SOLUTION OF WHITEHEAD EQUATION ON GROUPS

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Abstract. Let G be a group and H an abelian group. Let $J^*(G,H)$ be the set of solutions $f\colon G\to H$ of the Jensen functional equation $f(xy)+f(xy^{-1})=2f(x)$ satisfying the condition $f(xyz)-f(xzy)=f(yz)-f(zy)$ for all $x,y,z\in G$. Let $Q^*(G,H)$ be the set of solutions $f\colon G\to H$ of the quadratic equation $f(xy)+f(xy^{-1})=2f(x)+2f(y)$ satisfying the Kannappan condition $f(xyz)=f(xzy)$ for all $x,y,z\in G$. In this paper we determine solutions of the Whitehead equation on groups. We show that every solution $f\colon G\to H$ of the Whitehead equation is of the form $4f=2\varphi+2\psi$, where $2\varphi\in J^*(G,H)$ and $2\psi\in Q^*(G,H)$. Moreover, if H has the additional property that $2h=0$ implies $h=0$ for all $h\in H$, then every solution $f\colon G\to H$ of the Whitehead equation is of the form $2f=\varphi+\psi$, where $\varphi\in J^*(G,H)$ and $2\psi(x)=B(x,x)$ for some symmetric bihomomorphism $B\colon G\times G\to H$.

Keywords: homomorphism, Fréchet functional equation, Jensen functional equation, symmetric bihomomorphism, Whitehead functional equation

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1. Introduction

Let G be a group and H an abelian group. Let $f\colon G\to H$ be a function. The Cauchy difference of f, $C_f^{(1)}\colon G\times G\to H$, is given by

$$C_f^{(1)}(x,y)=f(xy)-f(x)-f(y) \tag{1.1}$$

for all $x,y\in G$. The Cauchy difference of f measures how much f deviates from being a group homomorphism of the group G into the group H. The second Cauchy difference of f, $C_f^{(2)}\colon G\times G\times G\to H$, is given by

$$C_f^{(2)}(x,y,z)=C_f(xy,z)-C_f(x,z)-C_f(y,z) \tag{1.2}$$

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for all $x, y, z \in G$. If $C_f^{(2)}(x, y, z) = 0$, then one arrives at the functional equation

$$f(xyz) + f(x) + f(y) + f(z) = f(xy) + f(xz) + f(yz) \tag{1.3}$$

for all $x, y, z \in G$. This equation first appeared in a paper [5] of J. H. C. Whitehead in 1950. In that paper he solved the functional equation (1.3) on abelian groups assuming that f satisfies the condition $f(x^{-1}) = f(x)$. On an AMS meeting professor Deeba of University of Houston asked for the solution of equation (1.3). Kannappan in [1] solved this equation for mappings $f\colon V \to K$ where V is a vector space and K is a field with characteristic different from 2.

The solution $f\colon G \to H$ of the functional equation

$$f(xyz) + f(x) + f(y) + f(z) = f(xy) + f(yz) + f(zx) \tag{1.4}$$

for all $x, y, z \in G$ can be found in the book [2] by Kannappan. The functional equation (1.4) is referred to as the Fréchet functional equation in [2]. On an abelian group G the functional equations (1.3) and (1.4) are equivalent. However, on an arbitrary group G, Whitehead and Fréchet functional equations are not equivalent.

It is easy to see that if $f\colon G \to H$ satisfies (1.3), then $f(e) = 0$, where e is the identity (or neutral) element of the group G. Let $W(G, H)$ be the set of all functions that satisfy the Whitehead functional equation (1.3). Let $J(G, H)$ be the set of all solutions of the Jensen functional equation

$$f(xy) + f(xy^{-1}) = 2f(x), \quad \forall\, x, y \in G \tag{1.5}$$

and let $J_0(G, H)$ denote the set of all solutions $f\colon G \to H$ of the Jensen functional equation (1.5) together with the normalization condition $f(e) = 0$. Moreover, denote by $J^*(G, H)$ the subspace of $J_0(G, H)$ consisting of functions φ satisfying the additional condition

$$\varphi(xyz) - \varphi(xzy) = \varphi(yz) - \varphi(zy) \tag{1.6}$$

for every $x, y, z \in G$. On groups, the Jensen functional equation was extensively studied by Ng in [3].

