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ON PERIODIC SOLUTIONS TO SECOND-ORDER DUFFING TYPE EQUATIONS

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Abstract. Sufficient and necessary conditions are found for the existence of a positive periodic solution to the Duffing type equation

$$u'' = p(t)u + q(t,u)u.$$

The results obtained are compared with facts well known for the autonomous Duffing equation

$$y'' + ay - by^3 = 0.$$

Uniqueness of solutions and possible generalisations are discussed, as well.

1. Introduction

The paper deals with the question on the existence and uniqueness of a **positive** solution to the periodic boundary value problem

$$u'' = p(t)u + q(t,u)u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega). \tag{1.1}$$

Here, $p \in L([0,\omega])$ and $q\colon [0,\omega] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Under a solution to problem (1.1), as usually, we understand a function $u\colon [0,\omega] \to \mathbb{R}$ which is absolutely continuous together with its first derivative, satisfies given equation almost everywhere, and verifies periodic conditions.

In mathematical models of various oscillators, one can find the following equation

$$y'' + \delta y' + ay - by^3 = \gamma \sin t, \tag{1.2}$$

where $a, b, \gamma \in \mathbb{R}$ and the damping constant satisfies $\delta \geq 0$. This equation is the central topis of the monograph [1] by Duffing published in 1918 and still bears his name today (see also [5]). Considering the equation of motion of the forced damped pendulum

$$y'' + \delta y' + \frac{g}{\ell} \sin y = \gamma \sin t, \tag{1.3}$$

the equation (1.2) with $a, b > 0$ appears when approximating the non-linearity in (1.3) by Taylor's polynomial of the third order with the centre at 0. A survey of results dealing with the analysis of the pendullum equation is given in [10]. The equation (1.2) can be also interpreted as the equation of motion of a forced oscillator with a spring whose restoring force is given as a third-order polynomial. The phase portrait of the free undamped equation (1.2) with $a, b > 0$, i. e., the equation

$$y'' + ay - by^3 = 0, \tag{1.4}$$

can be easily determined and it is illustrated on Fig. 1.

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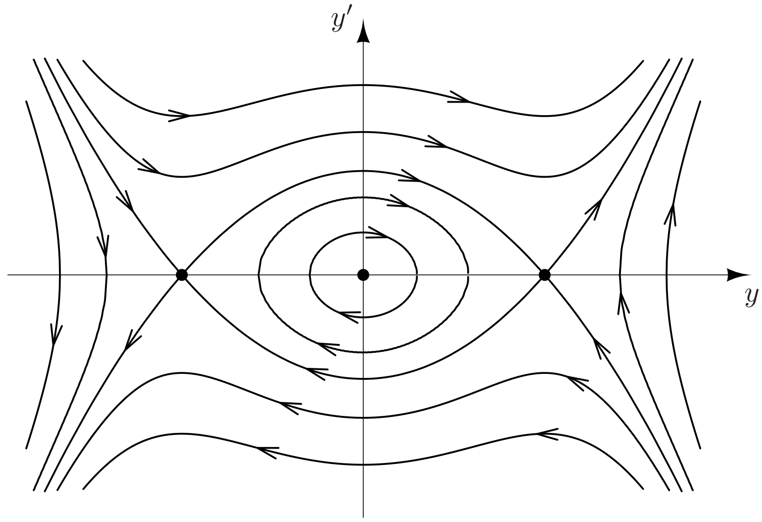

FIGURE 1. Phase portrait of equation (1.4) with $a, b > 0$.

Definition 1.1. A solution u to problem (1.1) is referred as a sign-constant solution if there exists $i \in \{0, 1\}$ such that

$$(-1)^i u(t) \geq 0 \quad \text{for } t \in [0, \omega],$$

and a sign-changing solution otherwise.

Let us summarize some well-known facts concerning periodic solutions to equation (1.4) (see, e. g., [5, 6]).

Proposition 1.2. *The following statements hold:*

(1) *For any $a \leq 0$ and $b > 0$, equation* (1.4) *has a unique equilibrium $y = 0$ and no other periodic solutions occur.*
(2) *For any $a, b > 0$, equation* (1.4) *has exactly three equilibria $y = 0$, $y = \sqrt{\frac{a}{b}}$, $y = -\sqrt{\frac{a}{b}}$ and no other non-trivial sign-constant periodic solutions occur.*
(3) *For any $a, b > 0$ and $T \leq \frac{2\pi}{\sqrt{a}}$, equation* (1.4) *has exactly three T-periodic solutions.*
(4) *For any $a, b > 0$ and $T > \frac{2\pi}{\sqrt{a}}$, equation* (1.4) *has a sing-changing periodic solution with the minimal period T.*

In the present paper, we generalise these assertions to a non-autonomous case and an arbitrary power of the super-linearity in (1.4) (see Corollaries 2.28, 2.31 and Remark 2.29). Therefore, for $\omega > 0$ we consider the non-autonomous periodic problem

$$u'' = p(t)u + h(t)|u|^\lambda \operatorname{sgn} u; \quad u(0) = u(\omega), \; u'(0) = u'(\omega), \tag{1.5}$$

where $p, h \in L([0, \omega])$ and $\lambda > 1$. It is natural to refer the equation in (1.5) as a non-autonomous Duffing type equation. The question on the existence and multiplicity of periodic solutions to the autonomous Duffing equations is studied very often in the existing literature and thus, plenty of interested results is known. As for a non-autonomous case, many existence results can be found for the equations with a sub-linear non-linearity. However, the Duffing type equations are characterized by a super-linearity in the equation and we have found only a few results covering this case (see, e. g., [2, 7, 8, 11–13] and references therein). Below, we establish effective conditions for the existence and uniqueness of a **positive** periodic solution to (1.1) and their consequences for problem (1.5) (with a non-autonomous Duffing type equation), which can be easily compared with the facts well known in the autonomous case (1.4). At last, we will show possible extensions for a more general problem than (1.5), namely, for the periodic problem with two super-linear terms

$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u + f(t)|u|^{\mu} \operatorname{sgn} u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega), \tag{1.6}$$

where $p, h, f \in L([0,\omega])$ and $\lambda, \mu > 1$. It is worth mentioning that Duffing type equations with two or more super-linear terms appear when approximating the non-linearity in the equation of pendulum (1.3) by Taylor's polynomials of higher orders than 3.

Throughout the paper, the linear spaces of Lebesgue integrable and continuous functions defined on an interval $I \subseteq \mathbb{R}$ are denoted by the standard symbols $L(I)$ and $C(I)$, respectively. Having $A \subseteq L(I)$, symbols $\overline{A}$ and $\operatorname{Int} A$ denote the closure and the interior of the set A in the sense of the standard integral norm in $L(I)$. Moreover, $AC^1([a,b])$ stands for the set of functions $u\colon [a,b] \to \mathbb{R}$ which are absolutely continuous together with their first derivatives. Furthermore, $AC_{\ell}([a,b])$ (resp. $AC_u([a,b])$) is the set of absolutely continuous functions $u\colon [a,b] \to \mathbb{R}$ such that u' admits the representation $u'(t) = \gamma(t) + \sigma(t)$ for a. e. $t \in [a,b]$, where $\gamma\colon [a,b] \to \mathbb{R}$ is absolutely continuous and $\sigma\colon [a,b] \to \mathbb{R}$ is a non-decreasing (resp. non-increasing) function whose derivative is equal to zero almost everywhere on $[a,b]$. Finally, for $x \in \mathbb{R}$, we put

$$[x]_+ = \frac{|x| + x}{2}\,, \quad [x]_- = \frac{|x| - x}{2}\,.$$

Definition 1.3 ([9, Definition 0.1])**.** We say that the function $p \in L([0,\omega])$ belongs to the set $\mathcal{V}^+(\omega)$ (resp. $\mathcal{V}^-(\omega)$) if for any function $u \in AC^1([0,\omega])$ satisfying

$$u''(t) \geq p(t)u(t) \quad \text{for a. e. } t \in [0,\omega], \quad u(0) = u(\omega), \quad u'(0) = u'(\omega), \tag{1.7}$$

the inequality

$$u(t) \geq 0 \quad \text{for } t \in [0,\omega] \quad \big(\text{resp. } u(t) \leq 0 \quad \text{for } t \in [0,\omega]\big)$$

is fulfilled.

Definition 1.4 ([9, Definition 0.2])**.** We say that the function $p \in L([0,\omega])$ belongs to the set $\mathcal{V}_0(\omega)$ if the problem

$$u'' = p(t)u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega) \tag{1.8}$$

has a non-trivial sign-constant solution.

Remark 1.5. Efficient conditions for p to belong to each of the sets $\mathcal{V}^+(\omega)$, $\mathcal{V}^-(\omega)$, and $\mathcal{V}_0(\omega)$ are given in [9].

2. Main results

In this section, we formulate all the results, their proofs are postponed till Section 4 below.

Theorem 2.1. *Let* $q(\cdot, 0) \equiv 0$,

$$p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega), \tag{2.1}$$

and there exists a function $\beta \in AC_u([0,\omega])$ *satisfying*

$$\beta(t) > 0 \quad \textit{for } t \in [0,\omega], \tag{2.2}$$

$$\beta''(t) \le p(t)\beta(t) + q(t,\beta(t))\beta(t) \quad \textit{for a. e. } t \in [0,\omega], \tag{2.3}$$

$$\beta(0) = \beta(\omega), \quad \beta'(0) \le \beta'(\omega). \tag{2.4}$$

Then problem (1.1) *has at least one positive solution* u *such that*

$$u(t) \le \beta(t) \quad \textit{for } t \in [0,\omega]. \tag{2.5}$$

Let us introduce the hypothesis:

$$\left.\begin{aligned} &q(t,x) \ge q_0(t,x) \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \ge 0,\\ &q_0\colon [0,\omega] \times [0,+\infty[\, \to \mathbb{R} \text{ is a Carathéodory function},\\ &q_0(t,\cdot)\colon [0,+\infty[\, \to \mathbb{R} \text{ is non-decreasing for a. e. } t \in [0,\omega]. \end{aligned}\right\} \tag{H_1}$$

Corollary 2.2. *Let* $q(\cdot, 0) \equiv 0$, *relation* (2.1) *hold, and at least one of the following conditions be fulfilled:*

(a) *There exists* $c > 0$ *such that*

$$p(t) + q(t,c) \ge 0 \quad \textit{for a. e. } t \in [0,\omega]. \tag{2.6}$$

(b) *Hypothesis* (H_1) *is satisfied and there exists* $r > 0$ *such that* $p + q_0(\cdot, r) \in \mathcal{V}^-(\omega)$.

Then problem (1.1) *has at least one positive solution.*

In the next statement, we give an effective condition guaranteeing that the assumption (b) of Corollary 2.2 is satisfied.

