POSITIVE SOLUTION TO A SINGULAR $(k, n-k)$ CONJUGATE BOUNDARY VALUE PROBLEM

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(Received October 12, 2009)

Abstract. The positive solution is studied for a $(k, n-k)$ conjugate boundary value problem. The nonlinear term is allowed to be singular with respect to both the time and space variables. By applying the approximation theorem for completely continuous operators and the Guo-Krasnosel'skii fixed point theorem of cone expansion-compression type, an existence theorem for a positive solution is established.

Keywords: singular ordinary differential equation, higher order boundary value problem, positive solution, existence theorem

MSC 2010: 34B16, 34B18

1. Introduction

Let $n \geqslant 2$ and $1 \leqslant k \leqslant n-1$ be two fixed integers. We study positive solutions of the nonlinear $(k, n-k)$ conjugate boundary value problem

$$
\text{(P)} \begin{cases} (-1)^{n-k} u^{(n)}(t) = h(t) f(t, u(t)), & 0 < t < 1, \\ u^{(i)}(0) = 0, & 0 \leqslant i \leqslant k-1, \\ u^{(j)}(1) = 0, & 0 \leqslant j \leqslant n-k-1. \end{cases}
$$

Here, we call the function $u^* \in C[0,1]$ a positive solution of the problem (P), if $u^*(t)$ is a solution of (P) and $u^*(t) > 0$, $0 < t < 1$. We will allow the nonlinear term $h(t) f(t, u)$ to be singular at $t = 0$, $t = 1$ and $u = 0$.

Because of widespread applications in physics and engineering (see [1], [2]) in the past 20 years, there has been much attention paid to the nonlinear higher order boundary value problems. Particularly, the nonlinear $(k, n-k)$ conjugate boundary value problem (P) has been studied by some authors, for example, see [3]–[8]. In 1997, Eloe and Henderson proved the following existence theorem for a positive solution (see Theorem 7 in [3]).

Theorem 1.1. *Assume that*

(a1) $h(t) \equiv 1$ *and* $f\colon (0,1) \times (0,+\infty) \to (0,+\infty)$ *is continuous;*

(a2) $f(t,u)$ *is decreasing in* u *for each fixed* $t \in (0,1)$*;*

(a3) $\int_0^1 f(t,u)\,\mathrm{d}t < +\infty$ *for each fixed* $u \in (0,+\infty)$*;*

(a4) $\lim\limits_{u\to+0} \min\limits_{t\in W} f(t,u) = +\infty$ *for each compact subset* $W \subset (0,1)$*;*

(a5) $\lim\limits_{u\to+\infty} \max\limits_{t\in W} f(t,u) = 0$ *for each compact subset* $W \subset (0,1)$*;*

(a6) *for each* $r > 0$, $\int_0^1 f(t, rq(t))\,\mathrm{d}t < +\infty$, *where*

$$q(t) = 2^k t^k, \quad 0 \leqslant t \leqslant \frac{1}{2};$$
$$q(t) = 2^{n-k}(1-t)^{n-k}, \quad \frac{1}{2} \leqslant t \leqslant 1.$$

Then problem (P) *has at least one positive solution* $u^* \in C[0,1]$ *and there exists* $\theta > 0$ *such that* $u^*(t) \geqslant \theta q(t)$, $0 \leqslant t \leqslant 1$.

In Theorem 1.1, $f(t,u)$ may be singular at $t = 0$, $t = 1$ and $u = 0$. This is an outstanding advantage. For the existence of a positive solution to the singular $(k, n-k)$ conjugate boundary value problem (P), Theorem 1.1 is a powerful tool.

The purpose of this paper is to improve Theorem 1.1 and prove a new existence theorem, that is, Theorem 3.1. In Theorem 3.1, the conditions (a1), (a2), (a4)–(a6) are relaxed. And since the condition (a3) can be derived from (a2) and (a6), we omit it. Particularly, Theorem 3.1 does not require that $f(t,u)$ be decreasing in u. Therefore, the improvement is essential. In Remark 4.1, we will show that Theorem 1.1 is a corollary of Theorem 3.1. Finally, we will illustrate that the improvement is true by Example 4.2.

