Quantifying Kottman's constant

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- The isomorphic Kottman constant of a Banach space, with J. M. F. Castillo, M. González, and P. L. Papini, PAMS 2020+ arXiv:1910.01626
- Symmetrically separated sequences in the unit sphere of a Banach space, with P. Hájek and T. Russo, JFA 2018, 3148–3168 arXiv:1711.05149

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The Elton–Odell theorem (1981). There exists $\varepsilon = \varepsilon(X) > 0$ s.t. S_X contains a $(1+\varepsilon)$-separated sequence.

Diestel's *Sequences and series in Banach spaces*

An Afterthought to Riesz's Theorem

(This could have been done by Banach!)

Thanks to Cliff Kottman a substantial improvement of the Riesz lemma can be stated and proved. In fact, *if X is an infinite-dimensional normed linear space, then there exists a sequence (x_n) of norm-one elements of X for which $\|x_m - x_n\| > 1$ whenever $m \neq n$.*

Kottman's original argument depends on combinatorial features that live today in any improvements of the cited result. In Chapter XIV we shall see how this is so; for now, we give a noncombinatorial proof of Kottman's result. We were shown this proof by Bob Huff who blames Tom Starbird for its simplicity. Only the Hahn-Banach theorem is needed.

We proceed by induction. Choose $x_1 \in X$ with $\|x_1\| = 1$ and take $x_1^* \in X^*$ such that $\|x_1^*\| = 1 = x_1^* x_1$.

Sphere vs ball

Folk lemma. Let $x, y \in B_X$ be non-zero vectors in B_X. If $\|x - y\| \geqslant 1$, then

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Idea: WLOG $\|x\| \geqslant \|y\|$. $g(t) = \|x - ty\|$ is convex. $g(0) \leqslant 1$, $g(1) \geqslant 1$. Thus $\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \geqslant g(\|x\|/\|y\|) \geqslant g(1)$.

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Uncountable sets of unit vectors that are separated by more than 1, with T. Kochanek, Studia Mathematica 232 (2016), 19–44. `arXiv:1503.08166`

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In general the problem is reducible to looking at subspaces (obvious) and quotients:

$M \subset X$ closed $\Rightarrow$ if (x_n) is δ-separated in X/M, you can lift it to a $(\delta-)$-separated sequence.

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Some easy cases:

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- in c_0, take $x_n = (1, 1, 1, 1, -1, 0, 0, 0, \ldots)$. Then, $(x_n)_{n=1}^{\infty}$ is 2-sep.
- when X contains an **isomorphic** copy of c_0 or ℓ_1, by James' non-distortion theorem, you can manufacture an almost isometric copy, so you'll get a $(2 - \varepsilon)$-sep. sequence.

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- Main ingredient: James' char. of (non-)reflexivity: $\forall \theta \in (0,1) \exists (x_n) \subset B_X \exists (f_n) \subset B_{X^*} \langle f_k, x_j \rangle = \theta \ (k \leqslant j)$ & $\langle f_k, x_j \rangle = 0 \ (k > j)$.

Problem (open).

- Can you prove a symmetric version of this theorem?

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- Can you prove a symmetric version of this theorem?
- Can you possibly improve the estimate?

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What about symmetric separation? ($\|x \pm y\| > \delta$)?

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Lower ℓ_q-estimates

A normalised basic sequence $(x_n)_{n=1}^{\infty}$ *satisfies a lower q-estimate* if there is a $c > 0$ such that

$$c \cdot \left(\sum_{i=n}^{N} |a_n|^q \right)^{1/q} \leqslant \left\| \sum_{n=1}^{N} a_n x_n \right\|$$

for every choice $(a_n)_{n=1}^{N}$ and $N \in \mathbb{N}$.

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Suppose X has a basis $(x_n)_{n=1}^{\infty}$. Set $X_n := \overline{\mathrm{span}}\{x_i\}_{i=n}^{\infty}$ ($n \in \mathbb{N}$). An operator $T\colon X \to Y$ is *bounded by a pair* (γ, ϱ), where $0 < \gamma \leqslant \varrho < \infty$, if $\|T\| \leqslant \varrho$ and $\|T|_{X_n}\| \geqslant \gamma$ for every $n \in \mathbb{N}$.

