) and (2.5) and put $\tilde{y}_j = (\tilde{x}_j, \tilde{u}_j)$, where $\tilde{x}_j(t) = \widetilde{X}(t)e_j$ and $\tilde{u}_j(t) = \widetilde{U}(t)e_j$ on $[a, \infty)$ for every $j \in \{1, \ldots, n\}$. By (2.3), the symmetric matrix

$$\Lambda(t) := \int_a^t \widetilde{X}^{-1}(\tau)B(\tau)\widetilde{X}^{*-1}(\tau)\,\mathrm{d}\tau$$

satisfies $\lim\limits_{t\to\infty} \Lambda^{-1}(t) = 0$. Define the dominant solution $\widehat{Y} = (\widehat{X}, \widehat{U})$ of (H) by

$$(3.7)\qquad \widehat{X}(t) := \widetilde{X}(t)\Lambda(t), \quad \widehat{U}(t) := \widetilde{U}(t)\Lambda(t) + \widetilde{X}^{*-1}(t).$$

Then simple calculations, which use the formulas $\widetilde{X}^*\widetilde{U} = \widetilde{U}^*\widetilde{X}$ and $(\widetilde{X}^{-1})' = -\widetilde{X}^{-1}\widetilde{X}'\widetilde{X}^{-1}$, show that $(\widehat{X}, \widehat{U})$ is a solution of (H) and that the conjoined bases $(\widetilde{X}, \widetilde{U})$ and $(\widehat{X}, \widehat{U})$ are normalized. This yields that (2.1) holds true, so that the matrix $\widehat{X}^{-1}\widetilde{X}$ is Hermitian.

For a fixed $s \in [a, \infty)$ we denote $Y^{[s]}(t) := \widetilde{Y}(t) - \widehat{Y}(t)\widehat{X}^{-1}(s)\widetilde{X}(s)$, i.e.,

$$\begin{aligned} X^{[s]}(t) &:= \widetilde{X}(t) - \widehat{X}(t)\widehat{X}^{-1}(s)\widetilde{X}(s), \\ U^{[s]}(t) &:= \widetilde{U}(t) - \widehat{U}(t)\widehat{X}^{-1}(s)\widetilde{X}(s). \end{aligned} \tag{3.8}$$

Then, since it is a linear combination of two solutions, $Y^{[s]} = (X^{[s]}, U^{[s]})$ is a solution of (H) which satisfies $X^{[s]}(s) = 0$ and $U^{[s]}(s) = -\widehat{X}^{*-1}(s)$. Next we show that

$$\lim_{s\to\infty} \begin{pmatrix} X^{[s]}(t) \\ U^{[s]}(t) \end{pmatrix} = \begin{pmatrix} \widetilde{X}(t) \\ \widetilde{U}(t) \end{pmatrix} \quad \text{for every } t \in [a, \infty), \tag{3.9}$$

i.e., the functions $Y^{[s]}(\cdot)$ converge pointwise as $s \to \infty$ to the recessive solution $\widetilde{Y}(\cdot)$. Since $\Lambda^{-1}(s) \to 0$ as $s \to \infty$, the definition of $\widehat{X}(t)$ in (3.7) yields for $s \to \infty$

$$\begin{aligned} X^{[s]}(t) &= \widetilde{X}(t)[I - \Lambda(t)\Lambda^{-1}(s)] \to \widetilde{X}(t), \\ U^{[s]}(t) &= \widetilde{U}(t)[I - \Lambda(t)\Lambda^{-1}(s)] - \widetilde{X}^{*-1}(t)\Lambda^{-1}(s) \to \widetilde{U}(t). \end{aligned}$$

Next we define for $j \in \{1, \ldots, n\}$ functions $y_j^{[s]} = (x_j^{[s]}, u_j^{[s]})$ by

$$y_j^{[s]}(t) := \begin{cases} Y^{[s]}(t)e_j, & \text{for } t \in [a, s), \\ 0, & \text{for } t \in [s, \infty). \end{cases}$$

Then $x_j^{[s]}(s) = X^{[s]}(s)e_j = 0$, so that $y_j^{[s]}$ is admissible. Note that the values of $y_j^{[s]}(a)$ are irrelevant (as we shall also see at the end of the proofs of Claim 1 and Claim 5 below), so that without loss of generality we can take them to be zero. Otherwise, as in [12, pp. 415–418], we can modify the function $y_j^{[s]}$ by a suitable function with compact support to obtain the desired value $y_j^{[s]}(a) = 0$. Hence, we have $y_j^{[s]} \in \mathcal{D}_{\min}$.