In order to establish the main result we will apply the approximation theorem for completely continuous operators, the Guo-Krasnosel'skii fixed point theorem of cone expansion-compression type and the localization method used in papers [9]–[14].

2. Preliminaries

Let $C[0,1]$ be Banach space with the norm $\|u\| = \max\limits_{0\leqslant t\leqslant 1} |u(t)|$. Let $p(t) = \min\{t^k, (1-t)^{n-k}\}$ and let

$$K = \{u \in C[0,1]\colon\ u(t) \geqslant \|u\|\, p(t),\ 0 \leqslant t \leqslant 1\}.$$

Then K is a cone of nonnegative functions in $C[0,1]$. Write

$$\begin{aligned} K(r) &= \{u \in K\colon\ \|u\| < r\},\\ \partial K(r) &= \{u \in K\colon\ \|u\| = r\},\\ K[r_1, r_2] &= \{u \in K\colon\ r_1 \leqslant \|u\| \leqslant r_2\}. \end{aligned}$$

Let $G(t,s)$ be the Green function of the problem (P) when $f(t,u) \equiv 0$. According to [7], the Green function $G(t,s)$ has the exact expression

$$G(t,s) = \begin{cases} \dfrac{\int_0^{t(1-s)} \tau^{k-1}(\tau + s - t)^{n-k-1}\,\mathrm{d}\tau}{(k-1)!(n-k-1)!}, & 0 \leqslant t \leqslant s \leqslant 1,\\ \dfrac{\int_0^{s(1-t)} \tau^{n-k-1}(\tau + t - s)^{k-1}\,\mathrm{d}\tau}{(k-1)!(n-k-1)!}, & 0 \leqslant s \leqslant t \leqslant 1. \end{cases}$$

So $G\colon\ [0,1] \times [0,1] \to [0,+\infty)$ is continuous and $G(t,s) > 0$, $0 < t, s < 1$.

Lemma 2.1. $G(t,s) \leqslant k^k(n-k)^{n-k}/n^n(k-1)!(n-k-1)!$, $0 \leqslant t, s \leqslant 1$.

P r o o f. By Lemma 1 in [7] we have

$$G(t,s) \leqslant \frac{s^{n-k}(1-s)^k}{(k-1)!(n-k-1)!}, \quad 0 \leqslant t, s \leqslant 1.$$

Simple computations give that

$$\max_{0\leqslant s\leqslant 1} s^{n-k}(1-s)^k = \left(\frac{n-k}{n}\right)^{n-k} \left(1 - \frac{n-k}{n}\right)^k = \frac{k^k(n-k)^{n-k}}{n^n}.$$

The conclusion is derived directly from these facts. □

By Theorem 4.1 in [4] we have

Lemma 2.2. *Assume that $u \in C^{(n-1)}[0,1] \cap C^{(n)}(0,1)$ is such that*

$$\begin{cases} (-1)^{n-k} u^{(n)}(t) \geqslant 0, & 0 \leqslant t \leqslant 1, \\ u^{(i)}(0) = 0, & 0 \leqslant i \leqslant k-1, \\ u^{(j)}(1) = 0, & 0 \leqslant j \leqslant n-k-1. \end{cases}$$

Then $u(t) \geqslant \|u\| p(t)$, $0 \leqslant t \leqslant 1$.

In order to prove the main result, we need the following approximation theorem for completely continuous operators and the Guo-Krasnosel'skii fixed point theorem of cone expansion-compression type.

Lemma 2.3. *Let X, Y be two Banach spaces, let $V \subset X$ be a closed bounded set, let $T_m\colon V \to Y$ be a completely continuous operator for each m, let an operator $T\colon V \to Y$ be given. If $\sup\limits_{u \in V} \|T_m u - Tu\| \to 0$, then $T\colon V \to Y$ is a completely continuous operator.*

Lemma 2.4. *Let X be a Banach space, let K be a cone in X, let Ω_1, Ω_2 be two bounded open subsets in K such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $T\colon \overline{\Omega}_2 \setminus \Omega_1 \to K$ be a completely continuous operator. Assume that one of the following conditions is satisfied:*

(1) $\|Tx\| \leqslant \|x\|$, $x \in \partial\Omega_1$ *and* $\|Tx\| \geqslant \|x\|$, $x \in \partial\Omega_2$,
(2) $\|Tx\| \geqslant \|x\|$, $x \in \partial\Omega_1$ *and* $\|Tx\| \leqslant \|x\|$, $x \in \partial\Omega_2$.

