ULTIMATE BOUNDEDNESS OF SOME THIRD ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. We prove the ultimate boundedness of solutions of some third order nonlinear ordinary differential equations using the Lyapunov method. The results obtained generalize earlier results of Ezeilo, Tejumola, Reissig, Tunç and others. The Lyapunov function used does not involve the use of signum functions as used by others.

Keywords: ultimate boundedness, complete Lyapunov function, differential equation of third-order

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1. Introduction

Motivation for this paper comes from the generalization of the works by Reissig [18] and Tejumola [21] by Ezeilo in [13], the recent works of Afuwape and Omeike [6] and Ademola et al [1].

In exciting work, Ezeilo [13] investigated the equation of the form

$$\dddot{x} + \{\varphi_1(\dot{x}) + \varphi_2(x, \dot{x})\}\ddot{x} + g_1(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x})$$

for ultimate boundedness generalizing the works of Reissig [18] on

$$\dddot{x} + \varphi_2(x, \dot{x})\ddot{x} + g_1(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x})$$

and that of Tejumola [21] on

$$\dddot{x} + \varphi_1(\dot{x})\ddot{x} + g(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}).$$

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We shall consider here the equation

$$\dddot{x} + \{f_1(\ddot{x}) + f_2(\dot{x}, \ddot{x})\} + g(x, \dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}) \tag{1.1}$$

where f_1, f_2, g, h, p depend on the arguments displayed. Our assumptions on f_1, f_2, g, h, and p shall allow us to generalize the results of Ademola et al [1], Afuwape [2], [3] and a particular case of Afuwape and Omeike [4] concerning

$$\dddot{x} + f_1(\ddot{x}) + g_1(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x});$$

and Qian [17] and a particular case of Tunç [22] concerning

$$\dddot{x} + f_2(x, \dot{x})\ddot{x} + g(x, \dot{x}) = p(t).$$

The assumptions will also give us an opportunity to discuss the ultimate boundedness results which generalize the earlier ones. A good record of ultimate boundedness results of these types is recorded in the book [19], and the papers Hara [16], Afuwape and Omeike [6] and references therein. Also, the recent excellent book of Hadddad [15] includes a good summary of the theoretical works on the subject.

Consider the third order nonlinear ordinary differential equation of the form (1.1), or its equivalent system form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -\{f_1(z) + f_2(y, z)\} - g(x, y) - h(x) + p(t, x, y, z), \end{aligned} \tag{1.2}$$

where f_1, f_2, g, h and p are continuous in their respective arguments, and the dots denote differentiation with respect to t.

The object of this paper is to discuss the ultimate boundedness of solutions of Eq. (1.1). It is well known that the ultimate boundedness is a very important problem in the theory and applications of differential equations. An effective method for studying the ultimate boundedness of nonlinear differential equations is still Lyapunov's direct method (see [1]–[11]). In [13], incomplete Lyapunov functions augmented with signum functions and with certain restrictive conditions on the nonlinear functions were used. Our aim in this paper is to study a more general Eq. (1.1) for ultimate boundedness of solutions, using a complete Lyapunov function with less restrictive conditions on the nonlinear functions f_1, f_2, g, h and p. In the process, we shall be able to generalize earlier results of [1], [2], [3] and a particular case of [4], [17], [22].

2. Main results

Our main result is the following theorem.

Theorem 2.1. *In addition to the basic assumptions on the functions f_1, f_2, g, h and p, assume that the following conditions are satisfied (a, b, c, ν and A being some positive constants):*

(i) $(f_1(z) + f_2(y,z))/z \geqslant a$ *for all* $y, z \neq 0$,

(ii) $g(x,y)/y \geqslant b$ *for all* $x, y \neq 0$,

(iii) $h(x)/x \geqslant \nu$ *for all* $x \neq 0$,

(iv) $h'(x) \leqslant c$,

(v) $ab > c$,

(vi) $|p(t,x,y,z)| \leqslant A < \infty$ *for all* $t \geqslant 0$ *and for all* x, y, z.

Then every solution $x(t)$ *of* (1.1) *ultimately satisfies*

$$|x(t)| \leqslant D, \quad |\dot{x}(t)| \leqslant D, \quad |\ddot{x}(t)| \leqslant D \tag{2.1}$$

where D is a constant depending only on a, b, c, ν and A.

