PARTITION SENSITIVITY FOR MEASURABLE MAPS

C. A. Morales, Rio de Janeiro

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Abstract. We study countable partitions for measurable maps on measure spaces such that, for every point x, the set of points with the same itinerary as that of x is negligible. We prove in nonatomic probability spaces that every strong generator (Parry, W., *Aperiodic transformations and generators*, J. London Math. Soc. 43 (1968), 191–194) satisfies this property (but not conversely). In addition, measurable maps carrying partitions with this property are aperiodic and their corresponding spaces are nonatomic. From this we obtain a characterization of nonsingular countable-to-one mappings with these partitions on nonatomic Lebesgue probability spaces as those having strong generators. Furthermore, maps carrying these partitions include ergodic measure-preserving ones with positive entropy on probability spaces (thus extending the result in Cadre, B., Jacob, P., *On pairwise sensitivity*, J. Math. Anal. Appl. 309 (2005), 375–382). Some applications are given.

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1. Introduction

In this paper we study countable partitions P for measurable maps $f\colon X \to X$ on measure spaces $(X, \mathcal{B}, \mu)$ such that, for every $x \in X$, *the set of points with the same itinerary as that of x is negligible.* In other words,

$$\mu(\{y \in X\colon f^n(y) \in P(f^n(x)), \ \forall n \in \mathbb{N}\}) = 0, \quad \forall x \in X, \tag{1.1}$$

where $P(x)$ stands for the element of P containing $x \in X$. For simplicity, we call these partitions *measure-sensitive partitions.*

We prove in nonatomic probability spaces that every *strong generator* is a measure-sensitive partition but not conversely (results about strong generators can be found in [4], [7], [9], [12], [13] and [14]). We also show examples of measurable maps in nonatomic probability spaces carrying measure-sensitive partitions which are not strong generators. Motivated by these examples we shall study measurable maps on measure spaces carrying measure-sensitive partitions (called *measure-expansive maps* for short). Indeed, we prove that every measure-expansive map is aperiodic and also, in the probabilistic case, that its corresponding space is nonatomic. From this we obtain a characterization of nonsingular countable-to-one measure-expansive mappings on nonatomic Lebesgue probability spaces as those having strong generators. Furthermore, we prove that every ergodic measure-preserving map with positive entropy in a probability space is measure-expansive (thus extending a result in [3]). As an application we obtain some properties for ergodic measure-preserving maps with positive entropy (c.f. corollaries 2.1 and 2.4).

2. Statements and proofs

Hereafter, the term *countable* will mean either finite or countably infinite.

A *measure space* is a triple $(X, \mathcal{B}, \mu)$ where X is a set, $\mathcal{B}$ is a σ-algebra of subsets of X and μ is a positive measure in $\mathcal{B}$. A *probability space* is a measure space for which $\mu(X) = 1$. A measure space is *nonatomic* if it has no *atoms*, i.e., measurable sets A of positive measure satisfying $\mu(B) \in \{0, \mu(A)\}$ for every measurable set $B \subset A$. A *partition* is a disjoint collection P of nonempty measurable sets whose union is X. We allow $\mu(\xi) = 0$ for some $\xi \in P$. For $f\colon X \to X$ measurable and $k \in \mathbb{N}$ we define $f^{-k}(P) = \{f^{-k}(\xi)\colon \xi \in P\}$, which is a (countable) partition if P is. A *strong generator* of f is a countable partition P for which the smallest σ-algebra of $\mathcal{B}$ containing $\bigcup\limits_{k\in\mathbb{N}} f^{-k}(P)$ equals $\mathcal{B}$ (mod 0) (see [12]).

The result below is the central motivation of this paper.

Theorem 2.1. *Every strong generator of a measurable map in a nonatomic probability space is a measure-sensitive partition.*

Proof. Recall that the join of finitely many partitions $P_0, \ldots, P_n$ is the partition defined by

$$\bigvee_{k=0}^{n} P_k = \left\{ \bigcap_{k=0}^{n} \xi_k\colon\ \xi_k \in P_k, \forall 0 \leqslant k \leqslant n \right\}.$$

Given partitions P and Q, we write $P \leqslant Q$ if each member of Q is contained in some member of P (mod 0). A sequence of partitions $\{P_n\colon\ n \in \mathbb{N}\}$ (or simply P_n) is *increasing* if $P_0 \leqslant P_1 \leqslant \ldots \leqslant P_n \leqslant \ldots$. Certainly

$$P_n = \bigvee_{k=0}^{n} f^{-k}(P), \quad n \in \mathbb{N}, \tag{2.1}$$

defines an increasing sequence of countable partitions satisfying

$$P_n(x) = \bigcap_{k=0}^{n} f^{-k}(P(f^k(x)), \quad \forall x \in X.$$

Since for all $x \in X$ we have

$$\{y \in X\colon\ f^n(y) \in P(f^n(x)),\ \forall n \in \mathbb{N}\} = \bigcap_{n=0}^{\infty} f^{-n}(P(f^n(x))) = \bigcap_{n=0}^{\infty} P_n(x),$$

we obtain that the identity

$$\lim_{n\to\infty} \sup_{\xi\in P_n} \mu(\xi) = 0 \tag{2.2}$$

implies (1.1).