Corollary 2.3. *Let* $q(\cdot, 0) \equiv 0$ *and hypothesis* (H_1) *be satisfied. Let, moreover, condition* (2.1) *hold and*

$$\lim_{x\to+\infty} \int_E q_0(s,x)\mathrm{d}s = +\infty \quad \textit{for every } E \subseteq [0,\omega],\ \operatorname{meas} E > 0. \tag{2.7}$$

Then problem (1.1) *has at least one positive solution.*

Remark 2.4. By using Lebesgue's domination theorem, one can show that for the function q_0 appearing in hypothesis (H_1), condition (2.7) holds if and only if

$$\lim_{x\to+\infty} q_0(t,x) = +\infty \quad \text{for a. e. } t \in [0,\omega]. \tag{2.8}$$

Remark 2.5. Assumption (2.7) in Corollary 2.3 is optimal and cannot be weakened to the assumption

$$\lim_{x\to+\infty} \int_0^\omega q_0(s,x)\mathrm{d}s = +\infty \tag{2.9}$$

(see Example 2.8 below). However, assuming (2.9) instead of (2.7), problem (1.1) may still have a positive solution under a more restrictive assumption on p than (2.1). More precisely, the following statement holds.

Corollary 2.6. *Let $q(\cdot,0) \equiv 0$ and hypothesis (H_1) be satisfied. Let, moreover, condition* (2.9) *hold and there exist $x_0 > 0$ such that*

$$q_0(t,x_0) \geq 0 \quad \textit{for a. e. } t \in [0,\omega]. \tag{2.10}$$

Then problem (1.1) *has at least one positive solution provided that the inclusion*

$$p \in \mathcal{V}^+(\omega) \tag{2.11}$$

holds.

Remark 2.7. By using Lebesgue's domination theorem, one can show that for the function q_0 appearing in hypothesis (H_1), condition (2.9) is satisfied if there exists $E \subseteq [0,\omega]$ such that $\operatorname{meas} E > 0$ and the equality

$$\lim_{x\to+\infty} q_0(t,x) = +\infty \quad \text{for every } t \in E$$

holds.

Example 2.8. Let $0 < a < b < \omega$, $\lambda > 1$,

$$p(t) := -\frac{\pi^2}{(b-a)^2} \quad \text{for } t \in [0,\omega], \quad h(t) := \begin{cases} 1 & \text{for } t \in [0,a[\cup]b,\omega], \\ 0 & \text{for } t \in [a,b], \end{cases}$$

and

$$q(t,x) := h(t)|x|^{\lambda-1} \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \in \mathbb{R}.$$

It is clear that $\int_0^\omega p(s)\mathrm{d}s < 0$ and thus, in view of [9, Remark 0.7, Proposition 10.8], condition (2.1) is fulfilled. Moreover, hypothesis (H_1) holds with $q_0(t,x) := h(t)x^{\lambda-1}$. Therefore, all the assumptions of Corollary 2.3 are satisfied except of (2.7), instead of which condition (2.9) holds. We shall show that problem (1.1) has no positive solution. Indeed, if u is a positive solution to (1.1) then the function u is a positive solution to the equation

$$v'' = -\frac{\pi^2}{(b-a)^2}\, v \tag{2.12}$$

on the interval $[a,b]$, as well. However, the function $v(t) := \sin\frac{\pi(t-a)}{b-a}$ for $t \in [a,b]$ is also a solution to (2.12) with $v(a) = 0$ and $v(b) = 0$, which is in a contradiction with Sturm's separation theorem.

Under the hypothesis

$$\left.\begin{array}{l} \text{for every } d > c > 0 \text{ there exists } h_{cd} \in L([0,\omega]) \text{ such that} \\ h_{cd}(t) \geq 0 \quad \text{for a. e. } t \in [0,\omega],\ h_{cd} \not\equiv 0, \\ q(t,x) \geq h_{cd}(t) \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \in [c,d], \end{array}\right\} \tag{H_2}$$

the assumption $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ in the above results is necessary as follows from the next proposition.

Proposition 2.9. *Let $p \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, hypothesis (H_2) hold, and u be a non-trivial solution to problem* (1.1)*. Then*

$$u(t) < 0 \quad \textit{for } t \in [0,\omega]. \tag{2.13}$$

If the function q is non-decreasing in the second variable, then the assumption that $p + q_0(t,r) \in \mathcal{V}^-(\omega)$ for some $r > 0$ in Corollary 2.2 is necessary, in a certain sense, for the existence of a positive solution to problem (1.1). More precisely, the following statement holds.

Corollary 2.10. *Let* $q(\cdot,0) \equiv 0$,

$$\text{the function } q(t,\cdot)\colon]0,+\infty[\, \to \mathbb{R} \text{ is non-decreasing for a. e. } t \in [0,\omega], \tag{2.14}$$

$$q(\cdot,x) \not\equiv 0 \quad \text{for } x > 0, \tag{2.15}$$

and

$$\text{the function } x \mapsto \int_0^\omega q(s,x)\mathrm{d}s \text{ is not constant in every neighbourhood of } +\infty. \tag{2.16}$$

Then problem (1.1) *has at least one positive solution if and only if* $p \notin \mathcal{V}_0(\omega) \cup \mathcal{V}^-(\omega)$ *and there exists a number* $r > 0$ *such that* $p + q(\cdot, r) \in \mathcal{V}^-(\omega)$.

Now we give two uniqueness type results for problem (1.1). Introduce the following hypothesis:

$$\left.\begin{array}{l}\text{For every } d > c > 0 \text{ and } e > 0, \text{ there exists } h_{cde} \in L([0,\omega]) \text{ such that}\\ h_{cde}(t) > 0 \quad \text{for a. e. } t \in [0,\omega],\\ q(t,x+e) - q(t,x) \geq h_{cde}(t) \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \in [c,d].\end{array}\right\} \tag{H_3}$$

Theorem 2.11. *Let* $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, $q(\cdot,0) \equiv 0$, *and hypothesis* (H_3) *hold. Then problem* (1.1) *has at most one positive solution.*

Quite a stronger assertion can be proved under the assumption that $p \in \mathcal{V}^+(\omega)$. On the other hand, hypothesis (H_3) can be slightly weakened in that case to the following one:

$$\left.\begin{array}{l}\text{For every } d > c > 0 \text{ and } e > 0, \text{ there exists } h_{cde} \in L([0,\omega]) \text{ such that}\\ h_{cde}(t) \geq 0 \quad \text{for a. e. } t \in [0,\omega], \ h_{cde} \not\equiv 0,\\ q(t,x+e) - q(t,x) \geq h_{cde}(t) \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \in [c,d].\end{array}\right\} \tag{H_3'}$$

Theorem 2.12. *Let* $p \in \mathcal{V}^+(\omega)$, *hypothesis* (H_3') *hold, and*

$$q(t,0) \geq 0 \quad \text{for a. e. } t \in [0,\omega]. \tag{2.17}$$

Then problem (1.1) *has at most one positive solution. Moreover, any non-trivial solution to this problem is either positive or negative.*

Observe that, under the assumptions of Theorem 2.12, problem (1.1) has no sign-changing solutions. Another possibility how to ensure this property of solutions to problem (1.1) is presented in Theorem 2.16 below. Introduce the definition:

Definition 2.13. We say that the function $p \in L([0,\omega])$ belongs to the set $\mathcal{D}_1(\omega)$ if for any $a \in [0,\omega[$, the solution u to the initial value problem

$$u'' = \widetilde{p}(t)u; \quad u(a) = 0, \ u'(a) = 1 \tag{2.18}$$

has at most one zero in the interval $]a, a+\omega[$, where $\widetilde{p}$ is the ω-periodic extension of the function p to the whole real axis.

Definition 2.13 is meaningful as follows from the following statement, which is a consequence of two well-known results, namely, Corollary 5.2 stated in [4] and Sturm's comparison theorem (see, e. g., [4, Theorem 3.1]).

Proposition 2.14. *Let* $p \in L([0,\omega])$ *and either*

$$\int_0^\omega [p(s)]_- \mathrm{d}s \leq \frac{16}{\omega}, \tag{2.19}$$

or

$$p(t) \geq -\frac{4\pi^2}{\omega^2} \quad \textit{for a. e. } t \in [0, \omega]. \tag{2.20}$$

Then $p \in \mathcal{D}_1(\omega)$.

Remark 2.15. Observe that $\mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \cup \mathcal{V}^+(\omega) \subset \mathcal{D}_1(\omega)$. Indeed, by virtue of Definition 3.7 and Lemma 3.8 below, we derive $\mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \cup \mathcal{V}^+(\omega) \subseteq \mathcal{D}_1(\omega)$. Moreover, let $p(t) := k$ for $t \in [0, \omega]$. Then it is not difficult to verify that $p \in \mathcal{V}^-(\omega)$ iff $k > 0$, $p \in \mathcal{V}_0(\omega)$ iff $k = 0$, $p \in \mathcal{V}^+(\omega)$ iff $k \in [-\frac{\pi^2}{\omega^2}, 0[$, and $p \in \mathcal{D}_1(\omega)$ iff $k \geq -\frac{4\pi^2}{\omega^2}$. Consequently, $\mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \cup \mathcal{V}^+(\omega) \neq \mathcal{D}_1(\omega)$.

Theorem 2.16. *Let* $p \in \mathcal{D}_1(\omega)$,

$$q(t, x) \geq 0 \quad \textit{for a. e. } t \in [0, \omega] \textit{ and all } x \in \mathbb{R}, \tag{2.21}$$

and

$$\operatorname{meas}\Big(\cup_{n \in \mathbb{N}} \big\{ t \in [0, \omega] : q_n(t) > 0 \big\} \Big) > 0, \tag{2.22}$$

where

$$q_n(t) := \min \Big\{ q(t, x) : \tfrac{1}{n} \leq |x| \leq n \Big\} \quad \textit{for a. e. } t \in [0, \omega],\ n \in \mathbb{N}.$$

Then every non-trivial solution to problem (1.1) *is either positive or negative.*

Remark 2.17. It is easy to verify that condition (2.22) holds if and only if there exists $E \subseteq [0, \omega]$ such that $\operatorname{meas} E > 0$ and

$$q(t, x) > 0 \quad \text{for } t \in E,\ x \in \mathbb{R} \setminus \{0\}, \tag{2.23}$$

as well as, if and only if

$$\lim_{r \to 0+} \operatorname{meas} \big\{ t \in [0, \omega] : f(t, r) > 0 \big\} > 0,$$

where

$$f(t, r) := \min \Big\{ q(t, x) : r \leq |x| \leq \tfrac{1}{r} \Big\} \quad \text{for a. e. } t \in [0, \omega] \text{ and all } r \in]0, 1].$$

Remark 2.18. Assumption (2.22) in Theorem 2.16 is essential and cannot be omitted. Indeed, let

$$p(t) := -\frac{4\pi^2}{\omega^2}, \quad q(t, x) := \big[x - \sin \tfrac{2\pi t}{\omega} \big]_+ \quad \text{for } t \in [0, \omega],\ x \in \mathbb{R}.$$

Then condition (2.21) holds and, by virtue of Proposition 2.14, se get $p \in \mathcal{D}_1(\omega)$. Moreover, we have

$$q\big(t, \sin \tfrac{2\pi t}{\omega}\big) = 0 \quad \text{for } t \in [0, \omega]$$

and thus, it follows from Remark 2.17 that condition (2.22) is violated. Therefore, all the assumptions of Theorem 2.16 are satisfied except of inequality (2.22). However, the function

$$u(t) = \sin \tfrac{2\pi t}{\omega} \quad \text{for } t \in [0, \omega]$$

is a sign-changing solution to problem (1.1).

If q in (1.1) is a function with separated variables, we arrive at the problem

$$u'' = p(t)u + h(t)\varphi(u)u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega), \tag{2.24}$$

where $p, h \in L([0, \omega])$ and $\varphi \in C(\mathbb{R})$. This problem covers a rather wide class of periodic problems arising in applications and serves us as a model problem to illustrate the results stated above.

