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ALMOST $\tilde{g}_\alpha$-CLOSED FUNCTIONS AND SEPARATION AXIOMS

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Abstract. We introduce a new class of functions called almost $\tilde{g}_\alpha$-closed and use the functions to improve several preservation theorems of normality and regularity and also their generalizations. The main result of the paper is that normality and weak normality are preserved under almost $\tilde{g}_\alpha$-closed continuous surjections.

Keywords: topological space, $\tilde{g}$-closed set, $\tilde{g}_\alpha$-closed set, αg-closed set

MSC 2010: 54C10, 54C08, 54C05

1. Introduction

In topological spaces, it is well known that normality is preserved under closed continuous surjections. Many authors have tried to weaken the condition "closed" in this theorem. In 1978, Long and Herrington [12] used almost closedness due to Singal [33]. In 1982, Malghan [16] used g-closedness. In 1986, Greenwood and Reilly [6] used α-closedness due to Mashhour et al. [17]. In 1995, Yoshimura et al. [39] used almost g-closedness which is a generalization of both almost closedness and g-closedness. In 1999, Noiri [23] introduced almost αg-closedness using αg-closed sets [14]. Quite recently, Jafari et al. [8] have introduced the notion of $\tilde{g}_\alpha$-closed sets which are strictly weaker than both α-closed sets and $\tilde{g}$-closed sets. We use $\tilde{g}_\alpha$-closed sets to define a new class of functions called almost $\tilde{g}_\alpha$-closed functions. The purpose of the present paper is to improve preservation theorems of separation axioms, that is, normality, weak normality, mild normality, almost normality, regularity, almost regularity, quasi-regularity and strong s-regularity. The following properties are the main results of the present paper.

Theorem A. *Normality and weak normality are preserved under almost $\tilde{g}_\alpha$-closed continuous surjections.*

Theorem B. *Regularity and strong s-regularity are preserved under almost α-open almost $\tilde{g}_\alpha$-closed continuous surjections.*

2. Preliminaries

Throughout this paper (X, τ) and (Y, σ) (or X and Y) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), $\mathrm{cl}(A)$, $\mathrm{int}(A)$ and A^c denote the closure of A, the interior of A and the complement of A respectively.

We recall the following definitions which are useful in the sequel.

Definition 2.1. A subset A of a space (X, τ) is called

(1) a semi-open set [11] if $A \subseteq \mathrm{cl}(\mathrm{int}(A))$;
(2) an α-open set [20] if $A \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$;
(3) a regular open set [23] if $A = \mathrm{int}(\mathrm{cl}(A))$.

The complements of the above mentioned sets are called their respective closed sets.

The family of regular open (resp. regular closed) sets of a space (X, τ) is denoted by $\mathrm{RO}(X, \tau)$ ($\mathrm{RC}(X, \tau)$) or simply by $\mathrm{RO}(X)$ ($\mathrm{RC}(X)$, respectively).

The family of α-open sets of a space (X, τ) is denoted by τ^α. It is known [20] that $\tau \subset \tau^\alpha$ and τ^α is a topology for X. The closure and interior of a subset A of X with respect to τ^α are denoted by $\alpha\,\mathrm{cl}(A)$ and $\alpha\,\mathrm{int}(A)$, respectively. It is known in [1] that $\alpha\,\mathrm{cl}(A) = A \cup \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$ and $\alpha\,\mathrm{int}(A) = A \cap \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$ for any subset A of a space (X, τ).

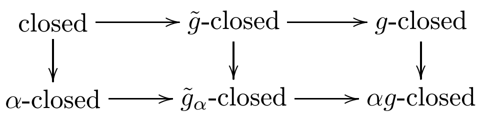
The semi-closure of a subset A of X is denoted by $s\,\mathrm{cl}(A)$, and defined as the intersection of all semi-closed sets of X containing A.

Definition 2.2. A subset A of a space (X, τ) is called

(1) a generalized closed (briefly g-closed) set [10] if $\mathrm{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ). The complement of g-closed set is called g-open set;
(2) an α-generalized closed (briefly αg-closed) set [14] if $\alpha\,\mathrm{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ). The complement of αg-closed set is called αg-open set;
(3) a $\hat{g}$-closed set [35] if $\mathrm{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in (X, τ). The complement of a $\hat{g}$-closed set is called a $\hat{g}$-open set;
(4) a ${}^*g$-closed set [36] if $\mathrm{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is $\hat{g}$-open in (X, τ). The complement of a ${}^*g$-closed set is called a ${}^*g$-open set;
(5) a ${}^\sharp g$-semi-closed (briefly ${}^\sharp gs$-closed) set [37] if $s\,\mathrm{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is ${}^*g$-open in (X, τ). The complement of a ${}^\sharp gs$-closed set is called a ${}^\sharp gs$-open set;

(6) a $\tilde{g}$-closed set [7] if $\mathrm{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is $^\sharp gs$-open in (X, τ). The complement of $\tilde{g}$-closed set is called a $\tilde{g}$-open set;
(7) a $\tilde{g}_\alpha$-closed set [8] if $\alpha\,\mathrm{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is $^\sharp gs$-open in (X, τ). The complement of a $\tilde{g}_\alpha$-closed set is called a $\tilde{g}_\alpha$-open set;
(8) a $r\alpha g$-closed [23] if $\alpha\,\mathrm{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in (X, τ). The complement of $r\alpha g$-closed set is called $r\alpha g$-open set;
(9) a generalized α-closed (briefly $g\alpha$-closed) set [13] if $\alpha\,\mathrm{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in (X, τ). The complement of a $g\alpha$-closed set is called $g\alpha$-open set.

R e m a r k 2.3. From Definitions 2.1 and 2.2, we have the following implications.

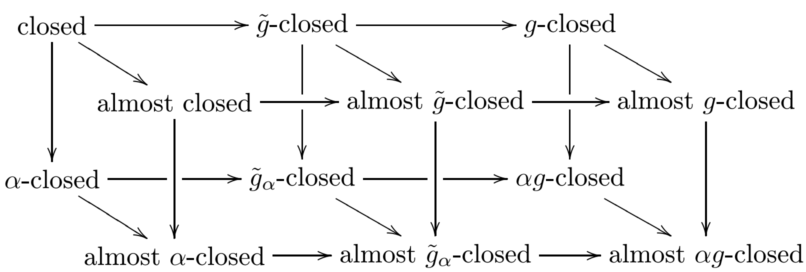


None of these implications is reversible as shown by the following examples and in the related papers [7] and [8].