Now suppose that P is a strong generator of a measurable map $f\colon X \to X$ in a nonatomic probability space $(X, \mathcal{B}, \mu)$. Then, the sequence (2.1) generates $\mathcal{B}$ (mod 0). From this and Lemma 5.2 p. 8 in [10] we obtain that the set of all finite unions of elements of these partitions is everywhere dense in the measure algebra associated to $(X, \mathcal{B}, \mu)$. Consequently, Lemma 9.3.3 p. 278 in [2] implies that the sequence (2.1) satisfies (2.2) and then (1.1) holds. □

We shall see later in Example 2.4 that the converse of this theorem is false, i.e., there are certain measurable maps in nonatomic probability spaces carrying measure-sensitive partitions which are not strong generators. These examples motivate the study of measure-sensitive partitions for measurable maps in measure spaces. For this we use the following auxiliary concept motivated by the notion of Lebesgue sequence of partitions (c.f. p. 81 in [10]).

Definition 2.1. A *measure-sensitive sequence of partitions* of $(X, \mathcal{B}, \mu)$ is an increasing sequence of countable partitions P_n such that $\mu\Big(\bigcap_{n\in\mathbb{N}} \xi_n\Big) = 0$ for all sequences of measurable sets ξ_n satisfying $\xi_n \in P_n$, $\forall n \in \mathbb{N}$. A *measure-sensitive space* is a measure space carrying a measure-sensitive sequence of partitions.

At first glance we observe that (2.2) is a sufficient condition for an increasing sequence P_n of countable partitions to be measure-sensitive (it is also necessary in probability spaces). On the other hand, the class of measure-sensitive spaces is broad enough to include all nonatomic standard probability spaces. Recall that a standard probability space is a probability space $(X, \mathcal{B}, \mu)$ whose underlying measurable space $(X, \mathcal{B})$ is isomorphic to a Polish space equipped with its Borel σ-algebra (e.g. [1]). Namely we have the following proposition.

Proposition 2.1. *All nonatomic standard probability spaces are measure-sensitive.*

P r o o f. As is well-known, for every nonatomic standard probability space $(X, \mathcal{B}, \mu)$ there are a measurable subset $X_0 \subset X$ with $\mu(X \setminus X_0) = 0$, and a sequence of countable partitions Q_n of X_0, such that $\bigcap_{n \in \mathbb{N}} \xi_n$ contains at most one point for every sequence of measurable sets ζ_n in X_0 satisfying $\zeta_n \in Q_n$, $\forall n \in \mathbb{N}$ (c.f. [10] p. 81). Defining $P_n = \{X \setminus X_0\} \cup Q_n$ we obtain an increasing sequence of countable partitions of $(X, \mathcal{B}, \mu)$. It suffices to prove that this sequence is measure-sensitive. For this we take a fixed (but arbitrary) sequence of measurable sets ξ_n of X with $\xi_n \in P_n$ for all $n \in \mathbb{N}$. It follows from the definition of P_n that either $\xi_n = X \setminus X_0$ for some $n \in \mathbb{N}$, or $\xi_n \in Q_n$ for all $n \in \mathbb{N}$. Then, the intersection $\bigcap_{n \in \mathbb{N}} \xi_n$ either is contained in $X \setminus X_0$ or reduces to a single measurable point. Since both $X \setminus X_0$ and the measurable points have measure zero (for nonatomic spaces are diffuse [2]), we obtain $\mu\Big(\bigcap_{n \in \mathbb{N}} \xi_n\Big) = 0$. Since ξ_n is arbitrary, we are done. $\square$

Although measure-sensitive probability spaces need not be standard, all of them are nonatomic. Indeed, we have the following result.

Proposition 2.2. *All measure-sensitive probability spaces are nonatomic.*

P r o o f. Suppose by contradiction that a measure-sensitive probability space $(X, \mathcal{B}, \mu)$ has an atom A. Take a measure-sensitive sequence of partitions P_n. Since A is an atom, we have that $\forall n \in \mathbb{N}\ \exists! \, \xi_n \in P_n$ such that $\mu(A \cap \xi_n) > 0$ (and so $\mu(A \cap \xi_n) = \mu(A)$). Notice that $\mu(\xi_n \cap \xi_{n+1}) > 0$ for, otherwise, $\mu(A) \geqslant \mu(A \cap (\xi_n \cup \xi_{n+1})) = \mu(A \cap \xi_n) + \mu(A \cap \xi_{n+1}) = 2\mu(A)$, which is impossible in probability spaces. Now observe that $\xi_n \in P_n$ and $P_n \leqslant P_{n+1}$, so there is $L \subset P_{n+1}$ such that