Now we proceed by proving the following claims.

Claim 1. For $r > s$ we have the formula

$$\mathcal{F}(y_j^{[s]} - y_j^{[r]}) = e_j^*[(\widehat{X}^{-1}\widetilde{X})(s) - (\widehat{X}^{-1}\widetilde{X})(r)]e_j. \tag{3.10}$$

To prove this we let $r > s \geqslant a$ and write

$$\begin{aligned} \mathcal{F}(y_j^{[s]} - y_j^{[r]}) &= \mathcal{F}(y_j^{[s]}) + \mathcal{F}(y_j^{[r]}) \\ &\quad - \int_a^\infty \{\Omega(y_j^{[s]}, y_j^{[r]}) + \Omega(y_j^{[r]}, y_j^{[s]})\}(t)\,\mathrm{d}t, \end{aligned} \tag{3.11}$$

where $\Omega(\cdot,\cdot)$ is defined in (2.6). Now the integral in $\mathcal{F}(y_j^{[s]})$ is just over $[a,s]$, so that by applying formula (2.7) we have

$$\begin{aligned}\mathcal{F}(y_j^{[s]}) &= (x_j^{[s]})^* u_j^{[s]}\big|_a^s + \int_a^s \{(x_j^{[s]})^*[Cx_j^{[s]} - A^* u_j^{[s]} - (u_j^{[s]})']\}(t)\,\mathrm{d}t \\ &= -[x_j^{[s]}(a)]^* u_j^{[s]}(a).\end{aligned} \tag{3.12}$$

Similarly, we obtain

$$\mathcal{F}(y_j^{[r]}) = -[x_j^{[r]}(a)]^* u_j^{[r]}(a). \tag{3.13}$$

Next, using the admissibility of $y_j^{[r]}$, identity $C^* = C$, and the product rule for $[(u_j^{[s]})^* x_j^{[r]}]'$ we get

$$\begin{aligned}\Omega(y_j^{[s]}, y_j^{[r]}) &= (x_j^{[s]})^* C x_j^{[r]} + (u_j^{[s]})^*[(x_j^{[r]})' - Ax_j^{[r]}] \\ &= [Cx_j^{[s]} - A^* u_j^{[s]} - (u_j^{[s]})']^* x_j^{[r]} + [(u_j^{[s]})^* x_j^{[r]}]'.\end{aligned} \tag{3.14}$$

Let $\{t_k\}_{k=1}^\infty \subseteq (a,s)$ be a sequence of points with $t_k \nearrow s$ as $k \to \infty$. Since $y_j^{[s]}$ is a solution of (H) on $[a,s)$, hence on $[a,t_k]$, and since $y_j^{[s]}(t) \equiv 0$ on $[s,\infty)$, it follows that

$$\begin{aligned}\int_a^\infty \Omega(y_j^{[s]}, y_j^{[r]})(t)\,\mathrm{d}t &\overset{(3.14)}{=} \lim_{k\to\infty} (u_j^{[s]})^* x_j^{[r]}\big|_a^{t_k} \\ &= \lim_{k\to\infty} e_j^*[U^{[s]}(t_k)]^* X^{[r]}(t_k) e_j - [u_j^{[s]}(a)]^* x_j^{[r]}(a) \\ &= e_j^*[U^{[s]}(s)]^* X^{[r]}(s) e_j - [u_j^{[s]}(a)]^* x_j^{[r]}(a),\end{aligned} \tag{3.15}$$