A function $f\colon G \to H$ is said to satisfy the Kannappan condition if for any $x, y, z \in G$, the relation

$$f(xyz) = f(xzy) \tag{1.7}$$

holds. The Kannappan condition on $f\colon G \to H$ is equivalent to f being a function on the abelian group $G/[G, G]$, where $[G, G]$ is the commutators subgroup of G.

The set $Q(G,H)$ will stand for the set of solutions $g\colon G \to H$ of the quadratic functional equation

$$g(xy) + g(xy^{-1}) = 2g(x) + 2g(y), \quad \forall\, x, y \in G. \tag{1.8}$$

Denote by $Q^*(G,H)$ the subset of $Q(G,H)$ consisting of functions satisfying the Kannappan condition (1.7). The quadratic functional equation on groups was studied in [4] and [6].

In Section 2 of the present paper we determine the general solutions of the equation (1.3) on arbitrary groups. Using these results proved for the Whitehead equation, in Section 3 we find the solutions of the Fréchet functional equation.

2. Solution of the Whitehead equation

In this section, we first present several lemmas that will be used to prove the main result of this paper.

Lemma 2.1. *Let G be a group and H an abelian group. If a function $f\colon G \to H$ satisfies the Whitehead equation* (1.3), *then it satisfies the following system of equations:*

$$\begin{aligned} &f(xy) + f(xy^{-1}) = 2f(x) + f(y) + f(y^{-1}),\\ &f(yx) + f(y^{-1}x) = 2f(x) + f(y) + f(y^{-1}),\\ &f(xyz) - f(xzy) = f(yz) - f(zy),\\ &f(zyx) - f(yzx) = f(zy) - f(yz). \end{aligned} \tag{2.1}$$

P r o o f. Letting $x = y = z = e$ in (1.3), we see that $f(e) = 0$. Now if we set $z = y^{-1}$ then from (1.3) we get

$$f(x) + f(x) + f(y) + f(y^{-1}) = f(xy) + f(xy^{-1}),$$

which is

$$f(xy) + f(xy^{-1}) = 2f(x) + f(y) + f(y^{-1}). \tag{2.2}$$

Now if we put $y = x^{-1}$ we get

$$f(z) + f(x) + f(x^{-1}) + f(z) = f(xz) + f(x^{-1}z)$$

and hence

$$f(xz) + f(x^{-1}z) = 2f(z) + f(x) + f(x^{-1}).$$

Now changing x to y and z to x we get

$$f(yx) + f(y^{-1}x) = 2f(x) + f(y) + f(y^{-1}). \tag{2.3}$$

Interchanging y with z in (1.3) we obtain

$$f(xzy) + f(x) + f(y) + f(z) = f(xy) + f(xz) + f(zy), \tag{2.4}$$

and subtracting (2.4) from the equation (1.3) we get

$$f(xyz) - f(xzy) = f(yz) - f(zy). \tag{2.5}$$

Similarly, we obtain the last equation of the system (2.1). Thus we see that if a function f satisfies equation (1.3) then it satisfies the system (2.1), and the proof of the lemma is now complete. □

Lemma 2.2. *Let G be a group and H an abelian group. If a function $f\colon G \to H$ satisfies the system of equations*

$$\begin{aligned} f(xy) + f(xy^{-1}) &= 2f(x) + f(y) + f(y^{-1}),\\ f(yx) + f(y^{-1}x) &= 2f(x) + f(y) + f(y^{-1}),\\ f(xyz) - f(xzy) &= f(yz) - f(zy),\\ f(zyx) - f(yzx) &= f(zy) - f(yz), \end{aligned} \tag{2.6}$$

then $2f = \varphi + \psi$ for some $\varphi \in J^*(G,H)$ and $\psi \in Q^*(G,H)$.

Proof. For any function $\phi\colon G \to H$, denote by ϕ^* the function defined by the rule $\phi^*(x) = \phi(x^{-1})$. It is clear that ϕ^* satisfies the system (2.6) if and only if ϕ satisfies this system.