$$\mu\Big(\xi_n \triangle \bigcup_{\zeta \in L} \zeta\Big) = 0. \tag{2.3}$$

If $\xi_{n+1} \cap \Big(\bigcup_{\zeta \in L} \zeta\Big) = \emptyset$ we would have $\xi_n \cap \xi_{n+1} = \xi_n \cap \xi_{n+1} \setminus \bigcup_{\zeta \in L} \zeta$ yielding

$$\mu(\xi_n \cap \xi_{n+1}) = \mu\Big(\xi_n \cap \xi_{n+1} \setminus \bigcup_{\zeta \in L} \zeta\Big) \leqslant \mu\Big(\xi_n \setminus \bigcup_{\zeta \in L} \zeta\Big) = 0$$

which is impossible. Hence $\xi_{n+1} \cap \Big(\bigcup_{\zeta \in L} \zeta\Big) \neq \emptyset$ and then $\xi_{n+1} \in L$, for P_{n+1} is a partition and $\xi_{n+1} \in P_{n+1}$. Using (2.3) we obtain $\xi_{n+1} \subset \xi_n$ (mod 0) so $A \cap \xi_{n+1} \subset$

$A \cap \xi_n$ (mod 0), for all $n \in \mathbb{N}^+$. From this and well-known properties of probability spaces we obtain

$$\mu\Big(A \cap \bigcap_{n\in\mathbb{N}} \xi_n\Big) = \mu\Big(\bigcap_{n\in\mathbb{N}} (A \cap \xi_n)\Big) = \lim_{n\to\infty} \mu(A \cap \xi_n) = \mu(A) > 0.$$

But P_n is measure-sensitive and $\xi_n \in P_n$, $\forall n \in \mathbb{N}$, so we have $\mu\Big(\bigcap_{n\in\mathbb{N}} \xi_n\Big) = 0$ yielding $\mu\Big(A \cap \bigcap_{n\in\mathbb{N}} \xi_n\Big) = 0$, which contradicts the above expression. This contradiction yields the proof. $\square$

An interesting question (posed by an anonymous referee) is if the converse of Proposition 2.2 is true for probability spaces, namely, if every nonatomic probability space is measure-sensitive. So far we have not found any counterexample for this question. Notice that such a counterexample (if it exists) must be nonstandard by Proposition 2.1.

The following equivalence relates measure-sensitive partitions and measure-sensitive sequences of partitions.

Lemma 2.1. *The following properties are equivalent for measurable maps $f\colon X \to X$ and countable partitions P on measure spaces $(X, \mathcal{B}, \mu)$:*
(i) *The sequence P_n in* (2.1) *is measure-sensitive.*
(ii) *The partition P is measure-sensitive.*
(iii) *The partition P satisfies*

$$\mu(\{y \in X\colon\ f^n(y) \in P(f^n(x)),\ \forall n \in \mathbb{N}\}) = 0, \quad \forall \mu\text{-a.e. } x \in X.$$

Proof. First we introduce some notation.

Given a partition P and $f\colon\ X \to X$ measurable we define

$$P_\infty(x) = \{y \in X\colon\ f^n(y) \in P(f^n(x)),\ \forall n \in \mathbb{N}\}, \quad \forall x \in X.$$

Notice that

$$P_\infty(x) = \bigcap_{n\in\mathbb{N}^+} P_n(x) \tag{2.4}$$

and

$$P_n(x) = \bigcap_{i=0}^{n} f^{-i}(P(f^i(x))), \tag{2.5}$$

so each $P_\infty(x)$ is a measurable set. For later use we keep the following identity

$$\left(\bigvee_{i=0}^{n} f^{-i}(P)\right)(x) = P_n(x), \quad \forall x \in X. \tag{2.6}$$

Clearly (1.1) (or (iii)) is equivalent to $\mu(P_\infty(x)) = 0$ for every $x \in X$ (or for μ-a.e. $x \in X$, respectively).

Next, we prove that (i) implies (ii). Suppose that the sequence (2.1) is measure-sensitive and fix $x \in X$. By (2.4) and (2.6) we have $P_\infty(x) = \bigcap_{n\in\mathbb{N}} \xi_n$ where $\xi_n = P_n(x) \in P_n$. Since the sequence P_n is measure-sensitive, we obtain $\mu(P_\infty(x)) = \mu\Big(\bigcap_{n\in\mathbb{N}} \xi_n\Big) = 0$, which proves (ii). Conversely, suppose that (ii) holds and let ξ_n be a sequence of measurable sets with $\xi_n \in P_n$ for all n. Take $y \in \bigcap_{n\in\mathbb{N}} \xi_n$. It follows that $y \in P_n(x)$ for all $n \in \mathbb{N}$ whence $y \in P_\infty(x)$ by (2.1). We conclude that $\bigcap_{n\in\mathbb{N}} \xi_n \subset P_\infty(x)$, therefore $\mu\Big(\bigcap_{n\in\mathbb{N}} \xi_n\Big) \leqslant \mu(P_\infty(x)) = 0$, which proves (i).

