GENERALIZED DEDUCTIVE SYSTEMS IN SUBREGULAR VARIETIES

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Abstract. An algebra $\mathscr{A} = (A, F)$ is subregular alias regular with respect to a unary term function g if for each $\Theta, \Phi \in \operatorname{Con} \mathscr{A}$ we have $\Theta = \Phi$ whenever $[g(a)]_\Theta = [g(a)]_\Phi$ for each $a \in A$. We borrow the concept of a deductive system from logic to modify it for subregular algebras. Using it we show that a subset $C \subseteq A$ is a class of some congruence on Θ containing $g(a)$ if and only if C is this generalized deductive system. This method is efficient (needs a finite number of steps).

Keywords: regular variety, subregular variety, deductive system, congruence class, difference system

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Let $\mathscr{A} = (A, F)$ be an algebra and $\emptyset \neq C \subseteq A$ a subset. The problem to decide whether C is a class of some congruence $\Theta \in \operatorname{Con} \mathscr{A}$ has been a problem of long standing. In general, it was solved by A. I. Mal'cev in 1954. However, his method is far from being effective. Essential progress was done for certain subsets of A for algebras having a constant 0. A. Ursini introduced a concept of an ideal in universal algebra [8] and it was shown by him and H.-P. Gumm [7] that in varieties permutable at 0 every 0-class of each congruence on $\mathscr{A}$ is just an ideal of $\mathscr{A}$ and vice versa. It turns out that for varieties which are permutable at 0 and weakly regular this method is effective, i.e. for a finite algebra of a finite type it can be decided by a finite number of steps of the corresponding algorithmical scheme. This method was extended for an arbitrary congruence class of algebra $\mathscr{A}$ of a regular and permutable variety and it was generalized by the author and R. Bělohlávek [2] to algebras in regular varieties. Recently, we have used another method, the so called deductive systems,

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to characterize 0-classes in weakly regular varieties (see [5]) or arbitrary congruence classes in algebras of regular varieties, see [3].

If the concept of regularity is weakened to the so called subregularity (see e.g. [1]), one can still use an effective method to characterize certain congruence classes. This is the aim of our paper.

Let us recall that an algebra $\mathscr{A} = (A, F)$ is *regular* if every $\Theta, \Phi \in \operatorname{Con} \mathscr{A}$ coincide whenever they have a class in common. An algebra $\mathscr{A}$ with a constant 0 is *weakly regular* if every $\Theta, \Phi \in \operatorname{Con} \mathscr{A}$ coincide whenever $[0]_\Theta = [0]_\Phi$.

These concepts have a common generalization.

Definition 1. Let g be a unary term function of an algebra $\mathscr{A} = (A, F)$. $\mathscr{A}$ is *regular with respect to* g if $\Theta = \Phi$ for $\Theta, \Phi \in \operatorname{Con} \mathscr{A}$ whenever $[g(a)]_\Theta = [g(a)]_\Phi$ for each $a \in A$. Let g be a unary term of variety $\mathscr{V}$. We say that $\mathscr{V}$ is *regular with respect to* g if each $\mathscr{A} \in \mathscr{V}$ has this property (with respect to the corresponding term function g^A).

Regularity with respect to g is known also under the name subregularity, see [1], provided the term g is implicitly given.

Let us mention that if $g(x) = x$ (the identity term) then regularity with respect to g is the regularity; if 0 is a constant of $\mathscr{A}$ and $g(x) = 0$ then regularity with respect to g is just the weak regularity.

Definition 2. Let g be a unary term of a variety $\mathscr{V}$. A finite set $\{p_1, \dots, p_n\}$ of ternary terms $p_1, \dots p_n$ of $\mathscr{V}$ is called a *g-difference system for $\mathscr{V}$ if*

$$[\,p_1(x, y, z) = g(z) \;\&\ldots\&\; p_n(x, y, z) = g(z)] \textit{ if and only if } x = y.$$

Example. If $g(z) = 0$ where 0 is a constant of $\mathscr{V}$ then the g-difference system is just the *Gödel equivalence system* as introduced in [4] (of course, then every $p_i(x, y, z)$ is independent of the last variable thus it is properly binary). If $g(z) = z$ then we have the difference system as introduced in [3].

If $g(z) = z$ and $\mathscr{V}$ is a variety of groups then for $p(x, y, z) = x - y + z$ the singleton $\{p\}$ is a g-difference system; if $\mathscr{V}$ is the variety of Boolean algebras then $\{p\}$ is a g-difference system for $p(x, y, z) = x \oplus y \oplus z$, where $\oplus$ denotes the so called symmetrical difference.