Remark 2.1. Theorem 2.1 generalizes the results of Ademola et al [1], if we set $f_2(\dot{x},\ddot{x}) = 0$.

Remark 2.2. In using Lyapunov's theory, Theorem 2.1 gives a different method of discussing the works of Afuwape [2], [3] who used the frequency domain methods, with $f_2(\dot{x},\ddot{x}) \equiv 0$.

Remark 2.3. Theorem 2.1 generalizes the results of Qian [17] and Tunç [22] if $f_1(\ddot{x}) \equiv 0$. This becomes obvious if we carry out some differentiations of $f_2(\dot{x},\ddot{x})$ and $g(x,\dot{x})$ with respect to their variables $x, \dot{x}, \ddot{x}$ to obtain the equivalent equation to that of [17] and [22]. However, the $p(t)$ will be replaced by $p(t,x,y,z)$, with the appropriate conditions on it.

3. Preliminaries

It is convenient here to consider, in place of Eq. (1.1), the system (1.2). In order to prove Theorem 2.1, we need to show that every solution $(x(t), y(t), z(t))$ of (1.2) satisfies

$$|x(t)| \leqslant D, \quad |y(t)| \leqslant D, \quad |z(t)| \leqslant D \tag{3.1}$$

for all sufficiently large t, where D is a suitable constant. Set $(x,y,z) \equiv (x(t), y(t), z(t))$.

Our proof of (3.1) rests entirely on two properties (stated in the lemma below) of the function $V(t) \equiv V(x,y,z)$ defined by

(3.2) $$2V(x,y,z) = \beta(1-\beta)b^2x^2 + b(\beta+\alpha a^{-1})y^2 + \alpha a^{-1}z^2 + [z + ay + (1-\beta)bx]^2$$

with $0 < \beta < 1$, and $\alpha > 0$.

Lemma 3.1. *Subject to the conditions of Theorem* 2.1, $V(0,0,0) = 0$ *and there is a positive constant* D_1 *depending only on* a, b, c, α *and* δ *such that*

(3.3) $$V(t) \equiv V(x,y,z) \geqslant D_1(x^2+y^2+z^2)$$

for all x, y, z.

Let us set $V(t) \equiv V(x(t), y(t), z(t))$.

Furthermore, there are finite constants D_2, D_3 *dependent only on* a, b, c, A, ν, δ *and* α *such that for any solution* $(x(t), y(t), z(t))$ *of* (1.2),

(3.4) $$\frac{\mathrm{d}}{\mathrm{d}t}V \equiv \frac{\mathrm{d}}{\mathrm{d}t}V(x(t),y(t),z(t)) \leqslant -D_2$$

provided that $x^2+y^2+z^2 \geqslant D_3$.

P r o o f of Lemma 3.1. Clearly, $V(0,0,0) = 0$. Also, by rearranging (3.2) and choosing

$$D_1 \geqslant \min\{\beta(1-\beta)b^2; b(\beta+\alpha a^{-1}); \alpha a^{-1}\}$$

we have (3.3).

To prove (3.4), we find that the derivative of V with respect to t along the solution path of (1.2), (after simplifications) gives

(3.5) $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V(t)|_{(1.2)} = {} & -b(1-\beta)xh(x) - \{ab\beta y^2 + a[yg(x,y) - by^2]\}\\ & - \{(\alpha a^{-1}+1)z(f_1+f_2) - az^2\}\\ & + \{b^2(1-\beta)xy - ayh(x) - b(1-\beta)xg(x,y)\}\\ & + \{[b(\alpha a^{-1}+1) + a^2]yz - (\alpha a^{-1}+1)zg(x,y) - ay(f_1+f_2)\}\\ & + \{ab(1-\beta)xz - (\alpha a^{-1}+1)zh(x) - b(1-\beta)x(f_1+f_2)\}\\ & + \{[b(1-\beta)x + ay + (\alpha a^{-1}+1)z]p(t,x,y,z)\}.\end{aligned}$$

A rearrangement of this shows that for $x \neq 0$; $y \neq 0$; $z \neq 0$ we have