Then T has a fixed point in $\overline{\Omega}_2 \setminus \Omega_1$.

3. Main result

We obtain the following existence theorem for a positive solution.

Theorem 3.1. *Assume that*

(b1) *$h\colon (0,1) \to [0,+\infty)$, $f\colon (0,1) \times (0,+\infty) \to [0,+\infty)$ are continuous and $0 < \int_0^1 h(t)\,\mathrm{d}t < +\infty$;*

(b2) *there exist functions $\varphi(t,u)$ and $g(t,u)$ such that*

$$f(t,u) \leqslant \varphi(t,u) + g(t,u), \quad (t,u) \in (0,1) \times (0,+\infty),$$

where $\varphi\colon (0,1) \times (0,+\infty) \to [0,+\infty)$ and $g\colon [0,1] \times [0,+\infty) \to [0,+\infty)$ are continuous, $\varphi(t,\cdot)\colon (0,+\infty) \to [0,+\infty)$ is nonincreasing for each fixed $t \in (0,1)$;

(b3) *$\int_0^1 h(t)\varphi(t, rp(t))\,\mathrm{d}t < +\infty$ for any $r > 0$;*

(b4) $\liminf_{r\to+\infty} j(r)/r < (n^n(k-1)!(n-k-1)!/k^k(n-k)^{n-k})\bigl[\int_0^1 h(t)\,\mathrm{d}t\bigr]^{-1}$, *where*

$$j(r) = \max\{g(t,u)\colon\ (t,u)\in[0,1]\times[0,r]\};$$

(b5) *there exist* $0\leqslant\alpha<\beta\leqslant 1$ *such that* $\liminf_{u\to+0}\min_{\alpha\leqslant t\leqslant\beta} h(t)f(t,u)>0$.

Then problem (P) *has at least one positive solution* $u^*\in K$.

P r o o f. Define an operator T by

$$(Tu)(t) = \int_0^1 G(t,s)h(s)f(s,u(s))\,\mathrm{d}s,\quad 0\leqslant t\leqslant 1,\ u\in K\setminus\{0\}.$$

S t e p I. We prove that $T\colon K[r_1,r_2]\to K$ for any $0<r_1<r_2$.

Let $u\in K[r_1,r_2]$, then $r_1p(t)\leqslant u(t)\leqslant r_2$, $0\leqslant t\leqslant 1$. Applying Lemma 2.1 and the conditions (b1)–(b4), we get that

$$\begin{aligned}(Tu)(t) &\leqslant \frac{k^k(n-k)^{n-k}\int_0^1 h(s)[\varphi(s,u(s))+g(s,u(s))]\,\mathrm{d}s}{n^n(k-1)!(n-k-1)!}\\ &\leqslant \frac{k^k(n-k)^{n-k}\int_0^1 h(s)\varphi(s,r_1p(s))\,\mathrm{d}s}{n^n(k-1)!(n-k-1)!}+\frac{k^k(n-k)^{n-k}j(r_2)\int_0^1 h(s)\,\mathrm{d}s}{n^n(k-1)!(n-k-1)!}.\end{aligned}$$

Therefore, $\|Tu\| = \max_{0\leqslant t\leqslant 1}(Tu)(t)<+\infty$ and Tu is well defined for any $u\in K[r_1,r_2]$.