Now suppose that a function f satisfies the system (2.6). Then functions

$$\varphi(x) = f(x) - f^*(x), \quad \psi(x) = f(x) + f^*(x) \tag{2.7}$$

satisfy the same system and $2f = \varphi + \psi$. From the first equation of the system (2.6) it follows that $\varphi \in J(G,H)$ and $\psi \in Q(G,H)$. From the third equation of (2.6) it follows that $\varphi \in J^*(G,H)$. Now consider the function ψ. It is clear that $\psi(x) = \psi(x^{-1})$ and ψ satisfies the system (2.6) if and only if it satisfies the system

$$\begin{aligned} \psi(xy) + \psi(xy^{-1}) &= 2\psi(x) + 2\psi(y),\\ \psi(xyz) - \psi(xzy) &= \psi(yz) - \psi(zy). \end{aligned} \tag{2.8}$$

Let us verify that, for any $x, y \in G$, the function ψ satisfies the relation $\psi(xy) = \psi(yx)$. Indeed, we have

$$\psi(xy) + \psi(xy^{-1}) = 2\psi(x) + 2\psi(y),$$
$$\psi(yx) + \psi(yx^{-1}) = 2\psi(y) + 2\psi(x).$$

Subtracting the latter equation from the former and taking into account the relation $\psi(yx^{-1}) = \psi((xy^{-1})^{-1}) = \psi(xy^{-1})$, we obtain

$$\psi(xy) = \psi(yx)$$

for all $x, y \in G$. Hence from (2.8) we see that ψ satisfies the system

$$\begin{aligned} \psi(xy) + \psi(xy^{-1}) &= 2\psi(x) + 2\psi(y), \\ \psi(xyz) &= \psi(xzy). \end{aligned} \tag{2.9}$$

Therefore ψ is a quadratic function satisfying the Kannappan condition $\psi(xyz) = \psi(xzy)$ and thus $\psi \in Q^*(G, H)$. The proof of the lemma is now complete. □

For any abelian group H and any $n \in \mathbb{N}$, let $nH = \{g;\ g = nh,\ \forall\, h \in H\}$, that is, the subgroup nH consists of elements of the form $g = nh$.

Lemma 2.3. *Let G be a group and H an abelian group.*
(a) *If $\varphi \in J^*(G, H)$, then $2\varphi \in W(G, H)$.*
(b) *If $\psi \in Q^*(G, H)$, then $2\psi \in W(G, H)$.*

Proof. To prove (a), let $\varphi \in J^*(G, H)$. Hence ϕ satisfies the relations

$$\varphi(xy) + \varphi(xy^{-1}) = 2\varphi(x) \tag{2.10}$$

and

$$\varphi(yx) + \varphi(yx^{-1}) = 2\varphi(y).$$

Hence by adding the last two equations, we have

$$\varphi(xy) + \varphi(xy^{-1}) + \varphi(yx) + \varphi(yx^{-1}) = 2\varphi(x) + 2\varphi(y). \tag{2.11}$$

Since φ is an odd function, $\varphi(xy^{-1}) = -\,\varphi(yx^{-1})$, and thus (2.11) yields

$$\varphi(xy) + \varphi(yx) = 2\varphi(x) + 2\varphi(y) \tag{2.12}$$

for all $x, y \in G$.

Replacing x by xy and y by z in (2.10), we obtain

$$\varphi(xyz) + \varphi(xyz^{-1}) = 2\varphi(xy). \tag{2.13}$$

Similarly, again replacing x by xz in (2.10), we have

$$\varphi(xzy) + \varphi(xzy^{-1}) = 2\varphi(xz). \tag{2.14}$$

Adding last two equalities and using the fact that

$$\varphi\left(xyz^{-1}\right) + \varphi\left(xzy^{-1}\right) = \varphi\left(x(yz^{-1})\right) + \varphi\left(x(yz^{-1})^{-1}\right)$$

we have

$$\varphi\left(xyz\right) + \varphi\left(xzy\right) + \varphi\left(x(yz^{-1})\right) + \varphi\left(x(yz^{-1})^{-1}\right) = 2\varphi(xy) + 2\varphi(xz)$$

for all $x, y, z \in G$. Using the equation (2.10) in the last equation, we have

$$\varphi(xyz) + \varphi(xzy) + 2\varphi(x) = 2\varphi(xy) + 2\varphi(xz) \tag{2.15}$$

for all $x, y, z \in G$. Taking the sum of (1.6) and (2.15), we get

$$2\varphi(xyz) + 2\varphi(x) = 2\varphi(xy) + 2\varphi(xz) + \varphi(yz) - \varphi(zy). \tag{2.16}$$