To prove that (ii) and (iii) are equivalent we only have to prove that (iii) implies (i). Assume by contradiction that P satisfies (iii) but not (ii). Since μ is a probability and (3) holds, the set $X' = \{x \in X\colon\ \mu(P_\infty(x)) = 0\}$ has measure one. Since (ii) does not hold there is $x \in X$ such that $\mu(P_\infty(x)) > 0$. Since μ is a probability and X' has measure one, we have $P_\infty(x) \cap X' \neq \emptyset$, so there is $y \in P_\infty(x)$ such that $\mu(P_\infty(y)) = 0$. But clearly the collection $\{P_\infty(x)\colon\ x \in X\}$ is a partition (because P is), so $P_\infty(x) = P_\infty(y)$ whence $\mu(P_\infty(x)) = \mu(P_\infty(y)) = 0$, which is a contradiction. This ends the proof. $\square$

Recall that a measurable map $f\colon\ X \to X$ is *measure-preserving* if $\mu \circ f^{-1} = \mu$. Moreover, it is *ergodic* if every measurable invariant set A (i.e. $A = f^{-1}(A)$ (mod 0)) satisfies either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$; and *totally ergodic* if f^n is ergodic for all $n \in \mathbb{N}^+$.

Example 2.1. If f is a totally ergodic measure-preserving map of a probability space, then every countable partition P with $0 < \mu(\xi) < 1$ for some $\xi \in P$ is measure-sensitive with respect to f (this follows from the equivalence (iii) in Lemma 2.1 and Lemma 1.1 p. 208 in [10]).

Hereafter, we fix a measure space $(X, \mathcal{B}, \mu)$ and a measurable map $f\colon\ X \to X$. We shall not assume that f is measure-preserving unless otherwise stated.

Using the Kolmogorov-Sinai's entropy we obtain sufficient conditions for the measure-sensitivity of a given partition. Recall that the *entropy* of a finite partition P is defined by

$$H(P) = -\sum_{\xi\in P} \mu(\xi) \log \mu(\xi).$$

The *entropy* of a finite partition P with respect to a measure-preserving map f is defined by

$$h(f, P) = \lim_{n\to\infty} \frac{1}{n} H(P_{n-1}).$$

Then, we have the following lemma.

Lemma 2.2. *A finite partition P with finite positive entropy of an ergodic measure-preserving map $f\colon X \to X$ in a probability space $(X, \mathcal{B}, \mu)$ is measure-sensitive.*

Proof. Since the map f is ergodic, the Shannon-McMillan-Breiman Theorem (c.f. [10] p. 209) implies that the partition P satisfies

$$(2.7) \qquad -\lim_{n\to\infty} \frac{1}{n} \log(\mu(P_n(x))) = h(f, P), \quad \mu\text{-a.e. } x \in X,$$

where $P_n(x)$ is defined as in (2.5). On the other hand, $P_{n+1}(x) \subset P_n(x)$ for all n, so (2.4) implies

$$(2.8) \qquad \mu(P_\infty(x)) = \lim_{n\to\infty} \mu(P_n(x)), \quad \forall x \in X.$$

But $h(f, P) > 0$, so (2.7) implies that $\mu(P_n(x))$ goes to zero for μ-a.e. $x \in X$. This together with (2.8) implies that P satisfies the equivalence (iii) in Lemma 2.1, so P is measure-sensitive. □

In the sequel we study measurable maps carrying measure-sensitive partitions (we call them *measure-expansive maps* for short). It follows at once from Lemma 2.1 that these maps only exist on measure-sensitive spaces. Consequently, we obtain the following result from Proposition 2.2.

Theorem 2.2. *Every probability space carrying measure-expansive maps is nonatomic.*

A simple but useful example follows.

Example 2.2. The irrational rotations of the circle are measure-expansive maps with respect to the Lebesgue measure. This follows from Example 2.1 since all such maps are measure-preserving and totally ergodic.

On the other hand, it is not difficult to find examples of measure-expansive measure-preserving maps which are not ergodic. These examples together with Example 2.2 suggest the question whether an ergodic measure-preserving map is measure-expansive. However, the answer is negative by the following example.

E x a m p l e 2.3. A measure-sensitive partition has necessarily more than one element. Consequently, if $\mathcal{B} = \{X, \emptyset\}$ then no map is measure-expansive although they are all ergodic and measure-preserving.

In spite of this we still can give conditions for the measure-expansivity of ergodic measure-preserving maps.

Recall that the *entropy* (c.f. [10], [16]) of f is defined by

$$h(f) = \sup\{h(f, Q)\colon\ Q \text{ is a finite partition of } X\}.$$

We obtain a natural generalization of Theorem 3.1 in [3].