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Let $(x_n)_{n=1}^\infty$ be such a seq. WLOG $X = \overline{\mathrm{span}}\{x_i\}_{i=1}^\infty$. Then $Tx_n := e_n$ ($n \in \mathbb{N}$) defines an injection into ℓ_q.

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Set $\varrho_n = \| T|_{X_n} \|$ ($n \in \mathbb{N}$). $(\varrho_n)_{n=1}^{\infty}$ decreases and $\varrho_n \geqslant 1$. $T|_{X_k}$ ($k \in \mathbb{N}$) is bounded by $(\inf_{n \geqslant 1} \varrho_n, \varrho_k)$ and, of course, $\varrho_k \to \inf_{n \geqslant 1} \varrho_n$ as $k \to \infty$. WLOG $T \colon X \to \ell_q$ is bounded by (γ, ϱ) with $\frac{\gamma}{\varrho}$ as close to 1 as we wish (with $\frac{\gamma}{\varrho} < 1$).

Let $\tilde{\gamma} < \gamma$ be such that $\frac{\tilde{\gamma}}{\varrho} \cong 1$. Since $\|T\| > \tilde{\gamma}$, we can find a u.v. y_1 in $\mathrm{span}\{x_i\}_{i=1}^{\infty}$ s.t. $\|Ty_1\| > \tilde{\gamma}$. Having found $y_1, \ldots, y_n$ in $\mathrm{span}\{x_i\}_{i=1}^{\infty}$ such that $\|Ty_k\| > \tilde{\gamma}$ and the Ty_k have mutually disjoint supports. Then there is N s.t. $y_1, \ldots, y_n \in \mathrm{span}\{x_i\}_{i=1}^{N}$ and the fact that $\| T|_{X_{N+1}} \| > \tilde{\gamma}$ allows us to find a u.v. $y_{n+1} \in \mathrm{span}\{x_i\}_{i=N+1}^{\infty}$ s.t. $\|Ty_{n+1}\| > \tilde{\gamma}$.

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So

$$K^s(X) \geqslant \frac{\tilde{\gamma}}{\varrho} \cdot 2^{1/q}$$

.

Cotype business

If X has finite cotype $q(X)$, then S_X contains a $(2^{1/q(X)}-)$-separated sequence.

- If $q_X = \infty$, then the assertion follows immediately from the Riesz lemma, so WLOG $q_X < \infty$.
- If X is a Schur space, then by Rosental's ℓ_1-theorem X contains a copy of ℓ_1 and the James' non-distortion theorem even implies $K^s(X) = 2$.
- In the other case, there is a weakly null normalised basic sequence in X; it is known (see, Hájek–Johannis) that for every $r > q_X$ such a sequence admits a subsequence with a lower r-estimate, so the result follows from the previous proposition.

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Take a b.-o. seq. $(x_n, f_n) \in X \times X^*$ ($\langle f_k, x_j \rangle = \delta_{kj}$, $\|f_i\| = \|x_i\| = 1$) and set

$\nu(X) = \sup_{i \neq k} |\langle f_i, x \rangle| + |\langle f_k, x \rangle|$, $\|x\|' = \max\{\|x\|, \nu(x)\}$.

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- $\tilde{K}(X) = \inf\{K(\tilde{X})\colon X \cong \tilde{X}\}$ (isomorphic Kottman constant).

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- $\tilde{K}(X) = \inf\{K(\tilde{X})\colon X \cong \tilde{X}\}$ (isomorphic Kottman constant).
- $K_f(X) = \sup\{\sigma > 0\colon \forall N \in \mathbb{N}\ \exists (x_n)_{n=1}^N$ in B_X s.t. $\|x_n - x_m\| \geqslant \sigma$ for $n \neq m\}$ (finite Kottman constant)
- for a Banach lattice X, $K^{\perp}$ (Kottman for disjoint sequences).
- K_s, $\tilde{K}_s$ the symmetric versions.