Analogously, if $\mathscr{V}$ is the variety of pseudocomplemented semilattices and $g(x) = x^{**}$ then $\{p\}$ is a g-difference system for $p(x, y, z) = (x + y) + z$ where

$$x + y = ((x \wedge y^*)^* \wedge (x^* \wedge y)^*)^*.$$

An example of a difference system having more than one term was found for MV-algebras in [3].

The following useful result was proved in [1]:

Proposition 1. *Let g be a unary term of a variety $\mathscr{V}$. Then $\mathscr{V}$ is regular with respect to g if and only if there exist ternary terms $p_1, \ldots, p_m$ such that*

$$\{p_1, \ldots, p_m\} \text{ is a } g\text{-difference system of } \mathscr{V}.$$

Moreover, every variety $\mathscr{V}$ which is regular with respect to g is n-permutable for some $n \geqslant 2$.

Let us note that m and n in Proposition 1 need not coincide. E.g. for groups we have $n = 2$ and $m = 1$.

In the sequel we will use the following result which is considered to be a folklore but its formal proof can be found in [6]:

Proposition 2. *A variety $\mathscr{V}$ is n-permutable for some $n \geqslant 2$ if and only if for each $\mathscr{A} \in \mathscr{V}$ and every binary relation R on $\mathscr{A}$ the following implication holds: if R is reflexive, transitive and compatible then $R \in \operatorname{Con} \mathscr{A}$.*

Recall that a relation R on an algebra $\mathscr{A} = (A, F)$ is *compatible* (with respect to F) if for each n-ary $f \in F$ and $a_1, \ldots, a_n, b_1, \ldots b_n \in A$,

$$\langle a_i, b_i \rangle \in R \ (i = 1, \ldots, n) \Rightarrow \langle f(a_1, \ldots, a_n), f(b_1, \ldots, b_n) \rangle \in R;$$

in other words, R is compatible if it is a subalgebra of the square $\mathscr{A} \times \mathscr{A}$.

The crucial concept of our paper is the following one:

Definition 3. Let g be a unary term function of an algebra $\mathscr{A} = (A, F)$ and let $t_1, \ldots, t_n$ be ternary term functions of $\mathscr{A}$, $z \in A$. A subset $D \subseteq A$ is called a *(g, z)-deductive system of $\mathscr{A}$ with respect to* $\{t_1, \ldots, t_n\}$ *if*

(i) $g(z) \in D$,

(ii) $a \in D$ *and* $t_i(a, b, z) \in D$ *for* $i = 1, \ldots, n$ *imply* $b \in D$,

(iii) $a \in D$ *implies* $t_i(g(z), a, z) \in D$ *for* $i = 1, \ldots, n$.

Let us note that (i) and (ii) imply the converse of (iii), thus

$$a \in D \Leftrightarrow t_i(g(z), a, z) \in D \text{ for } i = 1, \ldots, n.$$

E x a m p l e. Let "$\Rightarrow$" be the connective implication of an arbitrary (e.g. classical, non-classical, intuitionistic, multiple-valued, etc.) logic and D the subset of "tautologies". Then for $g(z) = 1$ (the tautology) and $n = 1$, $t_1(x, y, z) := x \Rightarrow y$ we surely have

$1 \in D$,
$a \in D$ and $(a \Rightarrow b) \in D$ implies $b \in D$,
$a \in D$ implies $(1 \Rightarrow a) \in D$.

Let R be a binary relation on a set A and $x \in A$. Denote $[x]_R = \{a \in A; \langle a, x\rangle \in R\}$.

Definition 4. Let $t_1, \ldots, t_n$ be ternary term functions of an algebra $\mathscr{A} = (A, F)$ and $D \subseteq A$, $z \in A$. Define a binary relation $\Theta_{D,z}$ on A *induced by* $\{t_1, \ldots, t_n\}$ as follows:

$$(*) \qquad \langle a, b\rangle \in \Theta_{D,z} \text{ if and only if } t_i(b, a, z) \in D \text{ for } i = 1, \ldots, n.$$

We are ready to characterize the classes $[g(z)]_{\Theta_{D,z}}$ of $\Theta_{D,z}$.