3. Almost $\tilde{g}_\alpha$-closed functions

Definition 3.1. A function $f\colon (X, \tau) \to (Y, \sigma)$ is said to be
(1) α-closed [17], g-closed [16], αg-closed [23], $\tilde{g}$-closed, $\tilde{g}_\alpha$-closed if for each closed set F of $X, f(F)$ is α-closed, g-closed, αg-closed, $\tilde{g}$-closed, $\tilde{g}_\alpha$-closed, respectively;
(2) almost closed [33], almost α-closed [23], almost g-closed [22], almost αg-closed [23], almost $\tilde{g}$-closed, almost $\tilde{g}_\alpha$-closed if for each $F \in \mathrm{RC}(X, \tau)$, $f(F)$ is closed α-closed, g-closed, αg-closed, $\tilde{g}$-closed, $\tilde{g}_\alpha$-closed, respectively.

R e m a r k 3.2. We have the following diagram for properties of functions:



The following two examples show that almost $\tilde{g}$-closedness is strictly weaker than almost closedness and $\tilde{g}$-closedness.

Example 3.3. Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b,c\}, \{a,b,c\}, X\}$. Then the identity function $f\colon (X,\tau) \to (X,\sigma)$ is almost $\tilde{g}$-closed. However, it is not almost closed since the set $\{a,c,d\} \in \mathrm{RC}(X,\tau)$ is such that $f(\{a,c,d\}) = \{a,c,d\}$ is not closed in (X,σ).

Example 3.4. Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b,c\}, \{a,b,c\}, X\}$. Then the identity function $f\colon (X,\tau) \to (X,\sigma)$ is almost $\tilde{g}$-closed. However, it is not $\tilde{g}$-closed since the closed set $\{c\}$ of (X,τ) is such that $f(\{c\}) = \{c\}$ is not $\tilde{g}$-closed in (X,σ).

The following two examples show that almost g-closedness is strictly weaker than almost $\tilde{g}$-closedness and g-closedness.

Example 3.5. Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset, \{a\}, \{b,c\}, \{a,b,c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, X\}$. Then the identity function $f\colon (X,\tau) \to (X,\sigma)$ is almost g-closed. However, it is not almost $\tilde{g}$-closed since the set $\{a,d\} \in \mathrm{RC}(X,\tau)$ is such that $f(\{a,d\}) = \{a,d\}$ is not $\tilde{g}$-closed in (X,σ).

Example 3.6. Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset, \{d\}, \{b,c\}, \{b,c,d\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\}$. Then the identity function $f\colon (X,\tau) \to (X,\sigma)$ is almost g-closed. However, it is not g-closed since the closed set $\{a\}$ of (X,τ) is such that $f(\{a\}) = \{a\}$ is not g-closed in (X,σ).

The following three examples show that almost $\tilde{g}_\alpha$-closedness is strictly weaker than almost α-closedness, $\tilde{g}_\alpha$-closedness and almost $\tilde{g}$-closedness.

Example 3.7. Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b,c\}, \{a,b,c\}, X\}$. Then the identity function $f\colon (X,\tau) \to (X,\sigma)$ is almost $\tilde{g}_\alpha$-closed. However, it is not almost α-closed since the set $\{a,c,d\} \in \mathrm{RC}(X,\tau)$ is such that $f(\{a,c,d\}) = \{a,c,d\}$ is not α-closed in (X,σ).

Example 3.8. Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b,c\}, \{a,b,c\}, X\}$. Then the identity function $f\colon (X,\tau) \to (X,\sigma)$ is almost $\tilde{g}_\alpha$-closed. However, it is not $\tilde{g}_\alpha$-closed since the closed set $\{c\}$ of (X,τ) is such that $f(\{c\}) = \{c\}$ is not $\tilde{g}_\alpha$-closed in (X,σ).

Example 3.9. Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset, \{d\}, \{a,b,c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\}$. Then the identity function $f\colon (X,\tau) \to (X,\sigma)$ is almost $\tilde{g}_\alpha$-closed. However, it is not almost $\tilde{g}$-closed since the set $\{d\} \in \mathrm{RC}(X,\tau)$ is such that $f(\{d\}) = \{d\}$ is not $\tilde{g}$-closed in (X,σ).

The following three examples show that almost αg-closedness is strictly weaker than almost g-closedness, αg-closedness and almost $\tilde{g}_\alpha$-closedness.

E x a m p l e 3.10. Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, X\}$. Then the identity function $f\colon (X,\tau) \to (X,\sigma)$ is almost αg-closed. However, it is not almost $\tilde{g}_\alpha$-closed since the set $\{a,d\} \in \mathrm{RC}(X,\tau)$ is such that $f(\{a,d\}) = \{a,d\}$ is not $\tilde{g}_\alpha$-closed in (X,σ).

E x a m p l e 3.11. Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset, \{a\}, \{c\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b,c\}, \{a,b,c\}, X\}$. Then the identity function $f\colon (X,\tau) \to (X,\sigma)$ is almost αg-closed. However, it is not αg-closed since the closed set $\{c\}$ of (X,τ) is such that $f(\{c\}) = \{c\}$ is not αg-closed in (X,σ).

E x a m p l e 3.12. Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset, \{d\}, \{a,b,c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,d\}, \{a,b,d\}, X\}$. Then the identity function $f\colon (X,\tau) \to (X,\sigma)$ is almost αg-closed. However, it is not almost g-closed since the set $\{d\} \in \mathrm{RC}(X,\tau)$ is such that $f(\{d\}) = \{d\}$ is not g-closed in (X,σ).