Theorem 2.19. *Let $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, $\varphi(0) = 0$, and there exists $c > 0$ such that*

$$p(t) + h(t)\varphi(c) \geq 0 \quad \textit{for a. e. } t \in [0, \omega].$$

Then problem (2.24) *has at least one positive solution.*

Theorem 2.20. *Let $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, $\varphi(0) = 0$, the inequality*

$$h(t) > 0 \quad \textit{for a. e. } t \in [0, \omega] \tag{2.25}$$

holds, and

$$\lim_{x \to +\infty} \varphi(x) = +\infty. \tag{2.26}$$

Then problem (2.24) *has at least one positive solution. If, in addition, the function φ is increasing on $[0, +\infty[$ then problem* (2.24) *has a unique positive solution.*

Remark 2.21. If $\varphi(x) > 0$ for $x > 0$ then the assumption $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ in Theorem 2.20 is also necessary for the existence of a positive solution to problem (2.24) (see Proposition 2.9 with $q(t, x) := h(t)\varphi(x)$).

It is shown in [9, Remark 0.7, Proposition 10.8] that if inclusion $p \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ holds, then either $\int_0^\omega p(s)\mathrm{d}s > 0$ or $p \equiv 0$. Hence, Theorem 2.20 immediately yields

Corollary 2.22. *Let the functions h and φ satisfy the assumptions of Theorem 2.20. Let, moreover,*

$$\int_0^\omega p(s)\mathrm{d}s \leq 0, \qquad p \not\equiv 0. \tag{2.27}$$

Then problem (2.24) *has at least one positive solution. If, in addition, the function φ is increasing on $[0, +\infty[$ then problem* (2.24) *has a unique positive solution.*

Remark 2.23. It follows from [11, Corollary 4.1] that problem (2.24) has a positive solution provided that (2.25) holds,

$$p(t) \leq 0 \quad \text{for a. e. } t \in [0, \omega], \quad p \not\equiv 0, \quad \int_0^\omega |p(s)|\mathrm{d}s \leq \frac{4}{\omega}\,, \tag{2.28}$$

and

$$\varphi(0) = 0, \quad \varphi(x) \geq 0 \quad \text{for } x > 0, \quad \lim_{x \to +\infty} \varphi(x) = +\infty. \tag{2.29}$$

In Corollary 2.22, condition (2.28) is weakened to (2.27) and condition (2.29) is relaxed to $\varphi(0) = 0$ and (2.26). Therefore, Corollary 2.22 extends the results stated in [11].

Assumption (2.25) in Theorem 2.20 is optimal and cannot be weakened to the assumption

$$h(t) \geq 0 \quad \text{for a. e. } t \in [0, \omega], \quad h \not\equiv 0 \tag{2.30}$$

(see Example 2.8). However, under a stronger assumption on the function p, problem (2.24) still may have a positive solution as follows from the next statement.

Theorem 2.24. *Let $p \in \mathcal{V}^+(\omega)$, $\varphi(0) = 0$, and conditions* (2.26) *and* (2.30) *hold. Then problem* (2.24) *has at least one positive solution. If, in addition, the function φ is increasing on $[0, +\infty[$ then problem* (2.24) *has a unique positive solution and no sign-changing solution.*

If the function φ in (2.24) is even, problem (2.24) can be rewritten in the form

$$u'' = p(t)u + h(t)\varphi(|u|)u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega), \tag{2.31}$$

where $p, h \in L([0,\omega])$ and $\varphi \in C([0,+\infty[\,)$. Clearly, if u is a solution to problem (2.31) then the function $-u$ is its solution, as well.

Theorem 2.25. *Let $\varphi(0) = 0$, φ is increasing on $[0,+\infty[$, and relations* (2.25) *and* (2.26) *be fulfilled. Then the following assertions hold:*

1. *If $p \in \mathcal{V}^-(\omega)\cup\mathcal{V}_0(\omega)$, then problem* (2.31) *possesses only the trivial solution.*
2. *If $p \in \mathcal{D}_1(\omega) \setminus \big[\mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)\big]$, then problem* (2.31) *has exactly three solutions (positive, negative, and trivial).*
3. *If $p \notin \mathcal{D}_1(\omega)$, then problem* (2.31) *possesses exactly three sign-constant solutions (positive, negative, and trivial).*

In the next theorem, assumption (2.25) is relaxed to (2.30).

Theorem 2.26. *Let $\varphi(0) = 0$, φ is increasing on $[0,+\infty[$, and relations* (2.26) *and* (2.30) *be fulfilled. Then the following assertions hold:*

1. *If $p \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, then problem* (2.31) *has only the trivial solution.*
2. *If $p \in \mathcal{V}^+(\omega)$, then problem* (2.31) *possesses exactly three solutions (positive, negative, and trivial).*
3. *If $p \in \mathcal{D}_1(\omega) \setminus \big[\mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \cup \mathcal{V}^+(\omega)\big]$, then every non-trivial solution to problem* (2.31) *is either positive or negative.*

Remark 2.27. It follows from Remark 2.15 that assertions (2) and (3) of Theorem 2.25, as well as assertion (3) of Theorem 2.26 are meaningful.

Now we derive corollaries for a non-autonomous Duffing equation and compare the results obtained with the facts well known in the autonomous case.

Corollary 2.28. *Let $\lambda > 1$ and condition* (2.25) *be fulfilled. Then the following assertions hold:*

1. *If $p \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ then problem* (1.5) *possesses only the trivial solution.*
2. *If $p \in \mathcal{D}_1(\omega)\setminus\big[\mathcal{V}^-(\omega)\cup\mathcal{V}_0(\omega)\big]$, then problem* (1.5) *has exactly three solutions (positive, negative, and trivial).*
3. *If $p \notin \mathcal{D}_1(\omega)$, then problem* (1.5) *possesses exactly three sign-constant solutions (positive, negative, and trivial).*

Remark 2.29. It is clear that the Duffing equation (1.4) is a particular case of the equation in (1.5), where $\lambda := 3$ and

$$p(t) := -a, \quad h(t) := b \quad \text{for } t \in [0,\omega]. \tag{2.32}$$

One can easily derive that, in this case, $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ if and only if $a > 0$. Hence, Corollary 2.28(1) yields that for any $a \leq 0$ and $b, \omega > 0$, equation (1.4) has no non-trivial ω-periodic solution. This is in a compliance with assertion (1) of Proposition 1.2. On the other hand, it follows from Corollary 2.28(2),(3) and Remark 2.15 that for any $a, b, \omega > 0$, equation (1.4) has a unique positive (resp. negative) ω-periodic solution. This is in a compliance with assertion (2) of Proposition 1.2. Finally, we know that $p \in \mathcal{D}_1(\omega)$ provided $a \leq 4\pi^2/\omega^2$ (see Proposition 2.14) and thus, Corollary 2.28(2) yields that if $a, b > 0$ and y is a periodic solution to equation (1.4) corresponding to a closed orbit on Fig. 1, then the minimal period T of

y satisfies the estimate

$$T > \frac{2\pi}{\sqrt{a}}\,.$$

This is in a compliance with assertion (3) of Proposition 1.2.

Therefore, Corollary 2.28 naturally extends the basic facts concerning periodic solutions to the Duffing equation (1.4) to the non-autonomous case.

Remark 2.30. It follows from [13] that problem (1.5) with a continuous p and $h(t) := c$ has at least one ω-periodic solution if $c > 0$ and

$$p(t) \leq 0 \quad \text{for } t \in [0,\omega], \quad p \not\equiv 0. \tag{2.33}$$

In Corollary 2.28, condition (2.33) is weakened to (2.1), which is guaranteed, e. g., by assumption (2.27). Moreover, the uniqueness of a positive solution follows from Corollary 2.28.

In the next corollary, assumption (2.25) is relaxed to (2.30) which is possible in the non-autonomous case only (if $h(t) := b$ then assumptions (2.25) and (2.30) coincide).

Corollary 2.31. *Let $\lambda > 1$ and condition* (2.30) *be fulfilled. Then the following assertions hold:*

(1) *If $p \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ then problem* (1.5) *possesses only the trivial solution.*
(2) *If $p \in \mathcal{V}^+(\omega)$ then problem* (1.5) *has exactly three solutions (positive, negative, and trivial).*
(3) *If $p \in \mathcal{D}_1(\omega) \setminus \big[\mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \cup \mathcal{V}^+(\omega)\big]$, then every non-trivial solution to problem* (1.5) *is either positive or negative.*

Remark 2.32. It follows from Remark 2.15 that assertions (2) and (3) of Corollary 2.28, as well as assertion (3) of Corollary 2.31 are meaningful.

Finally, we consider problem (1.6) involving two super-linear terms. Clearly, if u is a solution to problem (1.6) then the function $-u$ is its solution, as well. Therefore, the following statements follow from Corollaries 2.3 and 2.6.

Theorem 2.33. *Let $\lambda > \mu > 1$, relation* (2.30) *hold, and there exist $c > 0$ such that*

$$[f(t)]_- \leq ch(t) \quad \textit{for a. e. } t \in [0,\omega]. \tag{2.34}$$

If, moreover, $p \in \mathcal{V}^+(\omega)$ then problem (1.6) *has at least three solutions (positive, negative, and trivial).*

Remark 2.34. If

$$f(t) \geq 0 \quad \text{for a. e. } t \in [0,\omega] \tag{2.35}$$

then inequality (2.34) is satisfied and, by virtue of Theorem 2.12, we can claim in Theorem 2.33 that problem (1.6) has exactly three solutions.

Theorem 2.35. *Let $\lambda > \mu > 1$, relation* (2.25) *hold, and*

$$[f]_-^{\frac{\lambda-1}{\lambda-\mu}} h^{-\frac{\mu-1}{\lambda-\mu}} \in L([0,\omega]). \tag{2.36}$$

If, moreover, $p \in \mathcal{V}^+(\omega)$ then problem (1.6) *has at least three solutions (positive, negative, and trivial).*

Remark 2.36. Observe that if there exists $c > 0$ such that inequality (2.34) holds then inclusion (2.36) is satisfied.

Moreover, it follows from Proposition 2.9 that if inequality (2.35) is fulfilled, then the assumptiuon $p \in \mathcal{V}^+(\omega)$ in Theorem 2.35 is necessary for the existence of a non-trivial solution to problem (1.6).

3. Auxiliary Statements

In this section, we establish several statements which we need to prove main results. First of all, for convenience of references, we recall some results proved in [9].

Lemma 3.1 ([9, Proposition 10.2])**.** *The equality* $\mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) = \overline{\mathcal{V}^-(\omega)}$ *holds.*

Lemma 3.2 ([9, Theorem 11.1])**.** *Let* $g \in L([0,\omega])$, $g \not\equiv 0$,

$$\int_0^\omega [g(s)]_- \mathrm{d}s < \frac{4}{\omega}\,, \tag{3.1}$$

and

$$\int_0^\omega [g(s)]_+ \mathrm{d}s \geq \frac{4}{\omega}\left(\frac{1}{1-\frac{\omega}{4}\int_0^\omega [g(s)]_- \mathrm{d}s} - 1\right). \tag{3.2}$$

Then $g \in \mathcal{V}^-(\omega)$.