where the last equality follows from the continuity of $U^{[s]}$ and $X^{[r]}$ at s. Similarly to (3.14), the admissibility of $y_j^{[s]}$, identity $C^* = C$, and the product rule for $[(u_j^{[r]})^* x_j^{[s]}]'$ yield

$$\begin{aligned}\Omega(y_j^{[r]}, y_j^{[s]}) &= (x_j^{[r]})^* C x_j^{[s]} + (u_j^{[r]})^*[(x_j^{[s]})' - Ax_j^{[s]}] \\ &= [Cx_j^{[r]} - A^* u_j^{[r]} - (u_j^{[r]})']^* x_j^{[s]} + [(u_j^{[r]})^* x_j^{[s]}]'.\end{aligned}$$

Since $y_j^{[r]}$ is a solution of (H) on $[a,r)$, hence on $[a,s]$ because $s < r$, and since $y_j^{[s]}(t) \equiv 0$ on $[s,\infty)$, it follows that

$$\begin{aligned}&\int_a^\infty \Omega(y_j^{[r]}, y_j^{[s]})(t)\,\mathrm{d}t \\ &\quad\overset{(2.7)}{=} \int_a^s \{(x_j^{[s]})^*[Cx_j^{[r]} - A^* u_j^{[r]} - (u_j^{[r]})']\}(t)\,\mathrm{d}t + (u_j^{[r]})^* x_j^{[s]}\big|_a^s \\ &\quad= -[u_j^{[r]}(a)]^* x_j^{[s]}(a).\end{aligned} \tag{3.16}$$

Hence, by using equations (3.12), (3.13), (3.15), (3.16) in formula (3.11) we get

$$\mathcal{F}(y_j^{[s]} - y_j^{[r]}) = -e_j^*[U^{[s]}(s)]^* X^{[r]}(s) e_j + \psi_j^{[s,r]}, \tag{3.17}$$

where

$$\psi_j^{[s,r]} := -[x_j^{[s]}(a)]^* u_j^{[s]}(a) - [x_j^{[r]}(a)]^* u_j^{[r]}(a) + [u_j^{[s]}(a)]^* x_j^{[r]}(a) + [u_j^{[r]}(a)]^* x_j^{[s]}(a).$$

The definition of $X^{[r]}(s)$ and formula $U^{[s]}(s) = -\widehat{X}^{*-1}(s)$ yield that

$$[U^{[s]}(s)]^* X^{[r]}(s) = -(\widehat{X}^{-1}\widetilde{X})(s) + (\widehat{X}^{-1}\widetilde{X})(r). \tag{3.18}$$

On the other hand, by using the definition of $y_j^{[s]}$ and $y_j^{[r]}$ through the normalized conjoined bases $(\widetilde{X}, \widetilde{U})$ and $(\widehat{X}, \widehat{U})$, their properties in (2.1), and $\widehat{X}(a) = 0$, we infer

$$\begin{aligned}
\psi_j^{[s,r]} &= e_j^* \big[-(X^{[s]})^* U^{[s]} - (X^{[r]})^* U^{[r]} + (U^{[s]})^* X^{[r]} + (U^{[r]})^* X^{[s]} \big](a) e_j \\
&\overset{(3.8)}{=} e_j^* \big\{ 2(\widetilde{U}^* \widetilde{X} - \widetilde{X}^* \widetilde{U})(a) \\
&\quad + (\widetilde{X}^* \widehat{U} - \widetilde{U}^* \widehat{X})(a)[(\widehat{X}^{-1}\widetilde{X})(s) + (\widehat{X}^{-1}\widetilde{X})(r)] \\
&\quad + [(\widehat{X}^{-1}\widetilde{X})(s) + (\widehat{X}^{-1}\widetilde{X})(r)](\widehat{X}^* \widetilde{U} - \widehat{U}^* \widetilde{X})(a) \\
&\quad - [(\widehat{X}^{-1}\widetilde{X})(s) - (\widehat{X}^{-1}\widetilde{X})(r)](\widehat{X}^* \widehat{U})(a) \\
&\quad \times [(\widehat{X}^{-1}\widetilde{X})(s) - (\widehat{X}^{-1}\widetilde{X})(r)] \big\} e_j \overset{(2.1)}{=} 0.
\end{aligned}$$

Therefore, upon inserting formula (3.18) into equation (3.17) we get equality (3.10), which we wanted to prove. Note that the above calculations leading to formula (3.10) are independent of the values $y_j^{[s]}(a)$ and $y_j^{[r]}(a)$, because these values cancel out in $\psi_j^{[s,r]} = 0$.