Theorem 3.13. *A surjection $f\colon X \to Y$ is almost $\tilde{g}_\alpha$-closed if and only if for each subset S of Y and each $U \in \mathrm{RO}(X)$ containing $f^{-1}(S)$ there exists a $\tilde{g}_\alpha$-open set V of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.*

P r o o f. Necessity. Suppose that f is almost $\tilde{g}_\alpha$-closed. Let S be a subset of Y and let $U \in \mathrm{RO}(X)$ containing $f^{-1}(S)$. Put $V = Y - f(X-U)$, then V is a $\tilde{g}_\alpha$-open set in Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

Sufficiency. Let F be any regular closed set of X. Then $f^{-1}(Y - f(F)) \subset X - F$ and $X - F \in \mathrm{RO}(X)$. There exists a $\tilde{g}_\alpha$-open set V in Y such that $Y - f(F) \subset V$ and $f^{-1}(V) \subset X - F$. Therefore, we have $f(F) \supset Y - V$ and $F \subset f^{-1}(Y - V)$. Hence, we obtain $f(F) = Y - V$ and $f(F)$ is $\tilde{g}_\alpha$-closed in Y. This shows that f is almost $\tilde{g}_\alpha$-closed. □

Corollary 3.14. *If $f\colon X \to Y$ is an almost $\tilde{g}_\alpha$-closed surjection, then for each $\sharp gs$-closed set F in Y and each $U \in \mathrm{RO}(X)$ containing $f^{-1}(F)$ there exists an α-open set V in Y such that $F \subset V$ and $f^{-1}(V) \subset U$.*

P r o o f. Let F be a $\sharp gs$-closed set in Y and let $U \in \mathrm{RO}(X)$ containing $f^{-1}(F)$. By Theorem 3.13, there exists a $\tilde{g}_\alpha$-open set W in Y such that $F \subset W$ and $f^{-1}(W) \subset U$. Since W is $\tilde{g}_\alpha$-open, we have $F \subset \alpha\operatorname{int}(W)$. Put $V = \alpha\operatorname{int}(W)$, then V is α-open in Y and $f^{-1}(V) \subset U$. □

4. Normal spaces

In this section we make use of $\tilde{g}_\alpha$-closed sets to obtain further characterizations and preservation theorems of normal spaces.

Theorem 4.1. *The following conditions are equivalent for a space X:*

(1) *X is normal;*

(2) *for any disjoint closed sets A and B there exist disjoint $\tilde{g}_\alpha$-open sets U, V such that $A \subset U$ and $B \subset V$;*

(3) *for any closed set A and any open set V containing A there exists a $\tilde{g}_\alpha$-open set U in X such that $A \subset U \subset \alpha\,\mathrm{cl}(U) \subset V$.*

Proof. (1) $\Rightarrow$ (2). This is obvious since every open set is $\tilde{g}_\alpha$-open.

(2) $\Rightarrow$ (3). Let A be a closed set and V an open set containing A. Then A and $X - V$ are disjoint closed sets. There exist disjoint $\tilde{g}_\alpha$-open sets U and W such that $A \subset U$ and $X - V \subset W$. Since $X - V$ is closed and hence $^\sharp gs$-closed, we have $X - V \subset \alpha\,\mathrm{int}(W)$ and $U \cap \alpha\,\mathrm{int}(W) = \emptyset$. Therefore, we obtain $\alpha\,\mathrm{cl}(U) \cap \alpha\,\mathrm{int}(W) = \emptyset$ and hence $A \subset U \subset \alpha\,\mathrm{cl}(U) \subset X - \alpha\,\mathrm{int}(W) \subset V$.

(3) $\Rightarrow$ (1). Let A, B be disjoint closed sets in X. Then $A \subset X - B$ and $X - B$ is open. There exists a $\tilde{g}_\alpha$-open set G in X such that $A \subset G \subset \alpha\,\mathrm{cl}(G) \subset X - B$. Since A is closed, we have $A \subset \alpha\,\mathrm{int}(G)$. Put $U = \mathrm{int}(\mathrm{cl}(\mathrm{int}(\alpha\,\mathrm{int}(G))))$ and $V = \mathrm{int}(\mathrm{cl}(\mathrm{int}(X - \alpha\,\mathrm{cl}(G))))$. Then U and V are disjoint open sets in X such that $A \subset U$ and $B \subset V$. Therefore, X is normal. □

Theorem 4.2. *If $f\colon X \to Y$ is a continuous almost $\tilde{g}_\alpha$-closed surjection and X is a normal space, then Y is normal.*

Proof. Let A and B be any disjoint closed sets in Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed sets in X. Since X is normal, there exist disjoint open sets U and V such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Let $G = \mathrm{int}(\mathrm{cl}(U))$ and $H = \mathrm{int}(\mathrm{cl}(V))$, then G and H are disjoint regular open sets in X such that $f^{-1}(A) \subset G$ and $f^{-1}(B) \subset H$. By Theorem 3.13, there exist $\tilde{g}_\alpha$-open sets K and L in Y such that $A \subset K$, $B \subset L$, $f^{-1}(K) \subset G$ and $f^{-1}(L) \subset H$. Since G and H are disjoint, so are K and L. It follows from Theorem 4.1 that Y is normal.

The following two corollaries are immediate consequences of Theorem 4.2.

Corollary 4.3 [12]**.** *If $f\colon X \to Y$ is a continuous almost closed surjection and X is a normal space, then Y is normal.*

Corollary 4.4 [6]. *If $f\colon X \to Y$ is a continuous α-closed surjection and X is a normal space, then Y is normal.*

Definition 4.5. A space X is said to be

(1) weakly normal [40] if for each decreasing sequence $\{F_n\}$ of closed sets in X such that $\bigcap\{F_n\colon n \in \mathbb{N}\} = \emptyset$ and each closed set H in X with $H \cap F_1 = \emptyset$ there exist $n \in \mathbb{N}$ and an open set U in X such that $F_n \subset U$ and $\mathrm{cl}(U) \cap H = \emptyset$;

(2) mildly normal [34] if for any disjoint regular closed sets A and B there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$;

(3) almost normal [32] if for every pair of disjoint sets A and B, one of which is closed and the other is regular closed, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

Lemma 4.6 [23]. *If A is an α-open set of a space X, then $\alpha\,\mathrm{cl}(A) = \mathrm{cl}(A) = \mathrm{cl}(\mathrm{int}(A))$.*