Lemma 3.3. *Let* $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$. *Then there exists* $h \in L([0,\omega])$ *such that*

$$h(t) \geq 0 \quad \textit{for a. e. } t \in [0,\omega] \tag{3.3}$$

and $p + h \in \operatorname{Int} \mathcal{V}^+(\omega)$.

Proof. It follows from Propositions 10.10 and 10.11 stated in [9]. □

Lemma 3.4 ([9, Theorem 16.2])**.** *Let* $g \in \mathcal{V}^-(\omega)$. *Then there exist* $\nu, \Delta > 0$ *such that for any non-positive function* $f \in L([0,\omega])$, *the problem*

$$u'' = g(t)u + f(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega) \tag{3.4}$$

has a unique solution u *and this solution satisfies*

$$\nu \int_0^\omega |f(s)| \mathrm{d}s \leq u(t) \leq \Delta \int_0^\omega |f(s)| \mathrm{d}s \quad \textit{for } t \in [0,\omega]. \tag{3.5}$$

Lemma 3.5 ([9, Theorem 16.4])**.** *Let* $g \in \operatorname{Int} \mathcal{V}^+(\omega)$. *Then there exist numbers* $\nu, \Delta > 0$ *such that for any non-negative function* $f \in L([0,\omega])$, *problem* (3.4) *has a unique solution* u *and this solution satisfies* (3.5).

Lemma 3.6 ([9, Theorem 8.3])**.** *Let* $g \in L([0,\omega])$. *Then the inclusion* $g \in \mathcal{V}^-(\omega)$ *holds if and only if there exist a positive function* $\gamma \in AC^1([0,\omega])$ *satisfying*

$$\gamma''(t) \leq g(t)\gamma(t) \quad \textit{for a. e. } t \in [0,\omega], \quad \gamma(0) \geq \gamma(\omega), \quad \frac{\gamma'(\omega)}{\gamma(\omega)} \geq \frac{\gamma'(0)}{\gamma(0)}\,,$$

and

$$\gamma(0) - \gamma(\omega) + \frac{\gamma'(\omega)}{\gamma(\omega)} - \frac{\gamma'(0)}{\gamma(0)} + \operatorname{meas}\big\{t \in [0,\omega] : \gamma''(t) < g(t)\gamma(t)\big\} > 0.$$

Definition 3.7 ([9, Definition 0.4]). We say that the function $p \in L([0,\omega])$ belongs to the set $\mathcal{D}(\omega)$ if the problem

$$u'' = \widetilde{p}(t)u; \quad u(a) = 0, \ u(b) = 0 \tag{3.6}$$

has no non-trivial solution for any $a, b \in \mathbb{R}$ satisfying $0 < b - a < \omega$, where $\widetilde{p}$ is the ω-periodic extension of the function p to the whole real axis.

Lemma 3.8. *$\mathcal{D}(\omega) = \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \cup \mathcal{V}^+(\omega)$ and $\operatorname{Int}\mathcal{D}(\omega) = \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \cup \operatorname{Int}\mathcal{V}^+(\omega)$.*

Proof. It follows from Propositions 2.1, 10.5, and 10.6 stated in [9]. □

Lemma 3.9 ([9, Proposition 2.2]). *Let $p \in L([0,\omega])$. Then the inclusion $p \in \operatorname{Int}\mathcal{D}(\omega)$ holds if and only if problem (3.6) has no non-trivial solution for any $a, b \in \mathbb{R}$ satisfying $0 < b - a \leq \omega$, where $\widetilde{p}$ is the ω-periodic extension of the function p to the whole real axis.*

Lemma 3.10. *Let $p \in \mathcal{D}(\omega)$. Then the inclusion $p + \ell \in \operatorname{Int}\mathcal{D}(\omega)$ holds for any function $\ell \in L([0,\omega])$ satisfying*

$$\ell(t) \geq 0 \quad \text{for } t \in [0,\omega], \quad \ell \not\equiv 0. \tag{3.7}$$

Proof. Extend the functions p and ℓ periodically to the whole real axis and denote them by the same symbols. According to Lemma 3.9, it is sufficient to show that the problem

$$u'' = \big(p(t) + \ell(t)\big)u; \quad u(a) = 0, \ u(b) = 0 \tag{3.8}$$

has no non-trivial solution for any $a < b$ satisfying $b - a \leq \omega$.

Let $a, b \in \mathbb{R}$ with $0 < b - a \leq \omega$ be arbitrary and let u be a solution to problem (3.8). Suppose that $u \not\equiv 0$ on $[a, b]$. Then we can assume without loss of generality that there exists $t_0 \in\,]a, b]$ such that

$$u(t) > 0 \quad \text{for } t \in\,]a, t_0[\,, \quad u(t_0) = 0, \quad u'(t_0) < 0. \tag{3.9}$$

Let v be a solution to the initial value problem

$$v'' = p(t)v; \quad v(a) = 0, \ v'(a) = 1. \tag{3.10}$$

Since we assume $p \in \mathcal{D}(\omega)$, it is clear that

$$v(t) > 0 \quad \text{for } t \in\,]a, a + \omega[\,. \tag{3.11}$$

Moreover, from (3.8) and (3.10) we get

$$\big(u'(t)v(t) - u(t)v'(t)\big)' = \ell(t)u(t)v(t) \quad \text{for a. e. } t \in [a, b].$$

Hence, by virtue of conditions (3.7), (3.8), (3.9), (3.10), and (3.11), it follows from the latter equality that

$$u'(t_0)v(t_0) = \int_a^{t_0} \ell(s)u(s)v(s)\mathrm{d}s \geq 0$$

and

$$u'(t_0)v(t_0) > 0 \quad \text{whenever } t_0 = b \text{ and } b = a + \omega.$$

Consequently, in view of the third condition in (3.9), we obtain $v(t_0) \leq 0$, where $v(t_0) < 0$ if $t_0 = a+\omega$, which contradicts (3.11). The contradiction obtained proves that $u \equiv 0$ on $[a, b]$ and thus, $p + \ell \in \operatorname{Int}\mathcal{D}(\omega)$. □

Lemma 3.11 ([9, Proposition 2.5]). *Let $g\colon \mathbb{R}\to\mathbb{R}$ be an ω-periodic function such that $g\in\mathcal{D}(\omega)$ (resp. $g\in\operatorname{Int}\mathcal{D}(\omega)$). Then for any $a<b$ and $w\in AC^1([a,b])$ satisfying $b-a<\omega$ (resp. $b-a\le\omega$) and*

$$w''(t)\ge g(t)w(t)\quad \textit{for a. e. } t\in[a,b],\quad w(a)\le 0,\quad w(b)\le 0,$$

the inequality

$$w(t)\le 0\quad \textit{for } t\in[a,b] \tag{3.12}$$

holds.

Now, we recall a classical result concerning the solvability of the periodic problem

$$u''=f(t,u);\quad u(a)=u(b),\ u'(a)=u'(b), \tag{3.13}$$

where $f\colon [a,b]\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function (see, e. g., [3]).

Lemma 3.12. *Let there exist functions $\alpha\in AC_\ell([a,b])$ and $\beta\in AC_u([a,b])$ satisfying*

$$\alpha(t)\le\beta(t)\quad \textit{for } t\in[a,b],$$

$$\alpha''(t)\ge f(t,\alpha(t))\quad \textit{for a. e. } t\in[a,b],\quad \alpha(a)=\alpha(b),\quad \alpha'(a)\ge\alpha'(b),$$

and

$$\beta''(t)\le f(t,\beta(t))\quad \textit{for a. e. } t\in[a,b],\quad \beta(a)=\beta(b),\quad \beta'(a)\le\beta'(b).$$

Then problem (3.13) *has at least one solution u such that*

$$\alpha(t)\le u(t)\le\beta(t)\quad \textit{for } t\in[a,b].$$

Proposition 3.13. *Let $p\in L([0,\omega];\mathbb{R})$ and $f\colon [0,\omega]\times\,]0,+\infty[\,\to\mathbb{R}$ be a locally Carathéodory function*[1] *such that*

$$\textit{the function } f(t,\cdot)\colon\,]0,+\infty[\,\to\mathbb{R} \textit{ is non-decreasing for a. e. } t\in[0,\omega] \tag{3.14}$$

and

$$\lim_{x\to+\infty}\int_E f(s,x)\mathrm{d}s=+\infty\quad \textit{for every } E\subseteq[0,\omega],\ \operatorname{meas}E>0. \tag{3.15}$$

Then there exists $K>0$ such that

$$p+f(\cdot,x)\in\mathcal{V}^-(\omega)\quad \textit{for } x\ge K. \tag{3.16}$$

Proof. Assume, in addition, that

$$f(t,x)\ge 0\quad \text{for a. e. } t\in[0,\omega] \text{ and all } x\ge 1. \tag{3.17}$$

We first show that

$$[p(t)+f(t,\cdot)]_-\colon\,]0,+\infty[\,\to\mathbb{R} \text{ is non-increasing for a. e. } t\in[0,\omega] \tag{3.18}$$

and

$$[p(t)+f(t,\cdot)]_+\colon\,]0,+\infty[\,\to\mathbb{R} \text{ is non-decreasing for a. e. } t\in[0,\omega]. \tag{3.19}$$

Indeed, in view of assumption (3.14), for any $x_2\ge x_1>0$ we have

$$\begin{aligned}[p(t)+f(t,x_2)]_- &= \frac{1}{2}\Big(|p(t)+f(t,x_2)|-(p(t)+f(t,x_2))\Big)\\ &\le \frac{1}{2}\Big(|p(t)+f(t,x_1)|+|f(t,x_2)-f(t,x_1)|-(p(t)+f(t,x_2))\Big)\end{aligned}$$

[1]It means that for any $[x_1,x_2]\subset\,]0,+\infty[$, the restriction of f to the set $[0,\omega]\times[x_1,x_2]$ is a Carathéodory function.

$$= \frac{1}{2}\Big(|p(t)+f(t,x_1)| - (p(t)+f(t,x_1))\Big) = [p(t)+f(t,x_1)]_-$$

for a. e. $t \in [0,\omega]$ and thus, relation (3.18) holds. On the other hand, using assumption (3.14) and the inequality

$$|y+z| \geq |y| - z \quad \text{for } y, z \in \mathbb{R},\ z \geq 0,$$

for any $x_2 \geq x_1 > 0$ we get

$$\begin{aligned} [p(t)+f(t,x_2)]_+ &= \frac{1}{2}\Big(|p(t)+f(t,x_2)| + (p(t)+f(t,x_2))\Big) \\ &\geq \frac{1}{2}\Big(|p(t)+f(t,x_1)| - (f(t,x_2)-f(t,x_1)) + (p(t)+f(t,x_2))\Big) \\ &= \frac{1}{2}\Big(|p(t)+f(t,x_1)| + (p(t)+f(t,x_1))\Big) = [p(t)+f(t,x_1)]_+ \end{aligned}$$

for a. e. $t \in [0,\omega]$ and thus, relation (3.19) holds, as well.