From (2.12), we have

$$2\varphi(y) + 2\varphi(z) = \varphi(yz) + \varphi(zy). \tag{2.17}$$

Taking the sum of (2.17) and (2.16), we obtain

$$2\varphi(xyz) + 2\varphi(x) + 2\varphi(y) + 2\varphi(z) = 2\varphi(xy) + 2\varphi(xz) + 2\varphi(yz).$$

Hence $2\varphi \in W(G, H)$. This completes the proof of (a).

To prove (b), let $\psi \in Q^*(G, H)$. Hence ψ is even, that is, $\psi(x^{-1}) = \psi(x)$ for all $x \in G$. Replacing y by yz in (1.8), we have

$$\psi(xyz) + \psi(xz^{-1}y^{-1}) - 2\psi(x) - 2\psi(yz) = 0. \tag{2.18}$$

Similarly, replacing x by xz^{-1} and y by y^{-1} in (1.8) and using the fact that ψ is even, we obtain

$$\psi(xz^{-1}y^{-1}) + \psi(xz^{-1}y) - 2\psi(xz^{-1}) - 2\psi(y) = 0. \tag{2.19}$$

Again replacing x by xy and y by z^{-1}, we see that

$$\psi(xyz^{-1}) + \psi(xyz) - 2\psi(xy) - 2\psi(z) = 0. \tag{2.20}$$

Finally, replacing y by z^{-1}, we obtain

$$2\psi(xz^{-1}) + 2\psi(xz) - 4\psi(x) - 4\psi(z) = 0. \tag{2.21}$$

Subtracting the sum of (2.19) and (2.21) form the sum of (2.18) and (2.20) and using the Kannappan condition (1.7), we obtain

$$2\psi(xyz) + 2\psi(x) + 2\psi(y) + 2\psi(z) - 2\psi(xy) - 2\psi(xz) - 2\psi(yz) = 0.$$

Therefore, $2\psi \in W(G,H)$. Further, it is easy to see that $2J^*(G,H)$ and $2Q^*(G,H)$ are subgroups of $W(G,H)$. This completes the proof of the lemma. □

When a group G is the direct sum of subgroups H and K, then symbolically we denote this by writing $G = H \oplus K$. The following theorem easily follows from Lemma 2.1, Lemma 2.2 and Lemma 2.3.

Theorem 2.4. *Suppose that G is a group and H is an abelian group. If $f \in W(G,H)$, then*

$$4f = 2\varphi + 2\psi, \tag{2.22}$$

where $\varphi \in J(G,H)$, $2\varphi \in J(G,H) \cap W(G,H) = J^*(G,H)$, $\psi \in Q(G,H)$, and $2\psi \in Q(G,H) \cap W(G,H) = Q^*(G,H)$. Therefore

$$4W(G,H) = 2J^*(G,H) \oplus 2Q^*(G,H). \tag{2.23}$$

Remark 2.1. If H has the property

$$2h = 0 \text{ implies } h = 0 \tag{2.24}$$

for all $h \in H$, then from (2.23) we get

$$2W(G,H) = J^*(G,H) \oplus Q^*(G,H). \tag{2.25}$$

Lemma 2.5. *Let G be a group and H an abelian group. Let $\omega \in Q(G,H) \cap W(G,H)$. Then there is a symmetric bimorphism $B\colon G \times G \to H$ such that $2\omega(x) = B(x,x)$.*