Lemma 4.7 [21]. *A space X is weakly normal if and only if for each decreasing sequence $\{F_n\}$ of closed sets in X such that $\bigcap\{F_n\colon n \in \mathbb{N}\} = \emptyset$ and each open set U in X such that $F_1 \subset U$ there exist $n \in \mathbb{N}$ and an open set G in X such that $F_n \subset G \subset \mathrm{cl}(G) \subset U$.*

Theorem 4.8. *If $f\colon X \to Y$ is an almost $\tilde{g}_\alpha$-closed continuous surjection and X is a weakly normal space, then Y is weakly normal.*

Proof. Let $\{F_n\}$ be any decreasing sequence of closed sets of Y with no common point and let V be any open set in Y such that $F_1 \subset V$. Then $\{f^{-1}(F_n)\}$ is a decreasing sequence of closed sets in X with no common point and $f^{-1}(V)$ is an open set in X such that $f^{-1}(F_1) \subset f^{-1}(V)$. Since X is weakly normal, by Lemma 4.7 there exist $n \in \mathbb{N}$ and an open set U in X such that $f^{-1}(F_n) \subset U \subset \mathrm{cl}(U) \subset f^{-1}(V)$. Therefore, $f^{-1}(F_n) \subset \mathrm{int}(\mathrm{cl}(U))$ and by Corollary 3.14 there exists an α-open set G in Y such that $F_n \subset G$ and $f^{-1}(G) \subset \mathrm{int}(\mathrm{cl}(U))$. Since $\mathrm{cl}(U)$ is regular closed and f is almost $\tilde{g}_\alpha$-closed, $f(\mathrm{cl}(U))$ is $\tilde{g}_\alpha$-closed in Y. Thus, we obtain $F_n \subset G \subset \alpha\,\mathrm{cl}(G) \subset \alpha\,\mathrm{cl}(f(\mathrm{cl}(U))) \subset V$. Let $H = \mathrm{int}(\mathrm{cl}(\mathrm{int}(G)))$, then by Lemma 4.6 we have $F_n \subset H \subset \mathrm{cl}(H) = \alpha\,\mathrm{cl}(G) \subset V$. It follows from Lemma 4.7 that Y is weakly normal.

Corollary 4.9 [21]. *Weak normality is preserved under almost closed continuous surjections.*

Lemma 4.10 [23].

(1) *A subset A of a space X is $r\alpha g$-open if and only if $F \subset \alpha\operatorname{int}(A)$ whenever $F \in \mathrm{RC}(X)$ and $F \subset A$.*

(2) *Every αg-closed set is $r\alpha g$-closed but not conversely.*

Theorem 4.11. *The following conditions are equivalent for a space X:*

(1) *X is mildly normal;*

(2) *for any disjoint H, $K \in \mathrm{RC}(X)$ there exist disjoint $\tilde{g}_\alpha$-open sets U, V such that $H \subset U$ and $K \subset V$;*

(3) *for any disjoint H, $K \in \mathrm{RC}(X)$ there exist disjoint αg-open sets U, V such that $H \subset U$ and $K \subset V$;*

(4) *for any disjoint H, $K \in \mathrm{RC}(X)$ there exist disjoint $r\alpha g$-open sets U, V such that $H \subset U$ and $K \subset V$;*

(5) *for any $H \in \mathrm{RC}(X)$ and any $V \in \mathrm{RO}(X)$ containing H there exists an $r\alpha g$-open set U of X such that $H \subset U \subset \alpha\operatorname{cl}(U) \subset V$;*

(6) *for any $H \in \mathrm{RC}(X)$ and any $V \in \mathrm{RO}(X)$ containing H there exists an α-open set U of X such that $H \subset U \subset \alpha\operatorname{cl}(U) \subset V$;*

(7) *for any disjoint H, $K \in \mathrm{RC}(X)$ there exist disjoint α-open sets U, V such that $H \subset U$ and $K \subset V$.*

Proof. It is obvious that (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4).

(4) $\Rightarrow$ (5). Let $H \in \mathrm{RC}(X)$ and $V \in \mathrm{RO}(X)$ containing H. There exist disjoint $r\alpha g$-open sets U, W such that $H \subset U$ and $X - V \subset W$. By Lemma 4.10, we have $X - V \subset \alpha\operatorname{int}(W)$ and $U \cap \alpha\operatorname{int}(W) = \emptyset$. Therefore, we obtain $\alpha\operatorname{cl}(U) \cap \alpha\operatorname{int}(W) = \emptyset$ and hence $H \subset U \subset \alpha\operatorname{cl}(U) \subset X - \alpha\operatorname{int}(W) \subset V$.

(5) $\Rightarrow$ (6). Let $H \in \mathrm{RC}(X)$ and $V \in \mathrm{RO}(X)$ contain H. There exists an $r\alpha g$-open set G in X such that $H \subset G \subset \alpha\operatorname{cl}(G) \subset V$. Since $H \in \mathrm{RC}(X)$, by Lemma 4.10, we have $H \subset \alpha\operatorname{int}(G)$. Put $U = \alpha\operatorname{int}(G)$, then U is α-open in X and $H \subset U \subset \alpha\operatorname{cl}(U) \subset V$.

(6) $\Rightarrow$ (7). Let H and K be any disjoint regular closed sets in X. Then, since $H \subset X - K$ and $X - K \in \mathrm{RO}(X)$, there exists an α-open set U in X such that $H \subset U \subset \alpha\operatorname{cl}(U) \subset X - K$. Put $V = X - \alpha\operatorname{cl}(U)$, then U and V are disjoint α-open sets in X such that $H \subset U$ and $K \subset V$.

(7) $\Rightarrow$ (1). Let H and K be any disjoint regular closed sets in X. Then there exist disjoint α-open sets A and B in X such that $H \subset A$ and $K \subset B$. Since A and B are disjoint, we have $\operatorname{int}(\operatorname{cl}(\operatorname{int}(A))) \cap \operatorname{int}(\operatorname{cl}(\operatorname{int}(B))) = \emptyset$. Now, put $U = \operatorname{int}(\operatorname{cl}(\operatorname{int}(A)))$ and $V = \operatorname{int}(\operatorname{cl}(\operatorname{int}(B)))$, then U and V are disjoint open sets in X such that $H \subset U$ and $K \subset V$. Therefore, X is mildly normal.