(3.6) $$\frac{\mathrm{d}}{\mathrm{d}t}V(t)|_{(1.2)} = -W_1 - W_2 - W_3 - W_4 - W_5 - W_6 - W_7 + W_p$$

where

$$
\begin{aligned}
W_1 &= \gamma_1 b(1-\beta)x^2\Big[\frac{h(x)}{x}\Big] + \Big\{\delta_1 a\Big[\frac{g(x,y)}{y} - b\Big]y^2 + \eta_1 ab\beta y^2\Big\} \\
&\quad + \Big\{\xi_1(\alpha a^{-1}+1)\Big[\frac{f_1+f_2}{z} - a\Big]z^2 + \mu_1\alpha z^2\Big\}, \\
W_2 &= \Big\{\gamma_2 b(1-\beta)x^2\Big[\frac{h(x)}{x}\Big] + axy\Big[\frac{h(x)}{x}\Big] + \eta_2 ab\beta y^2\Big\}, \\
W_3 &= \Big\{\gamma_3 b(1-\beta)x^2\Big[\frac{h(x)}{x}\Big] + b(1-\beta)\Big[\frac{g(x,y)}{y} - b\Big]xy + \delta_2 a\Big[\frac{g(x,y)}{y} - b\Big]y^2\Big\}, \\
W_4 &= \Big\{\delta_3 a\Big[\frac{g(x,y)}{y} - b\Big]y^2 + (\alpha a^{-1}+1)\Big[\frac{g(x,y)}{y} - b\Big]yz + \mu_2\alpha z^2\Big\}, \\
W_5 &= \Big\{\eta_3 ab\beta y^2 + a\Big[\frac{f_1+f_2}{z} - a\Big]yz + \xi_2(\alpha a^{-1}+1)\Big[\frac{f_1+f_2}{z} - a\Big]z^2\Big\}, \\
W_6 &= \Big\{\gamma_4 b(1-\beta)x^2\Big[\frac{h(x)}{x}\Big] + b(1-\beta)\Big[\frac{f_1+f_2}{z} - a\Big]xz \\
&\quad + \xi_2(\alpha a^{-1}+1)\Big[\frac{f_1+f_2}{z} - a\Big]z^2\Big\}, \\
W_7 &= \Big\{\gamma_5 b(1-\beta)x^2\Big[\frac{h(x)}{x}\Big] + (\alpha a^{-1}+1)xz\Big[\frac{h(x)}{x}\Big] + \mu_3\alpha z^2\Big\}, \\
W_p &= \{[b(1-\beta)x + ay + (\alpha a^{-1}+1)z]p(t,x,y,z)\}
\end{aligned}
$$

with

$$
\sum_{i=1}^{5}\gamma_i = 1;\quad \sum_{i=1}^{3}\delta_i = 1;\quad \sum_{i=1}^{3}\eta_i = 1;\quad \sum_{i=1}^{3}\xi_i = 1;
$$
$$
\gamma_i > 0,\quad \delta_j > 0,\quad \eta_j > 0,\quad \xi_j > 0,\quad i = 1,2,3,4,5;\quad j = 1,2,3.
$$

We note that for any two real numbers, u, v, and for $s \in \{-1,+1\}$ there exists a constant $k > 0$ such that

$$
(su)v = \Big(ksu + \frac{1}{2k}v\Big)^2 - \Big(k^2u^2 + \frac{1}{4k^2}v^2\Big) > -\Big(k^2u^2 + \frac{1}{4k^2}v^2\Big).
$$

Moreover, this inequality is retained when multiplied by any positive term.

By virtue of the conditions of the theorem on the nonlinear functions, we have a constant $k_2 > 0$ such that

$$
\begin{aligned}
W_2 &= \Big\{\gamma_2 b(1-\beta)x^2\Big[\frac{h(x)}{x}\Big] + axy\Big[\frac{h(x)}{x}\Big] + \eta_2 ab\beta y^2\Big\} \\
&\geqslant \{\gamma_2 b(1-\beta)\nu x^2 + a\nu xy + \eta_2 ab\beta y^2\} \\
&\geqslant \Big\{\nu[\gamma_2 b(1-\beta) - k_2^2 a]x^2 + a\Big[\eta_2 b\beta - \frac{\nu}{4k_2^2}\Big]y^2\Big\} \\
&\geqslant 0,\quad \forall x,y,z \text{ whenever } \frac{\nu}{4\eta_2 b\beta} \leqslant k_2^2 < \frac{\gamma_2 b(1-\beta)}{a}.
\end{aligned}
$$