Claim 2. We have $\mathcal{F}(y_j^{[s]} - y_j^{[r]}) \to 0$ as $s, r \to \infty$, $s < r$. This follows immediately from identity (3.10) in Claim 1 and from the definition of the recessive solution $(\widetilde{X}, \widetilde{U})$ in (2.2), which yields $(\widehat{X}^{-1}\widetilde{X})(\tau) \to 0$ as $\tau \to \infty$ for $\tau \in \{s, r\}$.

Claim 3. For $s < r$ we have the formula

$$\big\langle l(y_j^{[s]} - y_j^{[r]}),\, y_j^{[s]} - y_j^{[r]} \big\rangle = \mathcal{F}(y_j^{[s]} - y_j^{[r]}) \to 0 \quad \text{as } s, r \to \infty,\ s < r. \tag{3.19}$$

To show this we put $y := y_j^{[s]} - y_j^{[r]}$. Then $y = (x, u) \in \mathrm{AC_{loc}}$ is admissible, $y(t) \equiv 0$ for $t \geqslant r$ $(> s)$ (so that $y \in L^2$), and

$$\begin{aligned}
x^*(a)u(a) = e_j^*[(\widehat{X}^{-1}\widetilde{X})(s) - (\widehat{X}^{-1}\widetilde{X})(r)](\widehat{X}^*\widehat{U})(a) \\
\times [(\widehat{X}^{-1}\widetilde{X})(s) - (\widehat{X}^{-1}\widetilde{X})(r)] e_j = 0,
\end{aligned}$$

because $\widehat{X}(a) = 0$. Hence, the result in (3.19) follows directly from identity (2.8) in Lemma 2.2 and from Claim 2.

Claim 4. For any sequence of points $s_k \to \infty$ as $k \to \infty$ the sequence $\{y_j^{[s_k]}\}_{k=0}^{\infty}$ converges in L^2 to the j-th recessive solution $\tilde{y}_j$, which therefore belongs to L^2 (and hence $\tilde{y}_j \in \mathcal{D}_{\max}$). In order to prove this, we take an arbitrary sequence $s_k \to \infty$ as $k \to \infty$ and pick $r_k > s_k$ for every index k. By our assumption (3.5) and Claim 3 we have $\langle y_j^{[s_k]} - y_j^{[r_k]}, y_j^{[s_k]} - y_j^{[r_k]} \rangle \to 0$ as $k \to \infty$, so that $\|y_j^{[s_k]} - y_j^{[r_k]}\|_{L^2} \to 0$ as $k \to \infty$. It follows that $\{y_j^{[s_k]}\}_{k=0}^{\infty}$ is a Cauchy sequence in the Hilbert space L^2, and therefore it converges in L^2 to an element $y \in L^2$. However, since by (3.9), for each $t \in [a, \infty)$ we have $\lim\limits_{k\to\infty} y_j^{[s_k]}(t) = \tilde{y}_j(t)$, it follows that $y = \tilde{y}_j$. Hence, $y_j^{[s_k]} \to \tilde{y}_j$ in L^2 as $k \to \infty$ and $\tilde{y}_j \in L^2$. Moreover, since $l(\tilde{y}_j) = 0$ ($\tilde{y}_j$ being a solution of (H)), we have $\tilde{y}_j \in \mathcal{D}_{\max}$.