Proof. Let $B(x,y) = \omega(xy) - \omega(x) - \omega(y)$, then we have

$$\begin{aligned}
&B(x,yz) - B(x,y) - B(x,z)\\
&= \omega(xyz) - \omega(x) - \omega(yz) - \omega(xy) + \omega(x) + \omega(y) - \omega(xz) + \omega(x) + \omega(z)\\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
&B(xy,z) - B(x,z) - B(y,z)\\
&= \omega(xyz) - \omega(xy) - \omega(z) - \omega(xz) + \omega(x) + \omega(z) - \omega(yz) + \omega(y) + \omega(z)\\
&= 0.
\end{aligned}$$

Therefore $B(x,y)$ is a bimorphism. Since $\omega(xy) = \omega(yx)$ it follows that $B(x,y)$ is a symmetric bimorphism. Now since $\omega(x^2) = 4\omega(x)$, we get

$$B(x,x) = \omega(x^2) - 2\omega(x) = 4\omega(x) - 2\omega(x) = 2\omega(x).$$

This completes the proof of the lemma. □

Theorem 2.6. *Let G be a group and H an abelian group with the property that $2h = 0$ implies $h = 0$ for all $h \in H$. Then every solution $f\colon G \to H$ of the Whitehead equation (1.3) is of the form $2f = \varphi + \psi$, where $\varphi \in J^*(G,H)$ and $2\psi(x) = B(x,x)$ for some symmetric bihomomorphism $B\colon G \times G \to H$.*

3. Fréchet equation

In this section, we determine the solution of Fréchet equation using the results obtained in the previous section. Consider the Fréchet equation

$$f(xyz) + f(x) + f(y) + f(z) = f(xy) + f(yz) + f(zx), \tag{3.1}$$

where $f\colon G \to H$. The set of solutions of (3.1) let us denote by $F(G,H)$.

Lemma 3.1. *Let G be a group and H an abelian group.*

(a) $F(G,H) \subseteq W(G,H)$.

(b) *Let $f \in W(G,H)$, then $f \in F(G,H)$ if and only if it satisfies the condition $f(xy) = f(yx)$ for any $x, y \in G$.*

P r o o f. To prove (a), let $f \in F(G,H)$. Then f satisfies

$$f(zxy) + f(x) + f(y) + f(z) = f(zx) + f(xy) + f(yz). \tag{3.2}$$

Subtracting (3.2) from (3.1), we obtain

$$f(zxy) = f(xyz). \tag{3.3}$$

Putting $y = 1$, we get

$$f(zx) = f(xz). \tag{3.4}$$

It follows that every solution of (3.1) satisfies equation (1.3). Therefore $F(G,H) \subseteq W(G,H)$.

Next, we prove (b). Let $f \in W(G,H)$. It is clear that if f satisfies the condition $f(xy) = f(yx)$ for any $x, y \in G$, then $f \in F(G,H)$. Therefore $F(G,H)$ is a subgroup of $W(G,H)$ consisting of functions satisfying the condition (3.4). □

Lemma 3.2. *Let G be a group and H an abelian group. If $f \in J(G,H)$ and satisfies $f(xy) = f(yx)$ for all $x, y \in G$, then $2f \in \mathrm{Hom}(G,H)$. Moreover, if H has the property* (2.24), *then $f \in \mathrm{Hom}(G,H)$.*

P r o o f. Let $f \in J(G,H)$. Then f satisfies

$$f(xy) + f(xy^{-1}) = 2f(x)$$

for all $x, y \in G$. Interchanging x and y, we have

$$f(yx) + f(yx^{-1}) = 2f(y).$$

Taking a sum of these equations and using relations $f(xy) = f(yx)$ and $f(yx^{-1}) = -f(xy^{-1})$ we obtain

$$2f(xy) = 2f(x) + 2f(y)$$

for all $x, y \in G$, and hence $2f \in \mathrm{Hom}(G,H)$. □

The following theorem easily follows.

Theorem 3.3. *Let G be a group and H an abelian group. Suppose $f \in F(G,H)$. Then*

$$4f = \xi + 2\psi, \tag{3.5}$$

where $\xi \in \mathrm{Hom}(G,H)$, $2\psi \in Q(G,H) \cap W(G,H) = Q^*(G,H)$, and $\psi \in Q(G,H)$. Therefore $4W(G,H) = \mathrm{Hom}(G,H) \oplus 2Q(G,H)$.

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