- $K(c_0) = 2 = \tilde{K}_s(c_0)$,

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Given X, one may find an equiv. norm $\tilde{X}$ so that $S_{\tilde{X}}$ contains a 2-sep. sequence.

Take a b.-o. seq. $(x_n, f_n) \in X \times X^*$ $(\langle f_k, x_j \rangle = \delta_{kj}, \|f_i\| = \|x_i\| = 1)$ and set

$\nu(X) = \sup_{i \neq k} |\langle f_i, x \rangle| + |\langle f_k, x \rangle|$, $\|x\|' = \max\{\|x\|, \nu(x)\}$.

- $K(X) = \sup\{\sigma > 0\colon S_X \text{ contains a } \sigma\text{-separated sequence}\}$ (Kottman constant)
- $\tilde{K}(X) = \inf\{K(\tilde{X})\colon X \cong \tilde{X}\}$ (isomorphic Kottman constant).
- $K_f(X) = \sup\{\sigma > 0\colon \forall N \in \mathbb{N}\ \exists (x_n)_{n=1}^N \text{ in } B_X \text{ s.t. } \|x_n - x_m\| \geqslant \sigma \text{ for } n \neq m\}$ (finite Kottman constant)
- for a Banach lattice X, $K^{\perp}$ (Kottman for disjoint sequences).
- $K_s, \tilde{K}_s$ the symmetric versions.

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- Kryczka–Prus: $K(X) \geqslant \sqrt[5]{4}$ for any non-reflexive X.

Two words on the paper

T. Russo, A note on symmetric separation in Banach spaces, RACSAM (2019), arXiv:1904.12598.

Theorem (Russo). If B_X contains a weakly null $(1+\varepsilon)$-sep. seq, then S_X contains a symmetrically $\sqrt{1+\varepsilon}$-sep. seq.

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x and y are toroidally δ-separated, whenever $\|x - \theta y\| \geqslant \delta$ for every $|\theta| = 1$.

Preliminary observations

- For a countably incomplete ultrafilter $\mathscr{U}$ and a space X, we have

$$1 < K(X) \leqslant K_f(X) = K(X^{\mathscr{U}}) \leqslant 2,$$

where $X^{\mathscr{U}}$ stands for the ultrapower of X w.r.t. $\mathscr{U}$.

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  where $X^{\mathscr{U}}$ stands for the ultrapower of X w.r.t. $\mathscr{U}$.
- There exists a space Z for which

$$K(Z) < K(Z^{**}),$$

  and it is easy to check that this space also satisfies $K_s(Z) < K_s(Z^{**})$. The said space is a J-sum of ℓ_1^n ($n \in \mathbb{N}$) in the sense of Bellenot; it has the property that $K(Z) < 2$, yet Z^{**} admits a quotient map onto ℓ_1 so that $K_s(Z^{**}) = 2$.

Castillo–González–K.–Papini

For every space X, $2 \leqslant K(X) \cdot K(X^*)$.

Based on a simple application of Ramsey's theorem:

Lemma

Let (x_n) be a bounded sequence in a Banach space. Then there exists an infinite subset M of $\mathbb{N}$ such that $\|x_i - x_j\|$ converges as $i, j \in M$, $i, j \to \infty$.

Proof.

X contains a basic seq. with basis constant at most $1+\varepsilon$: $(x_n)_{n=1}^{\infty}$ in X and $(x_n^*)_{n=1}^{\infty}$ in X^* with $\|x_n\| = 1$ and $\|x_n^*\| \leqslant 1+\varepsilon$ ($n \in \mathbb{N}$) s.t. $\langle x_i^*, x_j \rangle = \delta_{ij}$. For $i \neq j$,

$$2 = \langle x_i^* - x_j^*, x_i - x_j \rangle \leqslant \|x_i^* - x_j^*\| \cdot \|x_i - x_j\|.$$

Let us set $y_n^* = (1+\varepsilon)^{-1} x_n^*$. (Passing to a subsequence) $\|y_i^* - y_j^*\|$ and $\|x_i - x_j\|$ converge to k^* and to k, resp. in the sense of the Lemma. Then

$$2(1+\varepsilon)^{-1} \leqslant k^* \cdot k \leqslant K(X^*) \cdot K(X),$$

hence $2 \leqslant K(X) \cdot K(X^*)$.