Now observe that

$$\int_0^\omega [p(s)+f(s,n)]_+ \mathrm{d}s \geq \int_0^\omega p(s)\mathrm{d}s + \int_0^\omega f(s,n)\mathrm{d}s \quad \text{for } n \in \mathbb{N}.$$

Hence, assumption (3.15) yields that

$$\lim_{n\to+\infty} \int_0^\omega [p(s)+f(s,n)]_+ \mathrm{d}s = +\infty. \tag{3.20}$$

Furthermore, we will show that

$$\lim_{n\to+\infty} \int_0^\omega f_n(s)\mathrm{d}s = 0, \tag{3.21}$$

where

$$f_n(t) := [p(t)+f(t,n)]_- \quad \text{for a. e. } t \in [0,\omega],\ n \in \mathbb{N}.$$

Indeed, let

$$A_n := \big\{t \in [0,\omega] : p(t)+f(t,n) \leq 0\big\} \quad \text{for } n \in \mathbb{N}, \qquad A_0 := \cap_{n=1}^{+\infty} A_n.$$

It is clear that A_n are measurable sets, $A_{n+1} \subseteq A_n$ for $n \in \mathbb{N}$ and thus, the set A_0 is also measurable and

$$\operatorname{meas} A_0 = \lim_{n\to+\infty} \operatorname{meas} A_n. \tag{3.22}$$

Moreover, we have

$$0 \leq \int_{A_n} f_n(s)\mathrm{d}s = -\int_{A_n} p(s)\mathrm{d}s - \int_{A_n} f(s,n)\mathrm{d}s \quad \text{for } n \in \mathbb{N} \tag{3.23}$$

which, in view of (3.17), yields that

$$\int_{A_0} f(s,n)\mathrm{d}s \leq \int_{A_n} f(s,n)\mathrm{d}s \leq \int_0^\omega |p(s)|\mathrm{d}s \quad \text{for } n \in \mathbb{N}.$$

Therefore, by virtue of assumption (3.15), we get

$$\operatorname{meas} A_0 = 0. \tag{3.24}$$

Now let $\varepsilon > 0$ be arbitrary. Since the Lebesgue integral has the so-called property of an absolutely continuous integral, there exists a number $\delta > 0$ such that

$$\int_B |p(s)|\mathrm{d}s < \varepsilon \quad \text{for every } B \subseteq [0,\omega],\ \operatorname{meas} B < \delta. \tag{3.25}$$

On the other hand, it follows from relations (3.22) and (3.24) that there is a number $n_0 \in \mathbb{N}$ such that

$$\operatorname{meas} A_n < \delta \quad \text{for } n \geq n_0. \tag{3.26}$$

Consequently, in view of (3.17), (3.23), (3.25), and (3.26), we get

$$0 \leq \int_0^\omega f_n(s)\mathrm{d}s = \int_{A_n} f_n(s)\mathrm{d}s \leq \int_{A_n} |p(s)|\mathrm{d}s < \varepsilon \quad \text{for } n \geq n_0$$

and thus, desired relation (3.21) holds.

Finally, in view (3.18), (3.19), (3.20), and (3.21), there exists $K > 0$ such that for any $x \geq K$, the inequalities

$$\int_0^\omega [p(s) + f(s,x)]_-\mathrm{d}s < \frac{4}{\omega}$$

and

$$\int_0^\omega [p(s) + f(s,x)]_+\mathrm{d}s \geq \frac{4}{\omega}\left(\frac{1}{1 - \frac{\omega}{4}\int_0^\omega [p(s) + f(s,x)]_-\mathrm{d}s} - 1\right)$$

are fulfilled. Consequently, by virtue of Lemma 3.2 with $g(t) := p(t) + f(t,x)$, condition (3.16) holds.

To finish the proof it is sufficient to mention that if condition (3.17) is violated, then we put

$$\widetilde{p}(t) := p(t) + f(t,1), \quad \widetilde{f}(t,x) := f(t,x) - f(t,1)$$

and the assertion of the proposition follows from the above proved with $\widetilde{p}$ and $\widetilde{f}$ instead of p and f because, in view of assumption (3.14), we have

$$\widetilde{f}(t,x) \geq 0 \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \geq 1.$$

$\square$

Proposition 3.14. *Let $p \in \mathcal{V}^+(\omega)$ and $f\colon [0,\omega]\times\,]0,+\infty[\,\to \mathbb{R}$ be a locally Carathéodory function*[2] *such that condition* (3.14) *holds. Let, moreover, there exists $x_0 > 0$ such that*

$$f(t,x_0) \geq 0 \quad \textit{for a. e. } t \in [0,\omega] \tag{3.27}$$

and

$$\lim_{x\to+\infty} \int_0^\omega f(s,x)\mathrm{d}s = +\infty. \tag{3.28}$$

Then there exists $K \geq x_0$ such that relation (3.16) *is satisfied.*

Proof. We first mention that, in view of assumption (3.14) and (3.27), we have

$$f(t,x) \geq 0 \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \geq x_0. \tag{3.29}$$

Put

$$x_1 := \sup\left\{x \geq x_0 : \int_0^\omega f(s,x)\mathrm{d}s = \int_0^\omega f(s,x_0)\mathrm{d}s\right\}.$$

Clearly, assumption (3.28) yields that $x_0 \leq x_1 < +\infty$. Moreover, since the function f satisfies condition (3.14), we have

$$f(t,x) \geq f(t,x_1) \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \geq x_1 \tag{3.30}$$

and

$$\operatorname{meas}\big\{t \in [0,\omega] : f(t,x) > f(t,x_1)\big\} > 0 \quad \text{for } x > x_1. \tag{3.31}$$

[2]It means that for any $[x_1,x_2] \subset\,]0,+\infty[$, the restriction of f to the set $[0,\omega]\times[x_1,x_2]$ is a Carathéodory function.

Let

$$f_0(t,x) := f(t,x) - f(t,x_1) \quad \text{for a.e. } t \in [0,\omega] \text{ and all } x \geq x_1.$$

Then, in view of conditions (3.30) and (3.31), it follows from Lemmas 3.8 and 3.10 with $\ell(t) := f_0(t,x)$ that

$$p + \frac{1}{2}\, f_0(\cdot,x) \in \operatorname{Int}\mathcal{D}(\omega) \quad \text{for } x > x_1. \tag{3.32}$$

We will show that there exists $x_2 > x_1$ such that

$$p + \frac{1}{2}\, f_0(\cdot,x_2) \in \operatorname{Int}\mathcal{V}^+(\omega) \tag{3.33}$$

holds. Indeed, assume on the contrary that (3.33) is violated for every $x_2 > x_1$. Then there is a sequence $\{y_n\}_{n=1}^{+\infty} \subset]x_1,+\infty[$ such that $\lim_{n\to+\infty} y_n = x_1$ and

$$p + \frac{1}{2}\, f_0(\cdot,y_n) \notin \operatorname{Int}\mathcal{V}^+(\omega) \quad \text{for } n \in \mathbb{N}.$$

In view of (3.32), it follows from Lemma 3.8 that $p + \frac{1}{2}\, f_0(\cdot,y_n) \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ for $n \in \mathbb{N}$ and thus, taking into account that the function f_0 is continuous in the second argument, we get

$$p + \frac{1}{2}\, f_0(\cdot,x_1) \in \overline{\mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)}.$$

However, $p + \frac{1}{2}\, f_0(\cdot,x_1) = p$. Consequently, Lemma 3.1 then yields $p \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ which contradicts the assumption $p \in \mathcal{V}^+(\omega)$. The contradiction obtained proves that (3.33) holds with some $x_2 > x_1$.

Now let $\nu > 0$ be the number appearing in the assertion of Lemma 3.5 with $g(t) := p(t) + \frac{1}{2}\, f_0(t,x_2)$. According to assumption (3.28), there exists $K \geq x_2$ such that

$$\int_0^\omega f_0(s,x)\mathrm{d}s \geq \frac{3}{\nu} \quad \text{for } x \geq K. \tag{3.34}$$

Let $x \geq K$ be arbitrary. Then, in view of (3.30) and (3.33), it follows from Lemma 3.5 that the problem

$$\gamma'' = \Big(p(t) + \frac{1}{2}\, f_0(t,x_2)\Big)\gamma + f_0(t,x); \quad \gamma(0) = \gamma(\omega),\ \gamma'(0) = \gamma'(\omega) \tag{3.35}$$

has a unique solution γ and

$$\gamma(t) \geq \nu \int_0^\omega f_0(s,x)\mathrm{d}s \quad \text{for } t \in [0,\omega].$$

Hence, on account of (3.34), we get $\gamma(t) \geq 3$ for $t \in [0,\omega]$. Therefore, taking (3.14), (3.29), and (3.31) into account, from (3.35) we obtain

$$\begin{aligned}
\gamma''(t) &\leq \Big(p(t) + \frac{1}{2}\, f_0(t,x_2) + \frac{1}{3}\, f_0(t,x)\Big)\gamma(t) \\
&\leq \big(p(t) + f_0(t,x)\big)\gamma(t) \\
&\leq \big(p(t) + f(t,x)\big)\gamma(t) \quad \text{for a.e. } t \in [0,\omega],
\end{aligned}$$

and

$$\operatorname{meas}\{t \in [0,\omega] : \gamma''(t) < (p(t) + f(t,x))\gamma(t)\} > 0.$$

The assertion of the proposition follows now from Lemma 3.6 with $g(t) := p(t) + f(t,x)$. $\square$

The last three statements deal with the existence of functions α and β appearing in Lemma 3.12 with $f(t,x) := p(t)x + q(t,x)x$, which are usually called lower and upper functions of problem (1.1), respectively.

Proposition 3.15. *Let $\ell \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$. Then for any $c > 0$, there exists a function $\alpha \in AC^1([0,\omega])$ such that*

$$0 < \alpha(t) \le c \quad \textit{for } t \in [0,\omega] \tag{3.36}$$

and

$$\alpha''(t) \ge \ell(t)\alpha(t) \quad \textit{for a. e. } t \in [0,\omega], \quad \alpha(0) = \alpha(\omega), \quad \alpha'(0) = \alpha'(\omega). \tag{3.37}$$

Proof. Let $c > 0$ be arbitrary. According to Lemma 3.3, there exists a non-negative function $h \in L([0,\omega])$ such that $\ell + h \in \operatorname{Int} \mathcal{V}^+(\omega)$. Let $\nu, \Delta > 0$ be numbers appearing in the assertion of Lemma 3.5 with $g(t) := \ell(t) + h(t)$. Then it follows from Lemma 3.5 that the problem

$$\alpha'' = \big(\ell(t) + h(t)\big)\alpha + \frac{c}{\Delta\omega}\,; \quad \alpha(0) = \alpha(\omega),\ \alpha'(0) = \alpha'(\omega)$$

has a unique solution α and

$$\frac{\nu}{\Delta}\, c \le \alpha(t) \le c \quad \text{for } t \in [0,\omega].$$

Consequently, in view of inequality (3.3), the function α satisfies relations (3.36) and (3.37). $\square$

Proposition 3.16. *Let $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ and $q\colon [0,\omega] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying*

$$q(t,0) = 0 \quad \textit{for a. e. } t \in [0,\omega]. \tag{3.38}$$

Then for any $c > 0$, there exists a function $\alpha \in AC^1([0,\omega])$ such that relation (3.36) *holds and*

$$\alpha''(t) \ge p(t)\alpha(t) + q(t,\alpha(t))\alpha(t) \quad \textit{for a. e. } t \in [0,\omega], \tag{3.39}$$

$$\alpha(0) = \alpha(\omega), \quad \alpha'(0) = \alpha'(\omega). \tag{3.40}$$

Proof. Since q is a Carathéodory function with property (3.38), there exist a non-negative function $h \in L([0,\omega])$ and a non-negative, non-decreasing function $\varphi \in C([0,+\infty[)$ such that $\varphi(0) = 0$ and

$$|q(t,x)| \le h(t)\varphi(|x|) \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \in \mathbb{R},\ |x| \le 1. \tag{3.41}$$

We first show that there is $\varepsilon_0 \in \,]0,1]$ such that

$$p + h\varphi(\varepsilon) \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \quad \text{for every } \varepsilon \in \,]0,\varepsilon_0]. \tag{3.42}$$

Indeed, assume on the contrary that there exists a sequence $\{\varepsilon_n\}_{n=1}^{+\infty}$ of numbers from the interval $]0,1]$ such that

$$p + h\varphi(\varepsilon_n) \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \quad \text{for } n \in \mathbb{N}, \qquad \lim_{n\to+\infty} \varepsilon_n = 0.$$

Since $\varphi(0) = 0$, it is clear that

$$\lim_{n\to+\infty} \int_0^\omega |h(s)\varphi(\varepsilon_n)|\mathrm{d}s = 0.$$

Consequently, we have $p \in \overline{\mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)}$ which, by virtue of Lemma 3.1, contradicts the assumption $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$.