Theorem 2.3. *Ergodic measure-preserving maps with positive entropy in probability spaces are measure-expansive.*

P r o o f. Let f be such a map with entropy $h(f) > 0$. We can assume that $h(f) < \infty$. It follows that there is a finite partition Q with $0 < h(f, Q) < \infty$. Taking $P = \bigvee\limits_{i=0}^{n-1} f^{-i}(Q)$ with n large we obtain a finite partition with finite positive entropy since $h(f, P) = h(f, Q) > 0$. It follows that P is measure-sensitive by Lemma 2.2 whence f is measure-expansive by definition. $\square$

The first consequence of the above result is that the converse of Theorem 2.1 is false.

E x a m p l e 2.4. Let $f\colon X \to X$ be a homeomorphism with positive topological entropy of a compact metric space X. By the variational principle [16] there is a Borel probability measure μ with respect to which f is an ergodic measure-preserving map with positive entropy. Then, by Theorem 2.3, f carries a measure-sensitive partition which, by Corollary 4.18.1 in [16], cannot be a strong generator. Consequently, there are measurable maps in certain nonatomic probability spaces carrying measure-sensitive partitions which are not strong generators.

On the other hand, it is also false that ergodic measure-expansive measure-preserving maps on probability spaces have positive entropy. The counterexamples are precisely the irrational circle rotations (c.f. Example 2.2). Theorems 2.2 and 2.3 imply the well-known result below.

Corollary 2.1. *Probability spaces carrying ergodic measure-preserving maps with positive entropy are nonatomic.*

In the sequel we analyse the aperiodicity of measure-expansive maps. According to [12], a measurable map f is *aperiodic* whenever for all $n \in \mathbb{N}^+$, if $f^n(x) = x$ on a measurable set A, then $\mu(A) = 0$. Let us extend this definition in the following way.

Definition 2.2. We say that f is *eventually aperiodic* whenever the following property holds for every $(n,k)\in\mathbb{N}^+\times\mathbb{N}$: If A is a measurable set such that for every $x\in A$ there is $0\leqslant i\leqslant k$ such that $f^{n+i}(x)=f^i(x)$, then $\mu(A)=0$.

It follows easily from the definition that an eventually aperiodic map is aperiodic. The converse is true for invertible maps but not in general (e.g. the constant map $f(x)=c$ where c is a measurable point of zero mass).

With this definition we can state the following result.

Theorem 2.4. *Every measure-expansive map is eventually aperiodic (and so aperiodic).*

Proof. Let f be a measure-expansive map of X. Take $(n,k)\in\mathbb{N}^+\times\mathbb{N}$ and a measurable set A such that for every $x\in A$ there is $0\leqslant i\leqslant k$ such that $f^{n+i}(x)=f^i(x)$. Then

$$A\subset\bigcup_{i=0}^{k}f^{-i}(\mathrm{Fix}(f^n)),\tag{2.9}$$

where $\mathrm{Fix}(g)=\{x\in X\colon\ g(x)=x\}$ denotes the set of fixed points of a map g.

Let P be a measure-sensitive partition of f. Then $\bigvee\limits_{m=0}^{k+n}f^{-m}(P)$ is a countable partition. Fix $x,y\in A\cap\xi$. In particular, $\xi=\Big(\bigvee\limits_{m=0}^{k+n}f^{-m}(P)\Big)(x)$ whence $y\in\Big(\bigvee\limits_{m=0}^{k+n}f^{-m}(P)\Big)(x)$. This together with (2.5) and (2.6) yields

$$f^m(y)\in P(f^m(x)),\quad\forall m,\ 0\leqslant m\leqslant k+n.\tag{2.10}$$

But $x,y\in A$, so (2.9) implies $f^i(x),f^j(y)\in\mathrm{Fix}(f^n)$ for some $i,j\in\{0,\dots,k\}$. We can assume that $j\geqslant i$ (otherwise we interchange the roles of x and y in the argument below).

Now take $m>k+n$. Then, $m>j+n$, so $m-j=pn+r$ for some $p\in\mathbb{N}^+$ and some integer $0\leqslant r<n$. Since $0\leqslant j+r<k+n$ (for $0\leqslant j\leqslant k$ and $0\leqslant r<n$), we get

$$\begin{aligned}f^m(y)&=f^{m-j}(f^j(y))=f^{pn+r}(f^j(y))=f^r(f^{pn}(f^j(y)))\\&=f^{j+r}(y)\overset{(2.10)}{\in}P(f^{j+r}(x)).\end{aligned}$$