For fixed $u\in K[r_1,r_2]$ consider the $(k,n-k)$ boundary value problem

$$\begin{cases}(-1)^{n-k}w^{(n)}(t) = h(t)f(t,u(t)), & 0<t<1,\\ w^{(i)}(0)=0, & 0\leqslant i\leqslant k-1,\\ w^{(j)}(1)=0, & 0\leqslant j\leqslant n-k-1.\end{cases}$$

By (b2) and (b3), $h(\cdot)f(\cdot,u(\cdot))\in L^1[0,1]$. By the property of the Green function $G(t,s)$, $w(t)$ has the unique expression

$$w(t) = \int_0^1 G(t,s)h(s)f(s,u(s))\,\mathrm{d}s = (Tu)(t),\quad 0\leqslant t\leqslant 1.$$

Therefore,

$$\begin{cases}(-1)^{n-k}(Tu)^{(n)}(t) = h(t)f(t,u(t))\geqslant 0, & 0<t<1,\\ (Tu)^{(i)}(0)=0, & 0\leqslant i\leqslant k-1,\\ (Tu)^{(j)}(1)=0, & 0\leqslant j\leqslant n-k-1.\end{cases}$$

By Lemma 2.2, $(Tu)(t)\geqslant\|Tu\|p(t)$, $0\leqslant t\leqslant 1$ and $Tu\in K$.

S t e p II. We construct a sequence $\{T_m\}_{m=1}^{\infty}$ of completely continuous operators in order to approximate the operator T.

Define functions f_m as follows:

$$f_m(t,u) = \begin{cases} \min\limits_{u \leqslant v \leqslant 1/m} f(t,v), & 0 \leqslant u \leqslant 1/m, \\ f(t,u), & 1/m \leqslant u < +\infty. \end{cases}$$

Then $0 \leqslant f_m(t,u) \leqslant f(t,u)$, $(t,u) \in (0,1) \times [0,+\infty)$. The function $h(t)f_m(t,u)$ has the following properties:

(p1) For each fixed $t \in (0,1)$, $h(t)f_m(t,\cdot)\colon [0,+\infty) \to [0,+\infty)$ is continuous.

(p2) For each fixed $u \in [0,+\infty)$, $h(\cdot)f_m(\cdot,u)\colon (0,1) \to [0,+\infty)$ is lower semicontinuous. Consequently, $h(\cdot)f_m(\cdot,u)\colon (0,1) \to [0,+\infty)$ is measurable.

(p3) For any $r > 0$ and $(t,u) \in (0,1) \times [0,r]$,

$$h(t)f_m(t,u) \leqslant h(t)f(t,u) \leqslant h(t)\left[\varphi\left(t, \frac{1}{m}p(t)\right) + \max\left\{j\left(\frac{1}{m}\right), j(r)\right\}\right].$$

For $u \in K$ and $0 \leqslant t \leqslant 1$, define operators T_m, A_m and B as follows:

$$\begin{aligned} (T_m u)(t) &= \int_0^1 G(t,s)h(s)f_m(s,u(s))\,\mathrm{d}s, \\ (A_m u)(t) &= h(t)f_m(t,u(t)), \\ (Bu)(t) &= \int_0^1 G(t,s)u(s)\,\mathrm{d}s. \end{aligned}$$

Then $T_m = B \circ A_m$.

Let $u_i, u_0 \in K$, $i = 1,2,\ldots$ and $\|u_i - u_0\| \to 0$. Then $\max\limits_{0 \leqslant t \leqslant 1} |u_i(t) - u_0(t)| \to 0$. By the property (p1), we have

$$|h(t)f_m(t,u_i(t)) - h(t)f_m(t,u_0(t))| \to 0 \ (i \to \infty), \quad 0 < t < 1.$$

Let $\bar{r} = \max\{\|u_i\|\colon i = 1,2,\ldots\}$. Then $0 \leqslant u_i(t) \leqslant \bar{r}$, $t \in [0,1]$, $i = 1,2,\ldots$. By (p3), we have

$$h(t)f_m(t,u_i(t)) \leqslant h(t)\left[\varphi\left(t, \frac{1}{m}p(t)\right) + \max\left\{j\left(\frac{1}{m}\right), j(\bar{r})\right\}\right], \quad 0 < t < 1.$$

Here $h(t)\left[\varphi\left(t, m^{-1}p(t)\right) + \max\left\{j\left(1/m\right), j(\bar{r})\right\}\right]$ is a nonnegative integrable function on $[0,1]$ by the conditions (b1)–(b4). By the Lebesgue dominated convergence theorem (see [15]), we get that

$$\begin{aligned}&\lim_{i\to\infty}\int_0^1 |h(t)f_m(t,u_i(t)) - h(t)f_m(t,u_0(t))|\,\mathrm{d}t\\&= \int_0^1 \lim_{i\to\infty} |h(t)f_m(t,u_i(t)) - h(t)f_m(t,u_0(t))|\,\mathrm{d}t = 0.\end{aligned}$$

This implies that $A_m\colon K \to L^1[0,1]$ is continuous.