Now let $c > 0$ be arbitrary and $\delta := \min\{\varepsilon_0, c\}$. Then, in view of (3.42), it follows from Proposition 3.15 with $\ell(t) := p(t)+h(t)\varphi(\delta)$ that there exists a function $\alpha \in AC^1([0,\omega])$ such that

$$0 < \alpha(t) \le \delta \quad \text{for } t \in [0,\omega]$$

and

$$\alpha''(t) \ge \big(p(t) + h(t)\varphi(\delta)\big)\alpha(t) \quad \text{for a. e. } t \in [0,\omega], \quad \alpha(0) = \alpha(\omega), \quad \alpha'(0) = \alpha'(\omega).$$

Taking now into account that the function φ is non-decreasing and inequality (3.41) holds, we easily conclude that the function α satisfies relations (3.36), (3.39), and (3.40). □

Proposition 3.17. *Let $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ and $q\colon [0,\omega] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying hypothesis (H_1). Let, moreover, there exist $r > 0$ such that*

$$p + q_0(\cdot, r) \in \mathcal{V}^-(\omega). \tag{3.43}$$

Then for any $c \ge r$, there exists a function $\beta \in AC^1([0,\omega])$ satisfying inequalities (2.3) *and*

$$\beta(0) = \beta(\omega), \quad \beta'(0) = \beta'(\omega), \tag{3.44}$$

$$\beta(t) \ge c \quad \textit{for } t \in [0,\omega]. \tag{3.45}$$

Proof. Let $\nu > 0$ be the number appearing in the assertion of Lemma 3.4 with $g(t) := p(t) + q_0(t, r)$ and let $c \ge r$ be arbitrary. Then, in view of inclusion (3.43), it follows from Lemma 3.4 that the problem

$$\beta'' = \big(p(t) + q_0(t, r)\big)\beta - \frac{c}{\nu\omega}\,; \quad \beta(0) = \beta(\omega),\ \beta'(0) = \beta'(\omega)$$

has a unique solution β and this solution satisfies inequality (3.45). Consequently, (3.44) holds and since $c \ge r$, hypothesis (H_1) guarantees that the function β satisfies condition (2.3), as well. □

4. Proofs of Main Results

Proof of Theorem 2.1. According to Proposition 3.16, there exists a function $\alpha \in AC_\ell([0,\omega])$ satisfying relations (3.39), (3.40), and

$$0 < \alpha(t) \le \beta(t) \quad \text{for } t \in [0,\omega]. \tag{4.1}$$

Consequently, all the assumptions of Lemma 3.12 with $f(t,x) := p(t)x + q(t,x)x$, $a := 0$, and $b := \omega$ are fulfilled and thus, problem (1.1) has at least one positive solution u such that relation (2.5) holds. □

Proof of Corollary 2.2. By virtue of Theorem 2.1, to prove the corollary it is sufficient to show that, in both cases (a) and (b), there exists a function $\beta \in AC_u([0,\omega])$ satisfying relations (2.2), (2.3), and (2.4).

If condition (a) is fulfilled then it is clear that the constant function $\beta(t) := c$ satisfies (2.2), (2.3), and (2.4).

On the other hand, if condition (b) holds then the existence of a function β fulfilling (2.2), (2.3), and (2.4) follows from Proposition 3.17. □

Proof of Corollary 2.3. Since the function $q_0(t,\cdot)$ in hypothesis (H_1) is non-decreasing for a. e. $t \in [0,\omega]$ and relation (2.7) holds, it follows from Proposition 3.13 with $f(t,x) := q_0(t,x)$ that there exists $r > 0$ such that $p + q_0(\cdot, r) \in \mathcal{V}^-(\omega)$. Consequently, the assertion of the corollary follows from Corollary 2.2(b). □

Proof of Corollary 2.6. Assume that condition (2.11) is satisfied. Since the function $q_0(t,\cdot)$ in hypothesis (H_1) is non-decreasing for a. e. $t \in [0,\omega]$ and relations (2.9) and (2.10) hold, it follows from Proposition 3.14 with $f(t,x) := q_0(t,x)$ that there exists $r \geq x_0$ such that $p + q_0(\cdot, r) \in \mathcal{V}^-(\omega)$. Consequently, the assertion of the corollary follows from Corollary 2.2(b). □

Proof of Proposition 2.9. Assume on the contrary that condition (2.13) is violated. Then it is clear that either

$$u(t) > 0 \quad \text{for } t \in [0,\omega],$$

or

$$\max\{u(t) : t \in [0,\omega]\} > 0, \quad \min\{u(t) : t \in [0,\omega]\} \leq 0, \tag{4.2}$$

or

$$u(t) \leq 0 \quad \text{for } t \in [0,\omega], \quad u(t_0) = 0 \quad \text{for some } t_0 \in [0,\omega], \quad u \not\equiv 0. \tag{4.3}$$

First assume that u is positive on $[0,\omega]$. Then there are numbers $u^* > u_* > 0$ such that

$$u_* \leq u(t) \leq u^* \quad \text{for } t \in [0,\omega]$$

and thus, by virtue of hypothesis (H_2), we have

$$q(t,u(t)) \geq h_{u_* u^*}(t) \geq 0 \quad \text{for a. e. } t \in [0,\omega]. \tag{4.4}$$

However, it means that u is a positive function satisfying relations (1.7) and consequently, $p \notin \mathcal{V}^-(\omega)$ as follows from Definition 1.3. Further, we show that $p \notin \mathcal{V}_0(\omega)$. Suppose on the contrary that problem (1.8) has a positive solution u_0. Then, by virtue of Fredholm's third theorem and condition (4.4), we get the contradiction

$$0 = \int_0^\omega q(s,u(s))u(s)u_0(s)\mathrm{d}s \geq c_0 \int_0^\omega h_{u_* u^*}(s)\mathrm{d}s > 0,$$

where $c_0 := \min\{u(t)u_0(t) : t \in [0,\omega]\}$. Hence, we have proved that $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ in this case, which contradicts the assumption $p \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ of the proposition.

Now assume that (4.2) holds. Extend the functions u, p, and $q(\cdot,x)$ periodically to the whole real axis and denote them by the same symbols. Then there are $a < b$ such that $b - a \leq \omega$ and

$$u(t) > 0 \quad \text{for } t \in]a,b[, \quad u(a) = 0, \quad u(b) = 0. \tag{4.5}$$

Moreover, it follows from hypothesis (H_2) that

$$q(t,x) \geq 0 \quad \text{for a. e. } t \in [a,b] \text{ an all } x \geq 0.$$

Consequently, from the equation in (1.1) we get

$$u''(t) = p(t)u(t) + q(t,u(t))u(t) \geq p(t)u(t) \quad \text{for a. e. } t \in [a,b].$$

Therefore, Lemmas 3.8 and 3.11 (with $g(t) := p(t)$) yield that the inequality

$$u(t) \leq 0 \quad \text{for } t \in [a,b]$$

holds which contradicts (4.5).

Finally assume that (4.3) is satisfies. Then the function u is a solution to the initial value problem

$$w'' = \big(p(t) + q(t, u(t))\big)w; \quad w(t_0) = 0,\ w'(t_0) = 0. \tag{4.6}$$

Consequently, we have $u \equiv 0$ which contradicts (4.3). □

Proof of Corollary 2.10. It is clear that hypothesis (H_1) holds with $q_0(t, x) := q(t, x)$. Therefore, if (2.1) is fulfilled and there exists $r > 0$ such that $p + q(\cdot, r) \in \mathcal{V}^-(\omega)$, then it follows from Corollary 2.2(b) that problem (1.1) has at least one positive solution.

Now suppose that problem (1.1) possesses a positive solution u. In view of assumptions (2.14), (2.15), and $q(\cdot, 0) \equiv 0$, the function q satisfies hypothesis (H_2). Hence, Proposition 2.9 guarantees that $p \notin \mathcal{V}_0(\omega) \cup \mathcal{V}^-(\omega)$. Moreover, by virtue of assumptions (2.14) and (2.16), there exists $r > 0$ such that

$$q(t, u(t)) \le q(t, r) \quad \text{for a. e. } t \in [0, \omega], \qquad q(\cdot, u(\cdot)) \not\equiv q(\cdot, r).$$

Therefore, from (1.1) we get

$$u''(t) \le \big(p(t) + q(t, r)\big)u(t) \quad \text{for a. e. } t \in [0, \omega],$$
$$\operatorname{meas}\big\{t \in [0, \omega] : u''(t) < \big(p(t) + q(t, r)\big)u(t)\big\} > 0$$

and thus, Lemma 3.6 with $g(t) := p(t) + q(t, r)$ and $\gamma(t) := u(t)$ yields that $p + q(\cdot, r) \in \mathcal{V}^-(\omega)$. □

Proof of Theorem 2.11. According to hypothesis (H_3), one can show that

$$\text{the function } q(t, \cdot)\colon [0, +\infty[\, \to \mathbb{R} \text{ is non-decreasing for a. e. } t \in [0, \omega]. \tag{4.7}$$

Assume on the contrary that u and w are positive solutions to (1.1) satisfying

$$u(t_0) > w(t_0) \tag{4.8}$$

for some $t_0 \in [0, \omega]$.