Definition 4.12. A function $f\colon X \to Y$ is said to be

(1) an R-map [2], almost-continuous [33] if $f^{-1}(V)$ is regular open, open, respectively, in X for every $V \in \mathrm{RO}(Y)$;
(2) almost open [33], almost α-open [23] if $f(U)$ is open, α-open, respectively, in Y for every regular open set U in X;
(3) α-open [17] if $f(U)$ is α-open in Y for every open set U in X.

Remark 4.13 [23]. Both almost-openness and α-openness imply almost α-openness but not conversely as the following example shows.

Example 4.14 [23]. Let $X = \{a,b,c,d\}$ and $\tau = \{\emptyset, \{c\}, \{d\}, \{a,c\}, \{c,d\}, \{a,c,d\}, X\}$. Let $Y = \{a,b,c\}$ and $\sigma = \{\emptyset, Y, \{a\}, \{a,b\}\}$. Then a function $f\colon (X,\tau) \to (Y,\sigma)$, defined as $f(a) = f(d) = a$, $f(b) = b$ and $f(c) = c$, is almost α-open. However, it is neither almost open nor α-open.

Theorem 4.15. *Let $f\colon X \to Y$ be an R-map and an almost αg-closed surjection. If X is a mildly normal space, then Y is mildly normal.*

Proof. Let A and B be any disjoint regular closed sets in Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint regular closed sets of X. Since X is mildly normal, there exist disjoint open sets U and V in X such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Put $G = \mathrm{int}(\mathrm{cl}(U))$ and $H = \mathrm{int}(\mathrm{cl}(V))$, then G and H are disjoint regular open sets in X such that $f^{-1}(A) \subset G$ and $f^{-1}(B) \subset H$. By Theorem 3.8 [23], there exist αg-open sets K and L in Y such that $A \subset K$, $B \subset L$, $f^{-1}(K) \subset G$ and $f^{-1}(L) \subset H$. Since G and H are disjoint, so are K and L. It follows from Theorem 4.11 that Y is mildly normal. □

Corollary 4.16. *Let $f\colon X \to Y$ be an R-map and an almost $\tilde{g}_\alpha$-closed surjection and let X be mildly normal. Then Y is mildly normal.*

Lemma 4.17 [23]. *If a function $f\colon X \to Y$ is almost continuous almost α-open and V is regular open in Y, then $f^{-1}(V)$ is regular open in X.*

Theorem 4.18. *If $f\colon X \to Y$ is an almost α-open almost αg-closed continuous surjection and X is an almost normal space, then Y is almost normal.*

Proof. Let B be any closed set of Y and $V \in \mathrm{RO}(Y)$ contain B. Since f is continuous and almost α-open, $f^{-1}(B)$ is closed and $f^{-1}(V) \in \mathrm{RO}(X)$ by Lemma 4.17. Since X is almost normal and $f^{-1}(B) \subset f^{-1}(V)$, there exists $U \in \mathrm{RO}(X)$ such that $f^{-1}(B) \subset U \subset \mathrm{cl}(U) \subset f^{-1}(V)$ ([32], Theorem 2.1). Since f is almost α-open and almost αg-closed, $f(U)$ is α-open and $f(\mathrm{cl}(U))$ is αg-closed in Y. Therefore, we obtain $B \subset f(U) \subset \alpha\,\mathrm{cl}(f(U)) \subset \alpha\,\mathrm{cl}(f(\mathrm{cl}(U))) \subset V$. Put $G = \mathrm{int}(\mathrm{cl}(\mathrm{int}(f(U))))$. Then G is open in Y and $\alpha\,\mathrm{cl}(f(U)) = \mathrm{cl}(\mathrm{int}(f(U))) = \mathrm{cl}(G)$

by Lemma 4.6. Therefore, we obtain $B \subset f(U) \subset G \subset \mathrm{cl}(G) \subset V$. It follows from ([32], Theorem 2.1) that Y is almost normal. □

Corollary 4.19 [39]**.** *Almost normality is preserved under almost open almost g-closed continuous surjections.*

5. Regular spaces

In this section, we improve preservation theorems of regularity, almost regularity, quasi-regularity.

Definition 5.1. A space X is said to be

(1) almost regular [31] if for each $F \in \mathrm{RC}(X)$ and each $x \in X - F$ there exist disjoint open sets U and V in X such that $x \in U$ and $F \subset V$;
(2) quasi-regular [28] if for every nonempty open set V of X, there exists a nonempty open set U in X such that $\mathrm{cl}(U) \subset V$;
(3) strongly s-regular [5] if for any closed set A in X and any point $x \in X - A$ there exists an $F \in \mathrm{RC}(X)$ such that $x \in F$ and $F \cap A = \emptyset$.

It is shown in ([5], Theorem 1) that a space X is strongly s-regular if and only if every open set in X is the union of regular closed sets. Strongly s-regular spaces are called $P_{\sum}$-spaces by Wang [38]. Ganster [5] showed that strong s-regularity is strictly weaker than regularity and is independent of almost regularity.

Theorem 5.2 [23]**.** *The following conditions are equivalent for a space (X, τ):*

(1) *(X, τ) is regular (almost regular);*
(2) *for each closed (regular closed) set F and each $x \in X - F$, there exist disjoint U, $V \in \tau^{\alpha}$ such that $x \in U$ and $F \subset V$;*
(3) *for each open (regular open, respectively) set V and $x \in V$, there exists $U \in \tau^{\alpha}$ such that $x \in U \subset \alpha\,\mathrm{cl}(U) \subset V$.*

Theorem 5.3. *If $f\colon X \to Y$ is an almost α-open almost $\tilde{g}_{\alpha}$-closed continuous surjection and X is a regular space, then Y is regular.*