But

$$\begin{aligned}P(f^{j+r}(x))&=P(f^{j+r-i}(f^i(x)))=P(f^{j+r-i}(f^{pn}(f^i(x))))\\&=P(f^{m-i}(f^i(x)))=P(f^m(x)),\end{aligned}$$

so

$$f^m(y) \in P(f^m(x)), \quad \forall m > k+n.$$

This together with (2.10) implies that $f^m(y) \in P(f^m(x))$ for all $m \in \mathbb{N}$ whence $y \in P_\infty(x)$. Consequently, $A \cap \xi \subset P_\infty(x)$. As P is measure-sensitive, Lemma 2.1 implies

$$\mu(A \cap \xi) = 0, \quad \forall \xi \in \bigvee_{i=0}^{k+n} f^{-i}(P).$$

On the other hand, $\bigvee_{i=0}^{k+n} f^{-i}(P)$ is a partition, so

$$A = \bigcup_{\xi \in \bigvee_{i=0}^{k+n} f^{-i}(P)} (A \cap \xi)$$

and then $\mu(A) = 0$ since $\bigvee_{i=0}^{k+n} f^{-i}(P)$ is countable. This ends the proof. $\square$

It follows from Lemma 2.1 that, in nonatomic probability spaces, every measurable map carrying strong generators is measure-expansive. This motivates the question whether every measure-expansive map has a strong generator. We give a partial positive answer for certain maps defined as follows. We say that f is *countable-to-one (mod 0)* if $f^{-1}(x)$ is countable for μ-a.e. $x \in X$. We say that f is *nonsingular* if a measurable set A has measure zero if and only if $f^{-1}(A)$ also does. All measure-preserving maps are nonsingular. A *Lebesgue probability space* is a complete measure space which is isomorphic to the completion of a standard probability space (c.f. [1], [2]).

Corollary 2.2. *The following properties are equivalent for nonsingular countable-to-one (mod 0) maps f on nonatomic Lebesgue probability spaces:*
(1) *f is measure-expansive.*
(2) *f is eventually aperiodic.*
(3) *f is aperiodic.*
(4) *f has a strong generator.*

Proof. Notice that (1) $\Rightarrow$ (2) by Theorem 2.4 and (2) $\Rightarrow$ (3) follows from the definitions. Moreover, (3) $\Rightarrow$ (4) by Parry's theorem (c.f. [12], [14], [13]) while (4) $\Rightarrow$ (1) by Lemma 2.1. $\square$

Recall that $\mathrm{Fix}(g) = \{x \in X\colon\, g(x) = x\}$ denotes the set of fixed points of a mapping g.

Corollary 2.3. *If $f^k = f$ for some integer $k \geqslant 2$, then f is not measure-expansive.*

P r o o f. If f were measure-expansive, then it would be eventually aperiodic by Theorem 2.4. Besides, if $x \in X$ then $f^k(x) = f(x)$, so $f^{k-1}(f^k(x)) = f^{k-1}(f(x)) = f^k(x)$, therefore $f^k(x) \in \mathrm{Fix}(f^{k-1})$ whence $X \subset f^{-k}(\mathrm{Fix}(f^{k-1}))$. But since f is eventually aperiodic, $n = k-1 \in \mathbb{N}^+$ and X measurable, we obtain from the definition of eventual aperiodicity that $\mu(X) = 0$, which is impossible. Then, the result follows by contradiction. $\square$

E x a m p l e 2.5. By Corollary 2.3, neither the identity $f(x) = x$ nor the constant map $f(x) = c$ are measure-expansive (because they satisfy $f^2 = f$). In particular, the converse of Theorem 2.4 is false, since the constant maps are eventually aperiodic but not measure-expansive.

It is not difficult to prove that *all ergodic measure-preserving maps of a nonatomic probability space are aperiodic*. Then, Corollary 2.1 implies the well-known fact that *all ergodic measure-preserving maps with positive entropy on probability spaces are aperiodic*. However, using Theorems 2.3 and 2.4 we obtain the following stronger result.

Corollary 2.4. *All ergodic measure-preserving maps with positive entropy on probability spaces are* eventually *aperiodic.*

Next, we study the following variant of aperiodicity introduced by Helmberg and Simons (c.f. [4], p. 180).

Definition 2.3. We say that f is *HS-aperiodic*[1] whenever for every measurable set of positive measure A and for every $n \in \mathbb{N}^+$ there is a measurable subset $B \subset A$ such that $\mu(B \setminus f^{-n}(B)) > 0$.

Notice that HS-aperiodicity implies the aperiodicity used in [7] or [15] (for further comparisons see p. 88 in [9]).

A measurable map f is *negative nonsingular* if $\mu(f^{-1}(A)) = 0$ whenever A is a measurable set with $\mu(A) = 0$. Some consequences of the HS-aperiodicity on negative nonsingular maps in probability spaces are given in [9]. Observe that every measure-preserving map is negative nonsingular.

Let us present two technical (but simple) results for later usage. Hereafter, a measurable set A satisfying $A \subset f^{-1}(A) \pmod 0$ will be referred to as a *positively invariant set* (mod 0). For completeness we introduce the following property of these sets.

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[1] called aperiodic in [4].