Claim 5. We have the formula

$$\mathcal{F}(\tilde{y}_j - y_j^{[s]}) = e_j^*(\widehat{X}^{-1}\widetilde{X})(s)e_j. \tag{3.20}$$

First we write

$$\mathcal{F}(\tilde{y}_j - y_j^{[s]}) = \mathcal{F}(\tilde{y}_j) + \mathcal{F}(y_j^{[s]}) - \int_a^{\infty} \big\{\Omega(\tilde{y}_j, y_j^{[s]}) + \Omega(y_j^{[s]}, \tilde{y}_j)\big\}(t)\,\mathrm{d}t. \tag{3.21}$$

Now by formula (2.7) in Lemma 2.2 and assumption (2.5) we have

$$\mathcal{F}(\tilde{y}_j) = -[\tilde{x}_j(a)]^*\tilde{u}_j(a), \tag{3.22}$$

while the value of $\mathcal{F}(y_j^{[s]})$ has been calculated in (3.12). In addition, since $\tilde{y}_j$ is a solution of (H) and $y_j^{[s]}$ is admissible, hence

$$\Omega(\tilde{y}_j, y_j^{[s]}) = (\tilde{u}_j' + A^*\tilde{u}_j)^* x_j^{[s]} + \tilde{u}_j^*[(x_j^{[s]})' - Ax_j^{[s]}] = \big(\tilde{u}_j^* x_j^{[s]}\big)'.$$

Consequently, by using $y_j^{[s]}(t) \equiv 0$ on $[s, \infty)$, we get

$$\int_a^{\infty} \Omega(\tilde{y}_j, y_j^{[s]})(t)\,\mathrm{d}t = -[\tilde{u}_j(a)]^* x_j^{[s]}(a). \tag{3.23}$$

Similarly, the admissibility of $\tilde{y}_j$ yields

$$\Omega(y_j^{[s]}, \tilde{y}_j) = (x_j^{[s]})^* C\tilde{x}_j + (u_j^{[s]})^*(\tilde{x}_j' - A\tilde{x}_j) = (Cx_j^{[s]} - A^*u_j^{[s]})^*\tilde{x}_j + (u_j^{[s]})^*\tilde{x}_j'.$$

The function $y_j^{[s]}$ is a solution of (H) on $[a, s)$, so that on this interval we have

$$\Omega(y_j^{[s]}, \tilde{y}_j) = [(u_j^{[s]})']^*\tilde{x}_j + (u_j^{[s]})^*\tilde{x}_j' = \big[(u_j^{[s]})^*\tilde{x}_j\big]'.$$

Take any sequence $\{t_k\}_{k=1}^{\infty} \subseteq (a,s)$ with $t_k \nearrow s$ as $k \to \infty$. Then

$$(3.24)\qquad \begin{aligned}\int_a^{\infty} \Omega(y_j^{[s]}, \tilde{y}_j)(t)\,\mathrm{d}t &= \lim_{k\to\infty} (u_j^{[s]})^* \tilde{x}_j\big|_a^{t_k} \\ &= \lim_{k\to\infty} e_j^* [U^{[s]}(t_k)]^* \widetilde{X}(t_k) e_j - [u_j^{[s]}(a)]^* \tilde{x}_j(a) \\ &= e_j^* [U^{[s]}(s)]^* \widetilde{X}(s) e_j - [u_j^{[s]}(a)]^* \tilde{x}_j(a),\end{aligned}$$

where we have used the continuity of $U^{[s]}$ and $\widetilde{X}$ at s in the last equality. Upon inserting formulas (3.22), (3.12), (3.23), and (3.24) into equation (3.21) and using the identity $U^{[s]}(s) = -\widehat{X}^{*-1}(s)$ we get

$$\begin{aligned}\mathcal{F}(\tilde{y}_j - y_j^{[s]}) &= e_j^* (\widehat{X}^{-1} \widetilde{X})(s) e_j - \varphi_j^{[s]}, \\ \varphi_j^{[s]} &:= [\tilde{x}_j(a) - x_j^{[s]}(a)]^* [\tilde{u}_j(a) - u_j^{[s]}(a)].\end{aligned}$$

Now the definition of $y_j^{[s]}$ in terms of the solutions $(\widetilde{X}, \widetilde{U})$ and $(\widehat{X}, \widehat{U})$ yields

$$\varphi_j^{[s]} = e_j^* (\widehat{X}^{-1} \widetilde{X})(s) (\widehat{X}^* \widehat{U})(a) (\widehat{X}^{-1} \widetilde{X})(s) e_j = 0,$$

because $\widehat{X}(a) = 0$. Therefore, formula (3.20) is established. Again note that this formula is independent of the values $\tilde{y}_j(a)$ and $y_j^{[s]}(a)$.