P r o o f. Let y be any point of Y and V any open neighbourhood of y. There exists a point $x \in X$ with $f(x) = y$. Since X is regular and f is continuous, there exists an open set U in X such that $x \in U \subset \mathrm{cl}(U) \subset f^{-1}(V)$. Therefore, we have $y \in f(U) \subset f(\mathrm{int}(\mathrm{cl}(U))) \subset f(\mathrm{cl}(U)) \subset V$ and $f(\mathrm{int}(\mathrm{cl}(U)))$ is α-open because $\mathrm{int}(\mathrm{cl}(U)) \in \mathrm{RO}(X)$ and f is almost α-open. Since $\mathrm{cl}(U) \in \mathrm{RC}(X)$ and f is almost $\tilde{g}_{\alpha}$-closed, $f(\mathrm{cl}(U))$ is $\tilde{g}_{\alpha}$-closed and hence $y \in f(\mathrm{int}(\mathrm{cl}(U))) \subset \alpha\,\mathrm{cl}(f(\mathrm{int}(\mathrm{cl}(U)))) \subset \alpha\,\mathrm{cl}(f(\mathrm{cl}(U))) \subset V$. It follows from Theorem 5.2 that Y is regular. □

Corollary 5.4 [23]**.** *Regularity is preserved under almost α-open almost αg-closed continuous surjections.*

Theorem 5.5. *If $f\colon X\to Y$ is an almost α-open almost αg-closed almost continuous surjection and X is an almost regular space, then Y is almost regular.*

P r o o f. Let y be any point of Y and let $V\in \mathrm{RO}(Y)$ contain y. Since f is almost α-open almost continuous, $f^{-1}(V)\in \mathrm{RO}(Y)$ by Lemma 4.17. Take a point $x\in f^{-1}(y)$. Since X is almost regular, there exists $U\in \mathrm{RO}(X)$ such that $x\in U\subset \mathrm{cl}(U)\subset f^{-1}(V)$ ([31], Theorem 2.2). Hence $y\in f(U)\subset f(\mathrm{cl}(U))\subset V$. Since f is almost α-open almost αg-closed, $f(U)$ is α-open in Y and $f(\mathrm{cl}(U))$ is αg-closed in Y and hence we have $y\in f(U)\subset \alpha\,\mathrm{cl}(f(U))\subset \alpha\,\mathrm{cl}(f(\mathrm{cl}(U)))\subset V$. It follows from Theorem 5.2 that Y is almost regular. □

Definition 5.6. A function $f\colon X\to Y$ is said to be

(1) feebly continuous [4] if $\mathrm{int}(f^{-1}(V))\neq\emptyset$ for every nonempty open set V in Y;

(2) feebly open [4] if $\mathrm{int}(f(U))\neq\emptyset$ for every nonempty open set U in X;

(3) almost feebly open [23] if $\mathrm{int}(f(U))\neq\emptyset$ for every nonempty $U\in \mathrm{RO}(X)$.

R e m a r k 5.7 [23]. It is obvious that every feebly open function is almost feebly open. However, the converse is not true in general as the following example shows.

E x a m p l e 5.8 [23]. Let $X=Y=\{a,b,c\}$, $\tau=\{\emptyset,\{a\},\{b\},\{a,b\},\{a,c\},X\}$ and $\sigma=\{\emptyset,\{a\},\{b\},\{a,b\},Y\}$. Let $f\colon (X,\tau)\to(Y,\sigma)$ be a function defined as follows: $f(a)=c$, $f(b)=a$ and $f(c)=b$. Then f is almost feebly open but it is not feebly open since we have $\mathrm{RO}(X,\tau)=\{\emptyset,\{b\},\{a,c\},X\}$ and $\mathrm{int}(f(\{a\}))=\emptyset$.

Theorem 5.9. *If $f\colon X\to Y$ is an almost feebly open feebly continuous almost $\tilde g_\alpha$-closed surjection and X is a quasi-regular space, then Y is quasi-regular.*

P r o o f. Let V be any nonempty open set in Y. Since f is feebly continuous, $\mathrm{int}(f^{-1}(V))\neq\emptyset$ and by the quasi-regularity of X there exists a nonempty open set U of X such that $U\subset\mathrm{cl}(U)\subset\mathrm{int}(f^{-1}(V))$. We have $f(\mathrm{int}(\mathrm{cl}(U)))\subset f(\mathrm{cl}(U))\subset V$. Since f is almost feebly open, $\mathrm{int}(f(\mathrm{int}(\mathrm{cl}(U))))\neq\emptyset$. Since f is almost $\tilde g_\alpha$-closed, $f(\mathrm{cl}(U))$ is $\tilde g_\alpha$-closed and hence $\alpha\,\mathrm{cl}(f(\mathrm{cl}(U)))\subset V$. Now, put $G=\mathrm{int}(f(\mathrm{int}(\mathrm{cl}(U))))$, then by Lemma 4.6 we obtain $\emptyset\neq G\subset\mathrm{cl}(G)=\alpha\,\mathrm{cl}(G)\subset\alpha\,\mathrm{cl}(f(\mathrm{cl}(U)))\subset V$. This shows that Y is quasi-regular. □

Corollary 5.10 [9]**.** *Quasi regularity is preserved under feebly open feebly continuous closed surjections.*

We conclude the section with a preservation theorem of strongly s-regular spaces.

Theorem 5.11. *If $f\colon X\to Y$ is an almost α-open almost $\tilde{g}_\alpha$-closed continuous surjection and X is a strongly s-regular space, then Y is strongly s-regular.*

Proof. Let V be any open set in Y and y any point of V. Since f is continuous, $f^{-1}(V)$ is open in X. For a point $x\in f^{-1}(y)$ there exists $F\in \mathrm{RC}(X)$ such that $x\in F\subset f^{-1}(V)$; hence $y=f(x)\in f(F)\subset V$. Since f is continuous, we have $f(F)=f(\mathrm{cl}(\mathrm{int}(F)))\subset \mathrm{cl}(f(\mathrm{int}(F)))$. Since f is almost $\tilde{g}_\alpha$-closed, $f(F)$ is $\tilde{g}_\alpha$-closed and $\alpha\,\mathrm{cl}(f(F))\subset V$. Moreover, f is almost α-open, $f(\mathrm{int}(F))$ is α-open in Y and by Lemma 4.6 we have $\mathrm{cl}(f(\mathrm{int}(F)))=\mathrm{cl}(\mathrm{int}(f(\mathrm{int}(F))))=\alpha\,\mathrm{cl}(f(\mathrm{int}(F)))\subset\alpha\,\mathrm{cl}(f(F))$. Therefore, we obtain $\mathrm{cl}(\mathrm{int}(f(\mathrm{int}(F))))\in\mathrm{RC}(Y)$ and $y\in f(F)\subset\mathrm{cl}(f(\mathrm{int}(F)))=\mathrm{cl}(\mathrm{int}(f(\mathrm{int}(F))))\subset\alpha\,\mathrm{cl}(f(F))\subset V$. It follows from ([5], Theorem 1) that Y is strongly s-regular. □

6. Minimal structures

Definition 6.1 [26]. Let X be a nonempty set and $\wp(X)$ the power set of X. A subfamily m_x of $\wp(X)$ is called a minimal structure (briefly m-structure) on X if $\emptyset\in m_x$ and $X\in m_x$.