We first show that there exist $t_1, t_2 \in [0, \omega]$ and a solution v to problem (1.1) such that $t_1 < t_2$ and

$$u(t) \ge v(t) > 0 \quad \text{for } t \in [0, \omega], \qquad u(t) > v(t) \quad \text{for } t \in [t_1, t_2]. \tag{4.9}$$

Indeed, it is clear that either

$$u(t) > w(t) \quad \text{for } t \in [0, \omega] \tag{4.10}$$

or

$$u(t_*) = w(t_*) \quad \text{for some } t_* \in [0, \omega]. \tag{4.11}$$

If condition (4.10) holds then inequalities (4.9) are obviously satisfied with $v(t) := w(t)$, $t_1 := 0$, and $t_2 := \omega$. Therefore, suppose that condition (4.11) is fulfilled. Extend the functions u, w, p, and $q(\cdot, x)$ periodically to the whole real axis and denote them by the same symbols. Then, in view of (4.8), there exist $a, \tau \in \mathbb{R}$ such that $a < t_0 < \tau \le a + \omega$ and

$$u(t) > w(t) \quad \text{for } t \in \,]a, \tau[\,, \qquad u(a) = w(a), \qquad u(\tau) = w(\tau). \tag{4.12}$$

Put

$$\beta_0(t) := \begin{cases} w(t) & \text{for } t \in [a, \tau[\,, \\ u(t) & \text{for } t \in [\tau, a + \omega]. \end{cases}$$

By virtue of (4.12), it is not difficult to verify that $\beta_0 \in AC_u([a, a+\omega])$,

$$\beta_0(a) = \beta_0(a+\omega), \quad \beta_0'(a) \le \beta_0'(a+\omega), \tag{4.13}$$

and

$$0 < \beta_0(t) \le u(t) \quad \text{for } t \in [a, a+\omega], \qquad \beta_0(t) < u(t) \quad \text{for } t \in]a, \tau[\,. \tag{4.14}$$

Moreover, we have

$$\beta_0''(t) = p(t)\beta_0(t) + q(t, \beta_0(t))\beta_0(t) \quad \text{for a. e. } t \in [a, a+\omega] \tag{4.15}$$

because both the functions u and v are solutions to problem (1.1). On the other hand, according to Proposition 3.16, there exists a function $\alpha \in AC^1([0, \omega])$ satisfying relations (3.39), (3.40), and

$$0 < \alpha(t) \le \min\{\beta_0(s) : s \in [a, a+\omega]\} \quad \text{for } t \in [0, \omega]. \tag{4.16}$$

Extend the function α periodically to the whole real axis and denote it by the same symbol. Then, by virtue of relations (3.39), (3.40), (4.13), and (4.15), it follows from Lemma 3.12 with $f(t, x) := p(t)x + q(t, x)x$, $\beta(t) := \beta_0(t)$, and $b := a+\omega$ that there exists a function $v \in AC^1([a, a+\omega])$ satisfying

$$v''(t) = p(t)v(t) + q(t, v(t))v(t) \quad \text{for a. e. } t \in [a, a+\omega],$$
$$v(a) = v(a+\omega), \qquad v'(a) = v'(a+\omega),$$

and

$$\alpha(t) \le v(t) \le \beta_0(t) \quad \text{for } t \in [a, a+\omega].$$

However, in view of (4.14) and (4.16), the latter relation yields that

$$0 < v(t) \le u(t) \quad \text{for } t \in [a, a+\omega], \qquad v(t) < u(t) \quad \text{for } t \in]a, \tau[\,.$$

If we extend the function v periodically to the whole real axis and denote it by the same symbols, we easily conclude that the restriction of v to the interval $[0, \omega]$ is a solution to problem (1.1) satisfying desired condition (4.9) with $t_1 := \max\{0, \frac{a+t_0}{2}\}$ and $t_2 := \min\{\omega, \frac{t_0+\tau}{2}\}$.

Now it follows from relation (4.9) that there exist positive numbers v_*, v^*, e_0 such that

$$u(t) \ge v(t) + e_0, \quad v_* \le v(t) \le v^* \quad \text{for } t \in [t_1, t_2]. \tag{4.17}$$

Therefore, in view of conditions (4.7), (4.9) and hypothesis (H_3), we get

$$q(t, u(t)) \ge q(t, v(t)) \quad \text{for a. e. } t \in [0, \omega] \tag{4.18}$$

and

$$q(t, u(t)) - q(t, v(t)) \ge q(t, v(t) + e_0) - q(t, v(t)) \ge h_{v_* v^* e_0}(t) \tag{4.19}$$

for a. e. $t \in [t_1, t_2]$. On the other hand, it follows immediately from the equation in (1.1) that u and v are periodic solutions, respectively, to the equations

$$\begin{aligned} z'' &= \big(p(t) + q(t, v(t))\big)z + \big[q(t, u(t)) - q(t, v(t))\big]u(t), \\ z'' &= \big(p(t) + q(t, v(t))\big)z \end{aligned} \tag{4.20}$$

and thus, by virtue of (4.17), (4.18), and (4.19), Fredholm's third theorem yields the contradiction

$$0 = \int_0^\omega \big[q(s, u(s)) - q(s, v(s))\big]u(s)v(s)\mathrm{d}s$$

$$\geq \int_{t_1}^{t_2} \big[q(s,u(s)) - q(s,v(s))\big]u(s)v(s)\mathrm{d}s \geq (v_* + e_0)v_* \int_{t_1}^{t_2} h_{v_* v^* e_0}(s)\mathrm{d}s > 0.$$

$\square$

Proof of Theorem 2.12. According to hypothesis (H_3'), one can show that condition (4.7) holds, which together with assumption (2.17) yields that

$$q(t,x) \geq 0 \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \geq 0. \tag{4.21}$$

We first show that problem (1.1) has at most one positive solution. Assume on the contrary that u and v are positive solutions to (1.1) such that $u(t_*) > v(t_*)$ for some $t_* \in [0,\omega]$. It is clear that either

$$u(t) > v(t) \qquad \text{for } t \in [0,\omega] \tag{4.22}$$

or

$$u(t_0) = v(t_0) \quad \text{for some } t_0 \in [0,\omega]. \tag{4.23}$$

Suppose that (4.22) is satisfied. Then there exist positive numbers v_*, v^*, e_0 such that

$$u(t) \geq v(t) + e_0, \quad v_* \leq v(t) \leq v^* \quad \text{for } t \in [0,\omega]. \tag{4.24}$$

Therefore, in view of condition (4.7) and hypothesis (H_3'), for a. e. $t \in [0,\omega]$ we get

$$q(t,u(t)) - q(t,v(t)) \geq q(t,v(t)+e_0) - q(t,v(t)) \geq h_{v_* v^* e_0}(t). \tag{4.25}$$

On the other hand, it follows immediately from the equation in (1.1) that u and v are periodic solutions, respectively, to equations (4.20) and thus, by virtue of (4.24) and (4.25), Fredholm's third theorem yields the contradiction

$$0 = \int_0^{\omega} \big[q(s,u(s)) - q(s,v(s))\big]u(s)v(s)\mathrm{d}s \geq (v_* + e_0)v_* \int_0^{\omega} h_{v_* v^* e_0}(s)\mathrm{d}s > 0.$$

Now suppose that (4.23) holds. Extend the functions u, v, p, and $q(\cdot,x)$ periodically to the whole real axis and denote them by the same symbols. Then either

(i) there exists $a \in [0,\omega[$ such that

$$u(t) > v(t) \quad \text{for } t \in]a, a+\omega[\,, \quad u(a) = v(a), \tag{4.26}$$

or

(ii) there are $a < b$ such that $b - a < \omega$, $u(a) = v(a)$, $u(b) = v(b)$, and

$$u(t) > v(t) \quad \text{for } t \in]a,b[\,. \tag{4.27}$$

Put

$$w(t) := u(t) - v(t) \quad \text{for } t \in \mathbb{R}.$$

In the case (i), in view of conditions (4.7) and (4.26), it follows from the equation in (1.1) that $w(a) = 0$, $w(a+\omega) = 0$,

$$\begin{aligned} w''(t) &= \big(p(t) + q(t,v(t))\big)w(t) + \big[q(t,u(t)) - q(t,v(t))\big]u(t) \\ &\geq \big(p(t) + q(t,v(t))\big)w(t) \quad \text{for a. e. } t \in [a, a+\omega], \end{aligned} \tag{4.28}$$

and

$$w(t) > 0 \quad \text{for }]a, a+\omega[\,. \tag{4.29}$$

Since the function v is positive, from (4.21) and hypothesis (H_3') we get

$$q(t,v(t)) \geq q(t,\delta) \geq q(t,\delta) - q(t,\delta/2) \geq h_{\frac{\delta}{2}\delta\frac{\delta}{2}}(t) \quad \text{for a. e. } t \in [0,\omega],$$

where $\delta := \min\{v(t) : t \in [0, \omega]\}$ and thus, we have

$$q(t, v(t)) \geq 0 \quad \text{for a. e. } t \in [0, \omega], \quad q(\cdot, v(\cdot)) \not\equiv 0.$$

Hence, Lemmas 3.8 and 3.10 (with $\ell(t) := q(t, v(t))$) yield that $p + q(\cdot, v(\cdot)) \in \operatorname{Int} \mathcal{D}(\omega)$. Consequently, in view of Lemma 3.11 (with $g(t) := p + q(\cdot, v(\cdot))$ and $b := a + \omega$), from (4.28) we get $w(t) \leq 0$ for $t \in [a, a + \omega]$ which is in a contradiction with (4.29).

In the case (ii), by virtue of conditions (4.7), (4.21), and (4.27), it follows from the equation in (1.1) that

$$w''(t) = p(t)w(t) + \big[q(t, u(t)) - q(t, v(t))\big]u(t) + q(t, v(t))\big[u(t) - v(t)\big] \geq p(t)w(t)$$

for a. e. $t \in [a, b]$. Consequently, taking Lemmas 3.8 and 3.11 (with $g(t) := p(t)$) into account, we get inequality (3.12) which is in a contradiction with (4.27).

It remains to show that any non-trivial solution to problem (1.1) is either positive or negative. Assume on the contrary that u is a non-trivial solution to problem (1.1) such that $u(t_0) = 0$ for some $t_0 \in [0, \omega]$. Extend the functions u, p, and $q(\cdot, x)$ periodically to the whole real axis and denote them by the same symbols. It is clear that either

(a) there exists $i \in \{0, 1\}$ such that

$$(-1)^i u(t) \geq 0 \quad \text{for } t \in \mathbb{R},$$

or

(b) there are $a < b$ such that $b - a < \omega$ and (4.5) holds.

In the case (a), the function u is a non-trivial solution to problem (4.6) which is a contradiction.

In the case (b), in view of conditions (4.5) and (4.21), from the equation in (1.1) we get

$$u''(t) = p(t)u(t) + q(t, u(t))u(t) \geq p(t)u(t) \quad \text{for a. e. } t \in [a, b].$$

Therefore, Lemmas 3.8 and 3.11 (with $g(t) := p(t)$) yield that the inequality

$$u(t) \leq 0 \quad \text{for } t \in [a, b]$$

holds which contradicts (4.5). □

Proof of Theorem 2.16. Let u be a non-trivial solution to problem (1.1). Assume on the contrary that u has a zero on the interval $[0, \omega]$. Then it is clear that either

(a) there exist $t_0 \in [0, \omega]$ and $i \in \{0, 1\}$ such that

$$(-1)^i u(t) \geq 0 \quad \text{for } t \in [0, \omega], \qquad u(t_0) = 0,$$

or

(b)

$$\max\{u(t) : t \in [0, \omega]\} > 0, \quad \min\{u(t) : t \in [0, \omega]\} < 0.$$

If condition (a) holds, then the function u is a non-trivial solution to problem (4.6) which is a contradiction.