Lemma 2.3. *If A is a positively invariant set (mod 0) of finite measure of a negative nonsingular map f, then*

$$\mu\Big(\bigcap_{n=0}^{\infty} f^{-n}(A)\Big) = \mu(A). \tag{2.11}$$

P r o o f. It follows by induction that

$$\mu\Big(\bigcap_{n=0}^{m} f^{-n}(A)\Big) = \mu(A), \quad \forall m \in \mathbb{N}. \tag{2.12}$$

On the other hand,

$$\bigcap_{n=0}^{\infty} f^{-n}(A) = \bigcap_{m=0}^{\infty}\bigcap_{n=0}^{m} f^{-n}(A)$$

and $\bigcap_{n=0}^{m+1} f^{-n}(A) \subset \bigcap_{n=0}^{m} f^{-n}(A)$. As $\mu(A) < \infty$, we conclude that

$$\mu\Big(\bigcap_{n=0}^{\infty} f^{-n}(A)\Big) = \lim_{m\to\infty} \mu\Big(\bigcap_{n=0}^{m} f^{-n}(A)\Big) \overset{(2.12)}{=} \lim_{m\to\infty} \mu(A) = \mu(A),$$

which proves (2.11). □

We only use this lemma to prove the proposition below.

Proposition 2.3. *Let P be a measure-sensitive partition of a negative nonsingular map f. Then, no $\xi \in P$ with positive finite measure is positively invariant (mod 0).*

P r o o f. Suppose by contradiction that there is $\xi \in P$ with $0 < \mu(\xi) < \infty$ which is positively invariant (mod 0). Taking $A = \xi$ in Lemma 2.3 we obtain

$$\mu\Big(\bigcap_{n=0}^{\infty} f^{-n}(\xi)\Big) = \mu(\xi). \tag{2.13}$$

As $\mu(\xi) > 0$, we conclude that $\bigcap_{n=0}^{\infty} f^{-n}(\xi) \neq \emptyset$, and so there is $x \in \xi$ such that $f^n(x) \in \xi$ for all $n \in \mathbb{N}$. As $\xi \in P$, we obtain $P(f^n(x)) = \xi$, and so $f^{-n}(P(f^n(x))) = f^{-n}(\xi)$ for all $n \in \mathbb{N}$. Using (2.5) we get

$$P_m(x) = \bigcap_{n=0}^{m} f^{-n}(\xi).$$

Then, (2.4) yields

$$P_\infty(x) = \bigcap_{m=0}^{\infty} P_m(x) = \bigcap_{m=0}^{\infty}\bigcap_{n=0}^{m} f^{-n}(\xi) = \bigcap_{n=0}^{\infty} f^{-n}(\xi),$$

and so $\mu(P_\infty(x)) = \mu(\xi)$ by (2.13). Then, $\mu(\xi) = 0$ by Lemma 2.1 since P is measure-sensitive, which is impossible. This contradiction proves the result. $\square$

We also need the following lemma resembling a well-known property of expansive maps.

Lemma 2.4. *If $k \in \mathbb{N}^+$, then f is measure-expansive if and only if f^k is.*

P r o o f. The notation $P^f_\infty(x)$ will indicate the dependence of $P_\infty(x)$ on f.

First suppose that f^k is a measure-expansive with a measure-sensitive partition P. Then, $\mu(P^{f^k}_\infty(x)) = 0$ for all $x \in X$ by Lemma 2.1. But by definition, we have $P^f_\infty(x) \subset P^{f^k}_\infty(x)$, so $\mu(P^f_\infty(x)) = 0$ for all $x \in X$. Therefore, f is measure-expansive with the measure-sensitive partition P. Conversely, suppose that f is measure-expansive with expansivity constant P. Consider $Q = \bigvee_{i=0}^{k} f^{-i}(P)$, which is a countable partition satisfying $Q(x) = \bigcap_{i=0}^{k} f^{-i}(P(f^i(x)))$ by (2.6). Now, take $y \in Q^{f^k}_\infty(x)$. In particular, $y \in Q(x)$ hence $f^i(y) \in P(f^i(x))$ for every $0 \leqslant i \leqslant k$. Take $n > k$, so $n = pk + r$ for some nonnegative integers p and $0 \leqslant r < k$. As $y \in Q^{f^k}_\infty(x)$, we have $f^{pk}(y) \in Q(f^{pk}(x))$ and then $f^n(y) = f^{pk+i}(y) = f^i(f^{pk}(y)) \in P(f^i(f^{pk}(x)) = P(f^n(x))$, which proves $f^n(y) \in P(f^n(x))$ for all $n \in \mathbb{N}$. Then, $y \in P_\infty(x)$ which yields $Q^{f^k}_\infty(x) \subset P_\infty(x)$. Thus, $\mu(Q^{f^k}_\infty(x)) = 0$ for all $x \in X$ by the equivalence (ii) in Lemma 2.1, since P is measure-sensitive. It follows that f^k is measure-expansive with the measure-sensitive partition Q. $\square$

With these definitions and preliminary results we obtain the following.