In a similar manner, we have constants $k_i > 0$, $i = 3, 4, \dots, 7$, such that

$$W_3 \geqslant \Big\{ b(1-\beta)\Big[\gamma_3\nu - k_3^2\Big[\frac{g(x,y)}{y} - b\Big]\Big]x^2 + \Big[\frac{g(x,y)}{y} - b\Big]\Big[\delta_2 a - \frac{b(1-\beta)}{4k_3^2}\Big]y^2\Big\}$$

$$\geqslant 0, \quad \forall x, y, z, \text{ whenever } \frac{b(1-\beta)}{4\delta_2 a} \leqslant k_3^2 < \frac{\gamma_3\nu}{[g(x,y)/y - b]};$$

$$W_4 \geqslant \Big\{ \Big[\frac{g(x,y)}{y} - b\Big][\delta_3 a - k_4^2(\alpha a^{-1} + 1)]y^2 + \Big[\mu_2\alpha - \frac{(\alpha a^{-1} + 1)[g(x,y)/y - b]}{4k_4^2}\Big]z^2\Big\}$$

$$\geqslant 0, \quad \forall x, y, z, \text{ whenever } \frac{(\alpha a^{-1} + 1)[g(x,y)/y - b]}{4\mu_2\alpha} \leqslant k_4^2 < \frac{\delta_3 a}{(\alpha a^{-1} + 1)};$$

$$W_5 \geqslant \Big\{ a\Big[\eta_3 b\beta - k_5^2\Big[\frac{f_1 + f_2}{z} - a\Big]\Big]y^2 + \Big[\frac{f_1 + f_2}{z} - a\Big]\Big[\xi_2(\alpha a^{-1} + 1) - \frac{a}{4k_5^2}\Big]z^2\Big\}$$

$$\geqslant 0, \quad \forall x, y, z, \text{ whenever } \frac{a}{4\xi_2(\alpha a^{-1} + 1)} \leqslant k_5^2 < \frac{\eta_3 b\beta}{[(f_1 + f_2)/z - a]};$$

$$W_6 \geqslant \Big\{ b(1-\beta)\Big[\gamma_4\nu - k_6^2\Big[\frac{f_1 + f_2}{z} - a\Big]\Big]x^2$$

$$+ \Big[\frac{f_1 + f_2}{z} - a\Big]\Big[\xi_3(\alpha a^{-1} + 1) - \frac{b(1-\beta)}{4k_6^2}\Big]z^2\Big\}$$

$$\geqslant 0, \quad \forall x, y, z, \text{ whenever } \frac{b(1-\beta)}{4\xi_3(\alpha a^{-1} + 1)} \leqslant k_6^2 < \frac{\gamma_4\nu}{[(f_1 + f_2)/z - a]}.$$

Moreover,

$$W_7 \geqslant \Big\{ \nu[\gamma_5 b(1-\beta) - k_7^2(\alpha a^{-1} + 1)]x^2 + \Big[\mu_3\alpha - \frac{(\alpha a^{-1} + 1)\nu}{4k_7^2}\Big]z^2\Big\}$$

$$\geqslant 0, \quad \forall x, y, z, \text{ whenever } \frac{(\alpha a^{-1} + 1)\nu}{4\mu_3\alpha} \leqslant k_6^2 < \frac{\gamma_5 b(1-\beta)}{(\alpha a^{-1} + 1)}.$$

Also, we have that

$$W_1 \geqslant \gamma_1 b(1-\beta)\nu x^2 + \delta_1 ab\beta y^2 + \mu_1\alpha z^2 \geqslant D_4(x^2 + y^2 + z^2)$$

where $0 < D_4 \leqslant \min\{\gamma_1 b(1-\beta)\nu; \delta_1 ab\beta; \mu_1\alpha\}$.

Moreover,

$$W_p \leqslant \{[b(1-\beta)|x| + a|y| + (\alpha a^{-1} + 1)|z|]|p(t,x,y,z)|\} \leqslant D_5(|x| + |y| + |z|)$$

where $D_5 = A\max\{b(1-\beta); a; (\alpha a^{-1} + 1)\}$.

