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 $\mathcal{F}_p(\mathbb{Z}^d)$ and $\mathcal{F}_p(\mathbb{R}^d)$ for $0 < p \leq 1$**

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STRUCTURE OF THE LIPSCHITZ FREE p -SPACES $\mathcal{F}_p(\mathbb{Z}^d)$ AND $\mathcal{F}_p(\mathbb{R}^d)$ FOR $0 < p \leq 1$

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ABSTRACT. Our aim in this article is to contribute to the theory of Lipschitz free p -spaces for $0 < p \leq 1$ over the Euclidean spaces \mathbb{R}^d and \mathbb{Z}^d . To that end, on one hand we show that $\mathcal{F}_p(\mathbb{R}^d)$ admits a Schauder basis for every $p \in (0, 1]$, thus generalizing the corresponding result for the case $p = 1$ by Hájek and Pernecká [19, Theorem 3.1] and answering in the positive a question that was raised in [6]. Explicit formulas for the bases of both $\mathcal{F}_p(\mathbb{R}^d)$ and its isomorphic space $\mathcal{F}_p([0, 1]^d)$ are given. On the other hand we show that the well-known fact that $\mathcal{F}(\mathbb{Z})$ is isomorphic to ℓ_1 does not extend to the case when $p < 1$, that is, $\mathcal{F}_p(\mathbb{Z})$ is not isomorphic to ℓ_p when $0 < p < 1$.

1. INTRODUCTION AND BACKGROUND

Suppose $0 < p \leq 1$. Given a pointed p -metric space \mathcal{M} it is possible to construct a unique p -Banach space $\mathcal{F}_p(\mathcal{M})$ in such a way that \mathcal{M} embeds isometrically in $\mathcal{F}_p(\mathcal{M})$ via a canonical map denoted $\delta_{\mathcal{M}}$, and for every p -Banach space X and every Lipschitz map $f: \mathcal{M} \rightarrow X$ with Lipschitz constant $\text{Lip}(f)$ that maps the base point 0 in \mathcal{M} to $0 \in X$ extends to a unique linear bounded map $T_f: \mathcal{F}_p(\mathcal{M}) \rightarrow X$ with $\|T_f\| = \text{Lip}(f)$. The space $\mathcal{F}_p(\mathcal{M})$ is known as the *Lipschitz free p -space* over \mathcal{M} . This class of p -Banach spaces provides a canonical linearization process of Lipschitz maps between p -metric spaces: any Lipschitz map f from a p -metric space \mathcal{M}_1 to a p -metric space \mathcal{M}_2

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which maps the base point in \mathcal{M}_1 to the base point in \mathcal{M}_2 extends to a continuous linear map $L_f: \mathcal{F}_p(\mathcal{M}_1) \rightarrow \mathcal{F}_p(\mathcal{M}_2)$ with $\|L_f\| \leq \text{Lip}(f)$.

Lipschitz free p -spaces were introduced in [3], where they were used to provide for every for $0 < p < 1$ a couple of *separable* p -Banach spaces which are Lipschitz-isomorphic without being linearly isomorphic. These spaces constitute a new family of p -Banach spaces which are easy to define but whose geometry is difficult to grasp. This task was undertaken by the authors in [5] and continued in the articles [6, 7].

Within this subject, it is specially interesting and challenging to understand the structure of Lipschitz free p -spaces over subsets of the metric space \mathbb{R} (or more generally over \mathbb{R}^d for $d \in \mathbb{N}$) endowed with the Euclidean distance (see [5, Comments at the end of section §5]). Although the papers [6, 7] do not focus on this kind of Lipschitz free p -spaces, they contain results that apply in particular to them. Let us gather the most significant advances concerning the geometry of Lipschitz free p -spaces over Euclidean spaces achieved in those papers. Some of them are explicitly stated in the articles [6, 7] (in which case we provide the reference), while others are straightforward consequences of more general results. In the list below we assume $0 < p \leq 1$.