Theorem 2.5. *Every measure-expansive negative nonsingular map in a probability space is HS-aperiodic.*

P r o o f. Suppose by contradiction that there is a measure-expansive map f which is negative nonsingular but not HS-aperiodic. Then, there are both a measurable set of positive measure A and $n \in \mathbb{N}^+$ such that $\mu(B \setminus f^{-n}(B)) = 0$ for every measurable subset $B \subset A$. It follows that every measurable subset $B \subset A$ is positively invariant (mod 0) with respect to f^n. By Lemma 2.4, we can assume $n = 1$.

Now, let P be a measure-sensitive partition of f. Clearly, since $\mu(A) > 0$, there is $\xi \in P$ such that $\mu(A \cap \xi) > 0$. Taking $\eta = A \cap \xi$ we obtain that η is positively invariant (mod 0) with positive measure. In addition, consider the new partition $Q = (P \setminus \{\xi\}) \cup \{\eta, \xi \setminus A\}$, which is clearly measure-sensitive (for P is). Since this partition also carries a positively invariant (mod 0) member of positive measure (say η), we obtain a contradiction by Proposition 2.3. $\square$

To finish, we compare the measure-expansivity with the notion of pairwise sensitivity in metric measure spaces introduced in p. 376 of [3]. Similar notions have been introduced elsewhere (c.f. [5], [8] or [6]).

Recall that a *metric measure space* is a triple (X, d, μ), where (X, d) is a metric space and μ is a measure in the corresponding Borel σ-algebra. Hereafter, the term *measurable* will mean *Borel measurable*. The product measure in $X \times X$ will be denoted by $\mu^{\otimes 2}$.

Definition 2.4. A measurable map $f\colon X \to X$ on a metric measure space (X, d, μ) is *pairwise sensitive* if there is $\delta > 0$ such that

$$\mu^{\otimes 2}\left(\{(x, y) \in X \times X\colon \exists n \in \mathbb{N},\ d(f^n(x), f^n(y)) \geqslant \delta\}\right) = 1.$$

By a *metric probability space* we mean a metric measure space of total mass one. Given a map $f\colon X \to X$ and $\delta > 0$ we define the dynamical δ-balls

$$\Phi_\delta(x) = \{y \in X\colon d(f^n(x), f^n(y)) \leqslant \delta, \forall n \in \mathbb{N}\}, \quad \forall x \in X.$$

The following characterization of pairwise sensitivity (similar to one in [11]) is in [8].

Lemma 2.5. *The following properties are equivalent for measurable maps f on metric probability spaces (X, d, μ):*

(1) *f is pairwise sensitive.*

(2) *There is $\delta > 0$ such that*

$$\mu(\Phi_\delta(x)) = 0, \quad \forall x \in X. \tag{2.14}$$

By a *separable probability space* we mean a metric probability space whose underlying metric space is separable.

Theorem 2.6. *All pairwise sensitive maps on separable probability spaces are measure-expansive.*

Proof. Let f be a pairwise sensitive map on a separable probability space (X, d, μ). By Lemma 2.5, there is $\delta > 0$ satisfying (2.14). Since (X, d) is separable, we can select a countable covering $\{B_k \colon k \in I\}$ of X consisting of balls of radius δ, where I is either $\mathbb{N}$ or $\{0, 1, \ldots, s\}$ for some $s \in \mathbb{N}$. As usual we can transform this covering into a countable partition $P = \{\xi_k \colon k \in I\}$ by taking $\xi_0 = B_0$ and $\xi_k = B_k \setminus \bigcup_{i=0}^{k-1} B_i$ for $k \geqslant 1$. Clearly this partition satisfies $P_\infty(x) \subset \Phi_\delta(x)$. Then, (2.14) implies $\mu(P_\infty(x)) \leqslant \mu(\Phi_\delta(x)) = 0$ for every $x \in X$, so P is measure-sensitive by Lemma 2.1. □

The following example shows that the converse of Theorem 2.6 is false.

Example 2.6. An irrational circle rotation is measure-expansive with respect to the Lebesgue measure (c.f. Example 2.2 or Corollary 2.2) but not pairwise sensitive with respect to that measure (c.f. [3] p. 378).

Recall that a map $f \colon X \to X$ on a metric space (X, d) is *expansive* if there is $\delta > 0$ such that $x = y$ whenever $x, y \in X$ and $d(f^n(x), f^n(y)) \leqslant \delta$ for all $n \in \mathbb{N}$.

Corollary 2.5. *Every measurable expansive map in a nonatomic separable probability space is measure-expansive.*

Proof. Notice that a map f is expansive if and only if there is $\delta > 0$ such that $\Phi_\delta(x) = \{x\}$ for every $x \in X$. Then, Lemma 2.5 implies that every expansive measurable map on a nonatomic metric measure space is pairwise sensitive. Now apply Theorem 2.6. □

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Author's address: *C. A. Morales*, Instituto de Matemática, Universidade Federal do Rio de Janeiro, P. O. Box 68530, 21945-970, Rio de Janeiro, Brazil, e-mail: `morales@impa.br`.