Each member of the minimal structure m_x is called m_x-open. The complement of an m_x-open set is said to be m_x-closed and the pair (X,m_x) is called an m-space.

Remark 6.2. Let (X,τ) be a topological space. Then, by Definition 2.2(7), the family of $\tilde{g}_\alpha$-open sets is an m-structure on X.

Definition 6.3 [15]. Let X be a nonempty set and m_x an m-structure on X. For a subset A of X, the m_x-closure of A and the m_x-interior of A are defined as follows:

(1) $m_x\text{-cl}(A)=\bigcap\{F\colon A\subset F, X-F\in m_x\}$,

(2) $m_x\text{-int}(A)=\bigcup\{U\colon U\subset A, U\in m_x\}$.

Theorem 6.4 [15]. *Let X be a nonempty set and m_x a minimal structure on X. For subsets A and B of X, the following assertions hold:*

(1) $m_x\text{-cl}(X-A)=X-(m_x\text{-int}(A))$ *and* $m_x\text{-int}(X-A)=X-(m_x\text{-cl}(A))$,

(2) *If* $X-A\in m_x$, *then* $m_x\text{-cl}(A)=A$ *and if* $A\in m_x$, *then* $m_x\text{-int}(A)=A$,

(3) $m_x\text{-cl}(\emptyset)=\emptyset$, $m_x\text{-cl}(X)=X$, $m_x\text{-int}(\emptyset)=\emptyset$ *and* $m_x\text{-int}(X)=X$,

(4) *If* $A\subset B$, *then* $m_x\text{-cl}(A)\subset m_x\text{-cl}(B)$ *and* $m_x\text{-int}(A)\subset m_x\text{-int}(B)$,

(5) $A\subset m_x\text{-cl}(A)$ *and* $m_x\text{-int}(A)\subset A$,

(6) $m_x\text{-cl}(m_x\text{-cl}(A))=m_x\text{-cl}(A)$ *and* $m_x\text{-int}(m_x\text{-int}(A))=m_x\text{-int}(A)$.

Definition 6.5 [15]. A minimal structure m_x on a nonempty set X is said to have the property ($\mathcal{B}$) if the union of any family of subsets belonging to m_x belongs to m_x.

Remark 6.6. Let (X, τ) be a topological space. Then, by Definition 2.2 (7), the family of $\tilde{g}_\alpha$-open sets is an m-structure on X having the property ($\mathcal{B}$).

Definition 6.7 [18]. Let (X, m_x) be a minimal structure and $A \subset X$. A subset A of X is called an αm_x-open set if $A \subseteq m_x\text{-int}(m_x\text{-cl}(m_x\text{-int}(A)))$.

The complement of an αm_x-open set is called an αm_x-closed set.

The family of all αm_x-open sets in X will be denoted by $\alpha M(X)$.

Definition 6.8 [18]. Let (X, m_x) be a minimal structure. For a subset A of X, the α-closure of A and the α-interior of A, denoted by $\alpha m_x\text{-cl}(A)$ and $\alpha m_x\text{-int}(A)$, respectively, are defined as follows:

(1) $\alpha m_x\text{-cl}(A) = \bigcap\{F\colon\ A \subset F,\ F \text{ is } \alpha m_x\text{-closed in } X\}$,

(2) $\alpha m_x\text{-int}(A) = \bigcup\{U\colon\ U \subset A,\ U \text{ is } \alpha m_x\text{-open in } X\}$.

Definition 6.9 [19]. Let (X, m_x) be a space with a minimal structure m_x on X and $A \subset X$. A subset A of X is called an m_x-semiopen set if $A \subseteq m_x\text{-cl}(m_x\text{-int}(A))$. The complement of an m_x-semiopen set is called an m_x-semiclosed set.

Definition 6.10 [19]. Let (X, m_x) be a space with a minimal structure m_x on X. For a subset A of X, the m_x-semi-closure of A and the m_x-semi-interior of A, denoted by $m_x\text{-}s\,\text{cl}(A)$ and $m_x\text{-}s\,\text{int}(A)$, respectively, are defined as follows:

(1) $m_x\text{-}s\,\text{cl}(A) = \bigcap\{F\colon\ A \subset F,\ F \text{ is } m_x\text{-semiclosed in } X\}$,

(2) $m_x\text{-}s\,\text{int}(A) = \bigcup\{U\colon\ U \subset A,\ U \text{ is } m_x\text{-semiopen in } X\}$.

Definition 6.11 [30]. Let (X, m_x) be a minimal structure and $A \subset X$. A subset A of X is called an m_x-regular open set if $A = m_x\text{-int}(m_x\text{-cl}(A))$.

The complement of an m_x-regular open set is called an m_x-regular closed set.

The family of all m_x-regular closed, m_x-regular open sets of (X, m_x) is denoted by $\text{RC}(X, m_x)$, $\text{RO}(X, m_x)$, respectively.

Proposition 6.12. *Every m_x-regular open set is m_x-open but not conversely.*

Proof. Let A be an m_x-regular open set in X. Since $A = m_x\text{-int}(m_x\text{-cl}(A))$ then $m_x\text{-int}(A) = m_x\text{-int}(m_x\text{-cl}(A))$. We have $A = m_x\text{-int}(A)$. Thus A is m_x-open. □

Example 6.13. Let $X = \{a, b, c\}$. Define the m-structure on X as follows: $m_x = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Then $\text{RO}(X, m_x) = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{b, c\}\}$. Here $A = \{a, c\}$ is m_x-open but not m_x-regular open.