Applying the Arzela-Ascoli theorem, we can prove that $B\colon L^1[0,1] \to C[0,1]$ is completely continuous. Imitating the proof in Step I, we have $T_m\colon K \to K$. Therefore, $T_m\colon K \to K$ is completely continuous.

Step III. We prove that $T\colon K[r_1,r_2] \to K$ is completely continuous for any $0 < r_1 < r_2$.

Let $E(rp,m) = \{t \in [0,1]\colon rp(t) \leqslant 1/m\}$ for $r > 0$. If $mr > 2^{\max\{k,n-k\}}$, then

$$E(rq,m) = \left[0, \frac{1}{\sqrt[k]{mr}}\right] \cup \left[1 - \frac{1}{\sqrt[n-k]{mr}}, 1\right].$$

Consequently, $E(rq,m) \to \{0,1\}$, $m \to \infty$.

By (b3), $\int_0^1 h(t)\varphi(t,rp(t))\,\mathrm{d}t < +\infty$. From the condition (b1), $\int_0^1 h(t)\,\mathrm{d}t < +\infty$. By the absolute continuity of the integral (see [15]), we have

$$\lim_{m\to\infty}\int_{E(rp,m)} h(t)\varphi(t,rp(t))\,\mathrm{d}t = 0,$$
$$\lim_{m\to\infty}\int_{E(rp,m)} h(t)\,\mathrm{d}t = 0.$$

Let $u \in K[r_1,r_2]$ and let $E(u,m) = \{t \in [0,1]\colon u(t) \leqslant 1/m\}$. Since $r_1p(t) \leqslant u(t) \leqslant r_2$, $0 \leqslant t \leqslant 1$, we have

$$E(u,m) \subset E(r_1p,m),$$
$$\varphi(t,u(t)) \leqslant \varphi(t,r_1p(t)), \qquad g(t,u(t)) \leqslant j(r_2).$$

Since $f(t,u) \geqslant f_m(t,u), (t,u) \in (0,1) \times (0,+\infty)$, we obtain that

$$\begin{aligned}
&\sup_{u\in K[r_1,r_2]} \|Tu - T_m u\| = \sup_{u\in K[r_1,r_2]} \max_{0\leqslant t\leqslant 1} [(Tu)(t) - (T_m u)(t)] \\
&= \sup_{u\in K[r_1,r_2]} \max_{0\leqslant t\leqslant 1} \int_0^1 G(t,s)h(s)[f(s,u(s)) - f_m(s,u(s))]\,\mathrm{d}s \\
&\leqslant \sup_{u\in K[r_1,r_2]} \max_{0\leqslant t\leqslant 1} \int_{E(u,m)} G(t,s)h(s)f(s,u(s))\,\mathrm{d}s \\
&\leqslant \sup_{u\in K[r_1,r_2]} \max_{0\leqslant t\leqslant 1} \int_{E(r_1 p,m)} G(t,s)h(s)f(s,u(s))\,\mathrm{d}s \\
&\leqslant \sup_{u\in K[r_1,r_2]} \max_{0\leqslant t\leqslant 1} \int_{E(r_1 p,m)} G(t,s)h(s)[\varphi(s,u(s)) + g(s,u(s))]\,\mathrm{d}s \\
&\leqslant \frac{k^k(n-k)^{n-k}}{n^n(k-1)!(n-k-1)!} \Big[\int_{E(r_1 p,m)} h(s)\varphi(s,r_1 p(s))\,\mathrm{d}s + j(r_2)\int_{E(r_1 p,m)} h(s)\,\mathrm{d}s\Big] \\
&\quad \to 0 \quad (m\to\infty).
\end{aligned}$$

By Lemma 2.3, $T\colon\ K[r_1,r_2] \to K$ is a completely continuous operator.

Step IV. We prove that the problem (P) has a positive solution $u^* \in K$.