Therefore, suppose that condition (b) is satisfied. We first show that the function u has a finite number of zeros in the interval $[0, \omega]$. Indeed, assume that $\{t_n\}_{n=1}^{+\infty} \subset [0, \omega]$ and $t_0 \in [0, \omega]$ are such that

$$t_n \neq t_{n+1}, \quad u(t_n) = 0 \quad \text{for } n \in \mathbb{N}, \qquad \lim_{n \to +\infty} t_n = t_0.$$

Then we have $u(t_0) = 0$, $u'(t_0) = 0$ and thus, the function u is a non-trivial solution to problem (4.6), which is a contradiction. Consequently, it follows from assumptions (2.21), (2.22) and Remark 2.17 that

$$q(t, u(t)) \geq 0 \quad \text{for a. e. } t \in [0, \omega], \qquad q(\cdot, u(\cdot)) \not\equiv 0 \quad \text{on } [0, \omega]. \tag{4.30}$$

Now we extend the functions u, p, and $q(\cdot, x)$ periodically to the whole real axis and denote them by the same symbols. Then there exist $a \in [0, \omega[$ and $\tau_1, \tau_2 \in]a, a+\omega[$ such that $\tau_1 \leq \tau_2$,

$$u(t) > 0 \quad \text{for } t \in]a, \tau_1[\,, \quad u(a) = 0, \quad u(\tau_1) = 0, \tag{4.31}$$

and

$$u(t) < 0 \quad \text{for } t \in]\tau_2, a+\omega[\,, \quad u(\tau_2) = 0, \quad u(a+\omega) = 0. \tag{4.32}$$

Let v_1 and v_2 be solutions to the equation

$$v'' = p(t)v \tag{4.33}$$

satisfying the initial conditions

$$v(a) = 0, \quad v'(a) = 1 \tag{4.34}$$

and

$$v(a+\omega) = 0, \quad v'(a+\omega) = 1, \tag{4.35}$$

respectively. It follows from (1.1) and (4.33) that

$$\Big(u'(t)v_k(t) - u(t)v_k'(t)\Big)' = q(t, u(t))u(t)v_k(t) \quad \text{for a. e. } t \in \mathbb{R},\ k = 1, 2. \tag{4.36}$$

We show that there exists $\zeta_1 \in]a, \tau_1]$ such that

$$v_1(t) > 0 \quad \text{for } t \in]a, \zeta_1[\,, \qquad v_1(\zeta_1) = 0. \tag{4.37}$$

Indeed, if $v_1(t) > 0$ for $t \in]a, \tau_1]$ then, in view of (4.30), (4.31), and (4.34), equality (4.36) yields that

$$u'(\tau_1)v_1(\tau_1) = \int_a^{\tau_1} q(s, u(s))u(s)v_1(s)\mathrm{d}s \geq 0.$$

However, from the latter inequality we get $u'(\tau_1) \geq 0$ which is in a contradiction with (4.31). Therefore, (4.37) holds and, moreover,

$$\text{if} \quad q(\cdot, u(\cdot)) \not\equiv 0 \text{ on } [a, \tau_1] \quad \text{then} \quad \zeta_1 < \tau_1. \tag{4.38}$$

Analogously one can show that there exists $\zeta_2 \in [\tau_2, a+\omega\,[$ such that

$$v_2(t) < 0 \quad \text{for } t \in]\zeta_2, a+\omega\,[\,, \qquad v_2(\zeta_2) = 0, \tag{4.39}$$

and

$$\text{if} \quad q(\cdot, u(\cdot)) \not\equiv 0 \text{ on } [\tau_2, a+\omega] \quad \text{then} \quad \tau_2 < \zeta_2. \tag{4.40}$$

Observe that, by virtue of conditions (4.30) (4.38), and (4.40), we have $\zeta_1 < \zeta_2$.

Therefore, if solutions v_1 and v_2 are not linearly independent then condition (4.39) yields that $v_1(\zeta_2) = 0$. On the other hand, if solutions v_1 and v_2 are linearly independent then, in view of condition (4.39), it follows from Sturm's separation theorem that there exists a point $\zeta_3 \in]\zeta_2, a+\omega[$ such that $v_1(\zeta_3) = 0$. Consequently, in both cases, the solution v_1 to problem (4.33), (4.34) has at least two zeros in the interval $]a, a+\omega[$, which contradicts the assumption $p \in \mathcal{D}_1(\omega)$. □

Proof of Theorem 2.19. The assertion of the theorem immediately follows from Corollary 2.2(a) with $q(t, x) := h(t)\varphi(x)$. □

Proof of Theorem 2.20. Let

$$q(t,x) := h(t)\varphi(x) \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \in \mathbb{R}. \tag{4.41}$$

It is clear that q is a Carathéodory function satisfying $q(\cdot,0) \equiv 0$.

Put

$$\psi(x) := \min\big\{\varphi(z) : z \in [x,+\infty[\,\big\} \quad \text{for } x \geq 0. \tag{4.42}$$

One can easily verify that the function ψ is well defined. Moreover, the function ψ is continuous, non-decreasing and satisfies

$$\varphi(x) \geq \psi(x) \quad \text{for } x \geq 0, \qquad \lim_{x\to+\infty} \psi(x) = +\infty. \tag{4.43}$$

Consequently, in view of assumption (2.25), the function q satisfies hypothesis (H_1) with $q_0(t,x) := h(t)\psi(x)$ and condition (2.7) is fulfilled. Therefore, Corollary 2.3 guarantees that problem (2.24) has a positive solution u. If, in addition, the function φ is increasing on $[0,+\infty[$ then q satisfies hypothesis (H_3) and thus, u is the unique positive solution to (2.24) as follows from Theorem 2.11. □

Proof of Theorem 2.24. Let the function q be defined by formula (4.41) and the function ψ by relation (4.42). One can easily verify that the function ψ is well defined, continuous, non-decreasing, and satisfies relations (4.43). Moreover, it is clear that there exists $x_0 > 0$ such that

$$\psi(x_0) \geq 0. \tag{4.44}$$

Consequently, in view of (2.30), (4.43), and (4.44), the function q satisfies hypothesis (H_1) with $q_0(t,x) := h(t)\psi(x)$ and conditions (2.9) and (2.10) hold. Therefore, Corollary 2.6 guarantees that problem (2.24) has a positive solution u. If, in addition, the function φ is increasing on $[0,+\infty[$ then q satisfies hypothesis (H_3'). Therefore, it follows from Theorem 2.12 that u is the unique positive solution and problem (2.24) has no sign-changing solution. □

Proof of Theorem 2.25. Observe that if u is a solution to problem (2.31) then the function $-u$ is its solution, as well. Hence, Theorem 2.20 yields that

$$\begin{gathered} \text{if } p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega), \text{ then problem (2.31) has a unique} \\ \text{positive solution and a unique negative solution.} \end{gathered} \tag{4.45}$$

Put

$$q(t,x) := h(t)\varphi(|x|) \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \in \mathbb{R}. \tag{4.46}$$

It is clear that, in view of (2.25), the function q satisfies hypothesis (H_2) and conditions (2.21) and (2.23) with $E := \{t \in [0,\omega] : h(t) > 0\}$. Therefore, assertion (1) can be easily derived from Proposition 2.9 and assertion (2) is a consequence of (4.45) and Theorem 2.16 (see also Remark 2.17). Finally, in view of Remark 2.15, assertion (3) follows immediately from (4.45). □

Proof of Theorem 2.26. Observe that if u is a solution to problem (2.31) then the function $-u$ is its solution, as well. Hence, assertion (2) follows immediately from Theorem 2.24.

Let the function q be defined by formula (4.46). It is clear that, in view of (2.30), the function q satisfies hypothesis (H_2) and conditions (2.21) and (2.23) with $E := \{t \in [0,\omega] : h(t) > 0\}$. Therefore, assertion (1) can be easily derived from Proposition 2.9 and assertion (3) is a consequence of Theorem 2.16 (see also Remark 2.17). □

Proof of Corollary 2.28. It is clear that problem (1.5) is a particular case of (2.31), where $\varphi(x) := x^{\lambda-1}$ for $x \geq 0$. Therefore, the assertions of the corollary follow immediately from Theorem 2.25. □

Proof of Corollary 2.31. It is clear that problem (1.5) is a particular case of (2.31), where $\varphi(x) := x^{\lambda-1}$ for $x \geq 0$. Therefore, the assertions of the corollary follow immediately from Theorem 2.26. □

Proof of Theorem 2.33. Put

$$q(t,x) := h(t)|x|^{\lambda-1} + f(t)|x|^{\mu-1} \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \in \mathbb{R}. \tag{4.47}$$

By virtue of assumptions (2.30) and (2.34), we have

$$\begin{aligned} q(t,x) &\geq x^{\mu-1}\big(h(t)x^{\lambda-\mu} - [f(t)]_-\big) \\ &\geq h(t)x^{\mu-1}\big(x^{\lambda-\mu} - c\big) \\ &\geq h(t)\psi(x) \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \geq 0, \end{aligned}$$

where

$$\psi(x) := \begin{cases} -\frac{\lambda-\mu}{\lambda-1}\left[\frac{\mu-1}{\lambda-1}\right]^{\frac{\mu-1}{\lambda-\mu}} c^{\frac{\lambda-1}{\lambda-\mu}} & \text{for } 0 \leq x \leq \left[\frac{\mu-1}{\lambda-1}\, c\right]^{\frac{1}{\lambda-\mu}}, \\ x^{\mu-1}\big(x^{\lambda-\mu} - c\big) & \text{for } x > \left[\frac{\mu-1}{\lambda-1}\, c\right]^{\frac{1}{\lambda-\mu}}. \end{cases}$$

Consequently, the function q satisfies $q(\cdot,0) \equiv 0$ and hypothesis (H_1) with $q_0(t,x) := h(t)\psi(x)$. Moreover, inequality (2.10) holds with $x_0 := c^{\mu-\lambda}$ and, in view of (2.30), condition (2.9) is fulfilled. Therefore, the assertion of the theorem follows from Corollary 2.6. □

Proof of Theorem 2.35. Let the function q be defined by formula (4.47). It is clear that $q(\cdot,0) \equiv 0$ and, in view of assumption (2.25), we get

$$q(t,x) \geq h(t)x^{\mu-1}\left(x^{\lambda-\mu} - \frac{[f(t)]_-}{h(t)}\right) \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \geq 0.$$

Now for a. e. $t \in [0,\omega]$ and all $x \geq 0$, we put

$$q_0(t,x) := \begin{cases} -\frac{\lambda-\mu}{\lambda-1}\left[\frac{\mu-1}{\lambda-1}\right]^{\frac{\mu-1}{\lambda-\mu}} [f(t)]_-^{\frac{\lambda-1}{\lambda-\mu}} h^{-\frac{\mu-1}{\lambda-\mu}}(t) & \text{if } 0 \leq x \leq \left[\frac{\mu-1}{\lambda-1}\,\frac{[f(t)]_-}{h(t)}\right]^{\frac{1}{\lambda-\mu}}, \\ h(t)x^{\lambda-1} - [f(t)]_- x^{\mu-1} & \text{if } x > \left[\frac{\mu-1}{\lambda-1}\,\frac{[f(t)]_-}{h(t)}\right]^{\frac{1}{\lambda-\mu}}. \end{cases}$$

Then, by virtue of assumption (2.36), one can verify that $q_0\colon [0,\omega] \times [0,+\infty[\, \to \mathbb{R}$ is a Carathéodory function and hypothesis (H_1) holds. Moreover, the function q_0 satisfies (2.8) which, in view of Remark 2.4, yields that (2.7) is fulfilled. Therefore, the assertion of the theorem follows from Corollary 2.3. □

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