Twisted sums

Theorem (Castillo–González–K.–Papini). For a short exact sequence of Banach spaces

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First idea: The Kottman constant is cts w.r.t the Banach–Mazur distance: $K(X) \leqslant K(Y) \cdot d_{\mathrm{BM}}(X, Y)^2$. This is not sufficient, though.

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Main idea: the constant is cts w.r.t. to the Kadets metric

$$d_K(M, N) = \inf \max \Big\{ \sup_{x \in iB_M} \mathrm{dist}(x, jB_N), \sup_{y \in jB_N} \mathrm{dist}(y, iB_M) \Big\},$$

where the inf is taken w.r.t all isometric embeddings i, j of M, N into common spaces.

Kadets metric

$d_K(M, N) = \inf \max \{ \sup_{x \in iB_M} \operatorname{dist}(x, jB_N), \sup_{y \in jB_N} \operatorname{dist}(y, iB_M) \}$

Claim. Let $M, N \subseteq Z$. Then $|K(M) - K(N)| \leqslant 2 \cdot g(M, N)$.

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Sketch. Say $K(M)$ is attained. Take $(a_n)_{n=1}^{\infty}$ in B_M s.t. $K(M) = \|a_n - a_m\|$.
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Twisted sums

Kalton–Peck: $0 \to \ell_2 \to Z_2 \to \ell_2 \to 0$ that does not split.

For $\Omega x = (x \log(|x_n|/\|x\|_2))_n$ $(x \in \ell_2)$, $\|(y,x)\| = \|y - \Omega x\|_2 + \|x\|_2$ $((y,x) \in Z_2)$ is a quasi-norm. Kalton: the convex hull of the unit ball of the preceding quasi-norm provides an equivalent Banach-space topology.

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Sketch. Again, there is no loss of generality in assuming that $\tilde{K}(X) = K(\tilde{X})$. Thus

$$|\tilde{K}(A) - \tilde{K}(B)| = |K(\tilde{A}) - K(\tilde{B})| \leqslant 2 \cdot g(\tilde{A}, \tilde{B}).$$

The space $Y \oplus_1 Z$ is a subspace of $X \oplus_1 Z$. For each positive ε, the subspace $X_\varepsilon = \{(\varepsilon x, qx)) : x \in X\}$ of $X \oplus_1 Z$ is isomorphic to X. Both equalities follow from $\lim_{\varepsilon \to 0} g(X_\varepsilon, Y \oplus_1 Z) = 0$, which follows from a lemma due to M. Ostrovskii.

Complex interpolation

Kalton and Ostrovskii proved that the Kadets metric is continuous with respect to the interpolation parameter, by showing that

$$d_{\mathrm{K}}(X_t, X_s) \leqslant 2\left|\frac{\sin\left(\pi(t-s)/2\right)}{\sin\left(\pi(t+s)/2\right)}\right|, \quad 0 < s, t < 1.$$

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Corollary. Let (X_0, X_1) be an interpolation couple. Then the (symmetric, finite) Kottman constant is continuous with respect to the interpolation parameter; precisely

$$|K(X_t) - K(X_s)| \leqslant 4\left|\frac{\sin\left(\pi(t-s)/2\right)}{\sin\left(\pi(t+s)/2\right)}\right|, \quad 0 < s, t < 1.$$

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Theorem. Let (X_0, X_1) be regular interpolation pair of Banach spaces with X_0 reflexive and let $0 < a < b < 1$. Then

$$K(X_{(1-\theta)a+\theta b}) \leqslant K(X_a)^{1-\theta} K(X_b)^{\theta} \quad \big(\theta \in (0,1)\big).$$

The inequality is valid for $K_s(\cdot)$ and $K_f(\cdot)$ as well.

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