Lemma 1. *Let $t_1, \ldots, t_n$ be ternary term functions of an algebra $\mathscr{A} = (A, F)$, let g be a unary term function of $\mathscr{A}$ and $z \in A$. If D is a (g, z)-deductive system of $\mathscr{A}$ with respect to $\{t_1, \ldots, t_n\}$ and $\Theta_{D,z}$ is induced by $\{t_1, \ldots, t_n\}$ then $D = [g(z)]_{\Theta_{D,z}}$.*

Proof. Let $a \in D$. By (iii) we have $t_i(g(z), a, z) \in D$ for $i = 1, \ldots, n$ and, by $(*)$, $\langle a, g(z)\rangle \in \Theta_{D,z}$ which yields $a \in [g(z)]_{\Theta_{D,z}}$. Conversely, if $a \in [g(z)]_{\Theta_{D,z}}$ then $\langle a, g(z)\rangle \in \Theta_{D,z}$, thus $t_i(g(z), a, z) \in D$ for $i = 1, \ldots, n$. Applying (i) we infer $g(z) \in D$ and, by virtue of (ii), also $a \in D$. Together, $D = [g(z)]_{\Theta_{D,z}}$. □

Lemma 2. *Let $t_1, \ldots, t_n$ be ternary term functions of an algebra $\mathscr{A} = (A, F)$, let g be a unary term function of $\mathscr{A}$ and $z \in A$, $D \subseteq A$. Let $\Theta_{D,z}$ be induced by $\{t_1, \ldots, t_n\}$. If $\Theta_{D,z}$ is reflexive and transitive and $D = [g(z)]_{\Theta_{D,z}}$ then D is a (g, z)-deductive system of $\mathscr{A}$ with respect to $\{t_1, \ldots, t_n\}$.*

Proof. Suppose $a \in D$ and $t_i(a, b, z) \in D$ for $i = 1, \ldots, n$. Then $\langle b, a\rangle \in \Theta_{D,z}$. Since $D = [g(z)]_{\Theta_{D,z}}$, also $\langle a, g(z)\rangle \in \Theta_{D,z}$. Due to transitivity of $\Theta_{D,z}$, we have $b \in [g(z)]_{\Theta_{D,z}}$, i.e. D satisfies (ii) of Definition 3. The condition (i) follows by reflexivity of $\Theta_{D,z}$ (since $g(z) \in [g(z)]_{\Theta_{D,z}} = D$).

If $a \in D$ then $\langle a, g(z)\rangle \in \Theta_{D,z}$, thus $t_i(g(z), a, z) \in D$ for $i = 1, \ldots, n$, i.e. D satisfies also (iii) and hence it is a (g, z)-deductive system of $\mathscr{A}$ w.r.t. $\{t_1, \ldots, t_n\}$. □

Since congruences are compatible relations on an algebra $\mathscr{A} = (A, F)$, we must respect also the substitution property (with respect to F) to describe their classes. Hence, we define:

Definition 5. Let g be a unary and $p_1, \ldots, p_n$ n-ary term functions of an algebra $\mathscr{A} = (A, F)$. We say that $D \subseteq A$ is a *compatible (g, z)-deductive system of $\mathscr{A}$*

with respect to $\{p_1, \ldots, p_n\}$ if D is a (g, z)-deductive system of $\mathscr{A}$ with respect to $\{p_1, \ldots, p_n\}$ and for each k-ary operation $f \in F$ and every $a_1, \ldots, a_k, b_1, \ldots, b_k \in A$ the following implication holds:

$$\begin{aligned}&\text{if } p_i(a_1, b_1, z) \in D, \ldots, p_i(a_k, b_k, z) \in D \text{ for } i = 1, \ldots, n\\&\text{then } p_i(f(a_1, \ldots, a_k), f(b_1, \ldots, b_k), z) \in D \text{ for } i = 1, \ldots, n.\end{aligned}$$

Theorem 1. *Let g be a unary term of a variety $\mathscr{V}$ and $\{p_1, \ldots, p_n\}$ a g-difference system for $\mathscr{V}$. Let $\mathscr{A} = (A, F) \in \mathscr{V}$, $\Theta \in \operatorname{Con} \mathscr{A}$, $z \in A$ and $D = [g(z)]_\Theta$. Then*
(a) $\Theta_{D,z} = \Theta$;
(b) *D is a compatible (g, z)-deductive system of $\mathscr{A}$ with respect to $\{p_1, \ldots, p_n\}$.*

P r o o f. If $\langle a, b\rangle \in \Theta_{D,z}$ then $p_i(b, a, z) \in D = [g(z)]_\Theta$ for $i = 1, \ldots, n$ and hence $\langle p_i(b, a, z), g(z)\rangle \in \Theta$. Applying Proposition 1, we infer $\langle b, a\rangle \in \Theta$, thus also $\langle a, b\rangle \in \Theta$ proving $\Theta_{D,z} \subseteq \Theta$.

Conversely, if $\langle a, b\rangle \in \Theta$ then $\langle b, a\rangle \in \Theta$ and, by Proposition 1 again, $\langle p_i(b, a, z), g(z)\rangle \in \Theta$ for $i = 1, \ldots, n$, thus $p_i(b, a, z) \in [g(z)]_\Theta = D$. By $(*)$ of Definition 4 we conclude $\langle a, b\rangle \in \Theta_{D,z}$ giving $\Theta \subseteq \Theta_{D,z}$. We have shown $\Theta = \Theta_{D,z}$.