Claim 6. We have the formula

$$(3.25)\qquad \langle l(\tilde{y}_j - y_j^{[s]}),\, \tilde{y}_j - y_j^{[s]} \rangle = \mathcal{F}(\tilde{y}_j - y_j^{[s]}) \to 0 \quad \text{as } s \to \infty.$$

To show this we put $y := \tilde{y}_j - y_j^{[s]}$. Then $y = (x,u) \in \mathrm{AC}_{\mathrm{loc}}$ is admissible, $y \in L^2$ by Claim 4, $\lim_{t\to\infty} y(t) = 0$ by our assumption (2.5), and $x^*(a)u(a) = \varphi_j^{[s]} = 0$. Hence, formula (3.25) follows from identity (2.8) in Lemma 2.2 and from the property of the recessive solution (2.2).

Claim 7. We have $\tilde{y}_j \in \mathcal{D}_F$. Let $\{s_k\}_{k=1}^{\infty}$ be a sequence of points converging to ∞. In Claim 4 we proved that $\tilde{y}_j \in \mathcal{D}_{\max}$ and $y_j^{[s_k]} \to \tilde{y}_j$ in L^2 as $k \to \infty$, while in Claim 6 we showed that $\langle l(\tilde{y}_j - y_j^{[s_k]}),\, \tilde{y}_j - y_j^{[s_k]} \rangle \to 0$ as $k \to \infty$. Hence, by the characterization of $\mathcal{D}_F$ in (3.6), we obtain $\tilde{y}_j \in \mathcal{D}_F$.

Claim 8. Finally we prove that the set

$$\widetilde{\mathcal{D}} := \{ y = (x,u) \in \mathcal{D}_{\max},\ x(a) = 0,\ [y, \tilde{y}_j]_\infty = 0 \text{ for all } j = 1, \dots, n \}$$

is equal to the domain $\mathcal{D}_F$ characterized in (3.6). Since the recessive solution $\widetilde{Y} = (\widetilde{X}, \widetilde{U})$ is a conjoined basis, we have $\widetilde{Y}^*(t) \mathcal{J} \widetilde{Y}(t) \equiv 0$ on $[a, \infty)$. Hence, $[\tilde{y}_i, \tilde{y}_j](t) =$

$\tilde{y}_i^*(t)\mathcal{J}\tilde{y}_j(t) \equiv 0$ on $[a,\infty)$ for every $i,j \in \{1,\dots,n\}$. Therefore, the set $\widetilde{\mathcal{D}}$ is a domain of a self-adjoint realization of $\mathcal{L}_{\min}$, see e.g. [19, Theorem 1]. That is, $\widetilde{\mathcal{D}} \subseteq \mathcal{D}_F$. Conversely, let $y \in \mathcal{D}_F$. Since we have already proved in Claim 7 that $\tilde{y}_j \in \mathcal{D}_F$, it follows that $[y,\tilde{y}_j]_\infty = 0$ for every $j \in \{1,\dots,n\}$. Hence, $y \in \widetilde{\mathcal{D}}$ and so $\mathcal{D}_F \subseteq \widetilde{\mathcal{D}}$. Therefore, $\widetilde{\mathcal{D}} = \mathcal{D}_F$ and the proof of Theorem 3.1 is complete. □

Remark 3.3. The theory of Friedrichs extension of linear operators is not devoted solely to the continuous time, i.e., to differential operators. For example, in [3] and [5] the Friedrichs extension is constructed for second order and higher order Sturm-Liouville difference operators. Extending these results to linear Hamiltonian difference systems is a subject of our present research.

Acknowledgements. The authors wish to thank Professor Ondřej Došlý for his time and consultations regarding the subject of this paper.

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