Definition 6.14 [30]. Let (X, m_x) be an m-space. We say that $A \subseteq X$ is

(1) an m_x-$\hat{g}$-closed set if m_x-cl$(A) \subseteq U$ whenever $A \subseteq U$ and U is m_x-semiopen in (X, m_x). The complement of an m_x-$\hat{g}$-closed set is called an m_x-$\hat{g}$-open set;

(2) an m_x-$^*g$-closed set if m_x-cl$(A) \subseteq U$ whenever $A \subseteq U$ and U is m_x-$\hat{g}$-open in (X, m_x). The complement of an m_x-$^*g$-closed set is called an m_x-$^*g$-open set;

(3) an m_x-$^\sharp g$-semi-closed (briefly m_x-$^\sharp gs$-closed) set if m_x-$s\,$cl$(A) \subseteq U$ whenever $A \subseteq U$ and U is m_x-$^*g$-open in (X, m_x). The complement of an m_x-$^\sharp gs$-closed set is called an m_x-$^\sharp gs$-open set.

Definition 6.15. Let (X, m_x) be an m-space. We say that $A \subseteq X$ is an m_x-$\tilde{g}_\alpha$-closed set if αm_x-cl$(A) \subseteq U$ whenever $A \subseteq U$ and U is m_x-$^\sharp gs$-open in (X, m_x). The complement of an m_x-$\tilde{g}_\alpha$-closed set is called an m_x-$\tilde{g}_\alpha$-open set.

Proposition 6.16. *Every m_x-closed set is m_x-$\tilde{g}_\alpha$-closed but not conversely.*

Proof. Let A be an m_x-closed set and G any m_x-$^\sharp gs$-open set containing A. Since A is m_x-closed, we have αm_x-cl$(A) \subseteq m_x$-cl$(A) = A \subseteq G$. Hence A is m_x-$\tilde{g}_\alpha$-closed. □

Example 6.17. Let $X = \{a, b, c\}$. Define the m-structure on X as follows: $m_x = \{\emptyset, X, \{a\}, \{b\}\}$. Then the sets in $\{\emptyset, X, \{c\}, \{a, c\}, \{b, c\}\}$ are called m_x-$\tilde{g}_\alpha$-closed and the sets in $\{\emptyset, X, \{a, c\}, \{b, c\}\}$ are called m_x-closed. Here $A = \{c\}$ is m_x-$\tilde{g}_\alpha$-closed but not m_x-closed.

Definition 6.18 [24]. A function $f\colon (X, m_x) \to (Y, m_y)$ is said to be M-closed if for each m_x-closed set F of X, $f(F)$ is m_y-closed in Y.

Theorem 6.19 [24]. *For a function $f\colon (X, m_x) \to (Y, m_y)$ where m_y has the property $(\mathcal{B})$, the following properties are equivalent:*

(1) *f is M-closed.*

(2) *For each subset F of Y and each $U \in m_x$ with $f^{-1}(F) \subset U$, there exists $V \in m_y$ such that $F \subset V$ and $f^{-1}(V) \subset U$.*

(3) *For each $y \in Y$ and each $U \in m_x$ with $f^{-1}(y) \subset U$, there exists $V \in m_y$ containing y such that $f^{-1}(V) \subset U$.*

Definition 6.20. A function $f\colon (X, m_x) \to (Y, m_y)$ is said to be almost-M-$\tilde{g}_\alpha$-closed if for each $F \in \mathrm{RC}(X, m_x)$, $f(F)$ is m_y-$\tilde{g}_\alpha$-closed in Y.

Remark 6.21. Every M-closed function is almost M-$\tilde{g}_\alpha$-closed but not conversely.

Example 6.22. Let $X = Y = \{a, b, c\}$. Define the m-structure on X and Y as follows: $m_x = \{\emptyset, X, \{a\}, \{b\}\}$ and $m_y = \{\emptyset, Y, \{a\}, \{a, b\}, \{b, c\}\}$. Then

$\mathrm{RC}(X, m_x) = \{\emptyset, X, \{a, c\}, \{b, c\}\}$; the sets in $\{\emptyset, X, \{a, c\}, \{b, c\}\}$ are called m_x-closed; the sets in $\{\emptyset, Y, \{a\}, \{c\}, \{b, c\}\}$ are called m_y-closed and the sets in $\{\emptyset, Y, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$ are called m_y-$\tilde{g}_\alpha$-closed. Then the identity function $f\colon (X, m_x) \to (Y, m_y)$ is almost M-$\tilde{g}_\alpha$-closed. However, it is not M-closed since $f(\{a, c\}) = \{a, c\}$ is not m_y-closed.

Theorem 6.23. *A surjection $f\colon (X, m_x) \to (Y, m_y)$ is almost M-$\tilde{g}_\alpha$-closed if and only if for each subset S of (Y, m_y) and each $U \in \mathrm{RO}(X, m_x)$ containing $f^{-1}(S)$ there exists an m-$\tilde{g}_\alpha$-open set V of (Y, m_y) such that $S \subset V$ and $f^{-1}(V) \subset U$.*

P r o o f. Necessity. Suppose that f is almost M-$\tilde{g}_\alpha$-closed. Let S be a subset of (Y, m_y) and let $U \in \mathrm{RO}(X, m_x)$ contain $f^{-1}(S)$. Put $V = Y - f(X - U)$, then V is an m_y-$\tilde{g}_\alpha$-open set of (Y, m_y) such that $S \subset V$ and $f^{-1}(V) \subset U$.

Sufficiency. Let F be any m_x-regular closed set in (X, m_x). Then $f^{-1}(Y - f(F)) \subset X - F$ and $X - F \in \mathrm{RO}(X, m_x)$. There exists an m_y-$\tilde{g}_\alpha$-open set V of (Y, m_y) such that $Y - f(F) \subset V$ and $f^{-1}(V) \subset X - F$. Therefore, we have $f(F) \supset Y - V$ and $F \subset f^{-1}(Y - V)$. Hence, we obtain $f(F) = Y - V$ and $f(F)$ is m_y-$\tilde{g}_\alpha$-closed in (Y, m_y). This shows that f is almost M-$\tilde{g}_\alpha$-closed.

R e m a r k 6.24. Theorem 3.13 is a particular case of Theorem 6.23 if $\tau = m_x$.

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