Hence, using (3.6) we have

$$\dot V \leqslant -D_4(x^2 + y^2 + z^2) + D_6(x^2 + y^2 + z^2)^{\frac{1}{2}} \tag{3.7}$$

where $D_6 = 3^{\frac{1}{2}} D_5$.

If we choose $(x^2+y^2+z^2)^{\frac{1}{2}} \geqslant D_7 = 2D_6D_4^{-1}$, inequality (3.7) implies that

$$\dot{V} \leqslant -\frac{1}{2}D_4(x^2+y^2+z^2).$$

We see at once that

$$\dot{V} \leqslant -D_8,$$

provided that $x^2+y^2+z^2 \geqslant 2D_8D_4^{-1}$; and this completes the verification of (3.4), (with $D_2 \equiv D_8$).

R e m a r k 3.1. We note that in the work of Tunç [22], using the Lyapunov method only ended up with

$$\dot{V} \leqslant -\overline{D}(y^2+z^2) + \overline{\overline{D}}(|y|+|z|)|p(t)|$$

which gave an incomplete nature of the function.

4. Proof of theorem 2.1

Let $(x(t), y(t), z(t))$ be any solution of (1.2). Then there is evidently a $t_0 \geqslant 0$ such that

$$x^2(t_0) + y^2(t_0) + z^2(t_0) < D_3,$$

where D_3 is the constant in the lemma; for otherwise, that is if

$$x^2(t) + y^2(t) + z^2(t) \geqslant D_3, \quad t \geqslant 0,$$

then, by (3.4),

$$\dot{V}(t) \leqslant -D_2 < 0, \quad t \geqslant 0,$$

and this in turn implies that $V(t) \to -\infty$ as $t \to \infty$, which contradicts (3.3). Hence to prove (3.4) it will suffice to show that if

$$x^2(t) + y^2(t) + z^2(t) < D_9 \quad \text{for } t = T, \tag{4.1}$$

where $D_9 \geqslant D_3$ is a finite constant, then there is a constant $D_{10} > 0$, depending on a, b, c, δ, α, ξ and D_9, such that

$$x^2(t) + y^2(t) + z^2(t) \leqslant D_{10} \quad \text{for } t \geqslant T. \tag{4.2}$$

Our proof of (4.2) is based essentially on an extension of an argument in the proof of [8; Lemma 1]. For any given constant $d > 0$, let $S(d)$ denote the surface

$x^2+y^2+z^2 = d$. Because V is continuous in x, y, z and tends to $+\infty$ as $x^2+y^2+z^2 \to \infty$, there is evidently a constant $D_{11} > 0$, depending on D_9 as well as on a, b, c, δ, ξ and α, such that

$$\min_{(x,y,z)\in S(D_{11})} V(x, y, z) > \max_{(x,y,z)\in S(D_9)} V(x, y, z). \tag{4.3}$$

It is easy to see from (4.1) and (4.3) that

$$x^2(t) + y^2(t) + z^2(t) < D_{11} \quad \text{for } t \geqslant T. \tag{4.4}$$

For suppose on the contrary that there is a $t > T$ such that

$$x^2(t) + y^2(t) + z^2(t) \geqslant D_{11}.$$

Then, by (4.1) and by the continuity of the quantities $x(t), y(t)$, $z(t)$ in the argument displayed, there exist $t_1, t_2, T < t_1 < t_2$ such that

$$x^2(t_1) + y^2(t_1) + z^2(t_1) = D_9, \tag{4.5}$$

$$x^2(t_2) + y^2(t_2) + z^2(t_2) = D_{11} \tag{4.6}$$

and such that

$$D_9 \leqslant x^2(t) + y^2(t) + z^2(t) \leqslant D_{11}, \quad t_1 \leqslant t \leqslant t_2. \tag{4.7}$$

But, writing $V(t) \equiv V\left(x(t), y(t), z(t)\right)$, since $D_9 \geqslant D_3$, (4.7) obviously implies [in view of (3.4)] that

$$V(t_2) < V(t_1),$$

and this contradicts the conclusion [from (4.3) and (4.6)]

$$V(t_2) > V(t_1).$$

Hence (4.4) holds. This completes the proof of (3.4), and the theorem now follows. $\square$

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