By Lemma 2, D is a (g, z)-deductive system of $\mathscr{A}$ with respect to $\{p_1, \ldots, p_n\}$. Since $\Theta \in \operatorname{Con} \mathscr{A}$ is compatible, it is an easy exercise to show that also D is compatible. $\square$

Theorem 2. *Let g be a unary term of a variety $\mathscr{V}$ and $\{p_1, \ldots, p_n\}$ a g-difference system for $\mathscr{V}$. Let $\mathscr{A} = (A, F) \in \mathscr{V}$, $z \in A$ and let D be a compatible (g, z)-deductive system of $\mathscr{A}$ with respect to $\{p_1, \ldots, p_n\}$. Then the relation $\Theta_{D,z}$ induced by $\{p_1, \ldots, p_n\}$ is a congruence on $\mathscr{A}$ and $D = [g(z)]_{\Theta_{D,z}}$.*

P r o o f. By Proposition 1, $\mathscr{V}$ satisfies $p_i(x, x, z) = g(z)$ for $i = 1, \ldots, n$ and hence the relation $\Theta_{D,z}$ induced by $\{p_1, \ldots, p_n\}$ is reflexive. Since the (g, z)-deductive system D is compatible, also $\Theta_{D,z}$ is compatible. Prove transitivity of $\Theta_{D,z}$: let $\langle a, b\rangle \in \Theta_{D,z}$ and $\langle b, c\rangle \in \Theta_{D,z}$. Then $p_i(c, b, z) \in D$ for $i = 1, \ldots, n$ and, by virtue of compatibility of $\Theta_{D,z}$,

$$\langle a, b\rangle \in \Theta_{D,z} \Rightarrow \langle p_i(c, a, z), p_i(c, b, z)\rangle \in \Theta_{D,z}$$

whence $p_j(p_i(c, b, z), p_i(c, a, z), z) \in D$ for $j = 1, \ldots, n$. However, D is a (g, z)-deductive system of $\mathscr{A}$ with respect to $\{p_1, \ldots, p_n\}$, thus, by (ii) of Definition 3, we conclude $p_i(c, a, z) \in D$ for $i = 1, \ldots, n$. Hence $\langle a, c\rangle \in \Theta_{D,z}$.

By Proposition 1, $\mathscr{V}$ is m-permutable for some $m \geqslant 2$ and, by Proposition 2, $\Theta_{D,z}$ is also symmetrical. Together, we have $\Theta_{D,z} \in \operatorname{Con} \mathscr{A}$. By Lemma 1 we conclude $D = [g(z)]_{\Theta_{D,z}}$. $\square$

Corollary 1. *Let $\mathscr{V}$ be a variety which is regular with respect to g. Then $\mathscr{V}$ has a g-difference system $\{p_1, \dots, p_n\}$ and for each $\mathscr{A} = (A, F) \in \mathscr{V}$, $z \in A$ and $D \subseteq A$, D is a congruence class containing $g(z)$ if and only if D is a (g, z)-deductive system of $\mathscr{A}$ with respect to $\{p_1, \dots, p_n\}$.*

Although the involved method of (g, z)-deductive systems enables us to characterize only the congruence classes containing $g(a)$ for some $a \in A$ and for $\mathscr{A} = (A, F)$ from a variety which is regular with respect to g, this method is effective in the following sense: if $\mathscr{A}$ is finite and of a finite type, we need to verify only a finite number of conditions of Definition 3 and Definition 5. Thus there exists an algorithmical scheme deciding whether a subset $C \subseteq A$ is a congruence class of $\mathscr{A}$ in a finite number of steps. This scheme depends on the computability of functions $p_1, \dots, p_n$. Applying the same reasoning and a computation as in [3], we obtain:

Corollary 2. *Let $\mathscr{V}$ be a variety regular with respect to g and of a finite type with k fundamental operation symbols. Let $\sigma(f_i)$ be the arity of the i-th operation symbol f_i. Let $\{p_1, \dots, p_n\}$ be its g-difference system. If $\mathscr{A} = (A, F) \in \mathscr{V}$ is finite and $C \subseteq A$, $a \in A$, $g(a) \in C$ and $|A| = m$, $|C| = r$ then there exists an algorithmical scheme for deciding whether C is a congruence class and this scheme needs*

$$n \sum_{i=1}^{k} m^{2\sigma_i(f_i)} + k \cdot m^2 \cdot n + r \cdot (m \cdot n + m + n)$$

steps.

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