Let $\varepsilon = \frac{1}{2}\big[n^n(k-1)!(n-k-1)!/k^k(n-k)^{n-k}\int_0^1 h(t)\,\mathrm{d}t - \liminf\limits_{r\to+\infty} j(r)/r\big]$. By (b4), $\varepsilon > 0$.

Let $\eta = \max\limits_{0\leqslant t\leqslant 1}\int_\alpha^\beta G(t,s)h(s)\,\mathrm{d}s$. By the inequality $G(t,s) > 0$, $0 < t,s < 1$ and the condition (b5), $\eta > 0$ and there exists $r_0 > 0$, $\gamma > 0$ such that $f(t,u) \geqslant \gamma$, $(t,u) \in [\alpha,\beta] \times (0,r_0]$.

Let $\bar r_1 = \min\{r_0, \gamma\eta\}$. If $u \in \partial K(\bar r_1)$, then $0 \leqslant u(t) \leqslant \bar r_1 \leqslant r_0$, $0 \leqslant t \leqslant 1$ and $f(t,u(t)) \geqslant \gamma$, $t \in [\alpha,\beta]$. It follows that

$$\begin{aligned}
\|Tu\| &\geqslant \max_{0\leqslant t\leqslant 1} \int_\alpha^\beta G(t,s)h(s)f(s,u(s))\,\mathrm{d}s \\
&\geqslant \gamma \max_{0\leqslant t\leqslant 1} \int_\alpha^\beta G(t,s)h(s)\,\mathrm{d}s = \gamma\eta \geqslant \bar r_1 = \|u\|.
\end{aligned}$$

On the other hand, if $r \geqslant 1$, then $\varphi(s,rp(s)) \leqslant \varphi(s,p(s))$ and

$$\begin{aligned}
&\lim_{r\to+\infty} \frac{1}{r} \sup_{u\in\partial K(r)} \max_{0\leqslant t\leqslant 1} \int_0^1 G(t,s)h(s)\varphi(s,u(s))\,\mathrm{d}s \\
&\leqslant \lim_{r\to+\infty} \frac{k^k(n-k)^{n-k}}{rn^n(k-1)!(n-k-1)!} \int_0^1 h(s)\varphi(s,rp(s))\,\mathrm{d}s \\
&\leqslant \lim_{r\to+\infty} \frac{k^k(n-k)^{n-k}}{rn^n(k-1)!(n-k-1)!} \int_0^1 h(s)\varphi(s,p(s))\,\mathrm{d}s = 0.
\end{aligned}$$

So, there exists $\bar r_2 > \max\{1, r_0\}$ such that

$$\sup_{u\in\partial K(\bar r_2)} \max_{0\leqslant t\leqslant 1} \int_0^1 G(t,s)h(s)\varphi(s,u(s))\,\mathrm{d}s \leqslant \frac{k^k(n-k)^{n-k}}{n^n(k-1)!(n-k-1)!}\,\varepsilon\bar r_2,$$

$$j(\bar r_2) \leqslant \left(\frac{n^n(k-1)!(n-k-1)!}{k^k(n-k)^{n-k}} - \varepsilon\right)\left[\int_0^1 h(t)\,\mathrm{d}t\right]^{-1} \bar r_2.$$

Consequently, $j(r_2)\int_0^1 h(t)\,\mathrm{d}t \leqslant (n^n(k-1)!(n-k-1)!/k^k(n-k)^{n-k} - \varepsilon)\bar r_2$.

If $u \in \partial K(\bar r_2)$, then

$$\begin{aligned}
\|Tu\| &\leqslant \max_{0\leqslant t\leqslant 1}\int_0^1 G(t,s)h(s)[\varphi(s,u(s)) + g(s,u(s))]\,\mathrm{d}s\\
&\leqslant \max_{0\leqslant t\leqslant 1}\int_0^1 G(t,s)h(s)\varphi(s,u(s))\,\mathrm{d}s + \max_{0\leqslant t,s\leqslant 1} G(t,s)j(\bar r_2)\int_0^1 h(s)\,\mathrm{d}s\\
&\leqslant \max_{0\leqslant t\leqslant 1}\int_0^1 G(t,s)h(s)\varphi(s,u(s))\,\mathrm{d}s + \frac{k^k(n-k)^{n-k}}{n^n(k-1)!(n-k-1)!} j(\bar r_2)\int_0^1 h(s)\,\mathrm{d}s\\
&\leqslant \frac{k^k(n-k)^{n-k}}{n^n(k-1)!(n-k-1)!}\,\varepsilon\bar r_2\\
&\quad + \frac{k^k(n-k)^{n-k}}{n^n(k-1)!(n-k-1)!}\left(\frac{n^n(k-1)!(n-k-1)!}{k^k(n-k)^{n-k}} - \varepsilon\right)\bar r_2\\
&= \bar r_2 = \|u\|\,.
\end{aligned}$$

By Lemma 2.4 there exists $u^* \in K[\bar r_1, \bar r_2] = \overline{K(\bar r_2)} \setminus K(\bar r_1)$ such that $Tu^* = u^*$. By the equivalence between the integral equation $Tu = u$ and the problem (P), u^* is a solution of (P). Since $u^*(t) \geqslant \bar r_1 p(t) > 0$, $0 < t < 1$, u^* is a positive solution. □

4. Remark and example

Remark 4.1. Theorem 1.1 is a simple corollary of Theorem 3.1.

Assume that the conditions (a1)–(a5) are satisfied. Then we have $h(t) \equiv 1$. In Theorem 3.1, let $g(t,u) \equiv 0$, $\varphi(t,u) = f(t,u)$. It is clear that the conditions (b1), (b2) and (b4) are satisfied. Moreover, the condition (b5) can be derived from (a4). Since $2^{-\max\{k,n-k\}}q(t) \leqslant p(t) \leqslant q(t)$, we have

$$\min\{2^k, 2^{n-k}\}p(t) \leqslant q(t) \leqslant \max\{2^k, 2^{n-k}\}p(t), \quad 0 \leqslant t \leqslant 1.$$

This shows that the condition (a6) is equivalent to (b3) with $h(t) \equiv 1$. Therefore, we can prove Theorem 1.1 by applying Theorem 3.1.

E x a m p l e 4.2. The example illustrates that Theorem 3.1 improves Theorem 1.1 even if $h(t) \equiv 1$.

Consider the $(2, 4-2)$ conjugate boundary value problem

$$\begin{cases} u^{(4)}(t) = \sqrt{|1-2t|u(t)} + \dfrac{|1-2t|(1+\sin(u(t)))}{\sqrt[3]{u(t)}}, \quad 0 < t < 1, \\ u(0) = u'(0) = u(1) = u'(1) = 0. \end{cases}$$

In this example $n = 4$, $k = 2$, $p(t) = \min\{t^2, (1-t)^2\}$, $n^n(k-1)!(n-k-1)!/k^k(n-k)^{n-k} = 16$, $h(t) \equiv 1$, $f(t,u) = \sqrt{|1-2t|u} + |1-2t|(1+\sin u)/\sqrt[3]{u}$.

Let $g(t,u) = \sqrt{|1-2t|u}$. Then

$$j(r) = \max\{\sqrt{|1-2t|u}\colon (t,u) \in [0,1] \times [0,r]\} = \sqrt{r}.$$

So $\lim\limits_{r\to+\infty} j(r)/r = 0$. Let $\varphi(t,u) = 2/\sqrt[3]{u}$. Then $f(t,u) \leqslant g(t,u) + \varphi(t,u)$ and $\varphi(t,u)$ is nonincreasing in u for each fixed $t \in [0,1]$. Moreover, for any $r > 0$,

$$\int_0^1 \varphi(t, rp(t))\,\mathrm{d}t \leqslant \int_0^1 \frac{2\,\mathrm{d}t}{\sqrt[3]{rp(t)}} = \frac{2}{\sqrt[3]{r}} \int_0^1 \frac{\mathrm{d}t}{\sqrt[3]{[\min\{t,(1-t)\}]^2}} < +\infty.$$

Therefore, the conditions (b1)–(b5) are satisfied. By Theorem 3.1, the problem has a positive solution $u^* \in K$.

However, $f(t,u)$ is not decreasing in u and the conditions (a4) and (a5) are not satisfied. The existence conclusion cannot be derived from Theorem 1.1.

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