- (A.1) For every $d \in \mathbb{N}$ and every net \mathcal{N} in \mathbb{R}^d , $\mathcal{F}_p(\mathcal{N}) \simeq \mathcal{F}_p(\mathbb{Z}^d)$ ([6, Proposition 3.6]).
- (A.2) The space $\mathcal{F}_p([0, 1])$ has a Schauder basis ([6, Theorem 5.7]).
- (A.3) For every $d \in \mathbb{N}$, the space $\mathcal{F}_p(\mathbb{Z}^d)$ has a Schauder basis ([6, Theorem 5.3]).
- (A.4) $\mathcal{F}_p(\mathbb{R}^d) \simeq \mathcal{F}_p([0, 1])^d \simeq \mathcal{F}_p(\mathbb{R}_+^d) \simeq \mathcal{F}_p(S^d) \simeq \ell_p(\mathcal{F}_p(\mathbb{R}^d))$ for every $d \in \mathbb{N}$ ([7, Theorem 4.15, Corollary 4.17 and Theorem 4.21]).
- (A.5) $\mathcal{F}_p(\mathbb{Z}^d) \simeq \mathcal{F}_p(\mathbb{N}^d) \simeq \ell_p(\mathcal{F}_p(\mathbb{Z}^d))$ for every $d \in \mathbb{N}$ ([7, Theorems 5.8 and 5.12]).
- (A.6) The spaces $\mathcal{F}_p(\mathbb{Z}^d)$ and $\mathcal{F}_p(\mathbb{R}^d)$, despite being non-isomorphic, have the same local structure ([7, Corollary 5.14]).
- (A.7) For every $d \in \mathbb{N}$ there is a constant $C = C(p, d)$ such that for every $\mathcal{M} \subset \mathcal{N} \subset \mathbb{R}^d$, the space $\mathcal{F}_p(\mathcal{M})$ is C -complemented in $\mathcal{F}_p(\mathcal{N})$ ([7, Corollary 5.3]).
- (A.8) For every $0 < p \leq 1$, every $d \in \mathbb{N}$ and every $\mathcal{M} \subset \mathbb{R}^d$, $\mathcal{F}_p(\mathcal{M})$ has the π -property ([7, Corollary 5.3]).
- (A.9) For every $\mathcal{M} \subset \mathbb{R}^d$ infinite, there is $\mathcal{N} \subset \mathcal{M}$ such that $\mathcal{F}_p(\mathcal{N}) \simeq \ell_p$ ([7, Theorem 3.2]).

Combining (A.7) with (A.4), and using Pełczyński's decomposition method (see, e.g., [4, Theorem 2.2.3]), yields that $\mathcal{F}_p(\mathcal{N}) \simeq \mathcal{F}_p(\mathbb{R}^d)$ whenever the subset \mathcal{N} of \mathbb{R}^d has non-empty interior. So, the research

on Lipschitz free p -spaces over subsets of Euclidean spaces reduces to the following two main topics (with non-empty intersection).

- (**Q.a**) The study of the geometry of the spaces $\mathcal{F}_p(\mathbb{R}^d)$.
- (**Q.b**) To contrast the spaces $\mathcal{F}_p(\mathcal{N})$ for different subsets $\mathcal{N} \subset \mathbb{R}^d$ with empty interior.

As far as the topic (**Q.a**) is concerned, the main question suggested by the previous work on the subject is whether (**A.2**) extends to higher dimensions. Within the scale of approximation properties, the existence of a Schauder basis is the most demanding one. Thus, this question also connects with (**A.8**), which states in particular that $\mathcal{F}_p([0, 1]^d)$ has the π -property.

The approximation properties of Lipschitz free spaces (i.e., Lipschitz free p -spaces for $p = 1$) has attracted a lot of attention from the specialists from the upsurge of interest in the theory back 2003. We refer the reader to [8, 10, 11, 13–17, 22, 25] for a non exhaustive list of papers containing positive answers to this question within different frameworks. In contrast, determining whether a given Lipschitz free space has a Schauder basis has shown to be a somewhat elusive task. To the best of our knowledge, the list of papers that achieve positive answers in this direction reduces to [9, 12, 18, 19].

As far as Lipschitz-free spaces over Euclidean spaces is concerned, it is known that $\mathcal{F}(\mathbb{R}^d)$ has a Schauder basis (see [19, Theorem 3.1]). One of the goals of this paper is to see whether analogous statements hold for the more general case of Lipschitz-free p -spaces for $p \in (0, 1]$. In this spirit, here we extend this result by proving that $\mathcal{F}_p(\mathbb{R}^d)$ admits a Schauder basis for every $p \in (0, 1]$, thus answering in the positive [6, Question 6.5]. Moreover, exact formulas for the basis of both $\mathcal{F}_p(\mathbb{R}^d)$ and its isomorphic space $\mathcal{F}_p([0, 1]^d)$ are provided. Section 3 will be devoted to take care of this (see Theorem 3.8 and Theorem 3.9).

As for the topic (**Q.b**), it is natural to initiate the study with uniformly separated subsets. Note that if $\mathcal{M} \subset \mathbb{R}^d$ is uniformly separated, then it is contained in a net \mathcal{N} . Consequently, by (**A.1**), (**A.9**) and (**A.7**), ℓ_p is complemented in $\mathcal{F}_p(\mathcal{M})$ and $\mathcal{F}_p(\mathcal{M})$ is complemented in $\mathcal{F}_p(\mathbb{Z}^d)$. Hence, if $\mathcal{F}_p(\mathbb{Z}^d)$ were isomorphic to ℓ_p , applying Pełczyński's decomposition method would give $\ell_p \simeq \mathcal{F}_p(\mathcal{M})$. So, our first task should be to determine whether $\mathcal{F}_p(\mathbb{Z}^d)$ is isomorphic to ℓ_p or not. It is known ([24]) that, for $d \geq 2$, $\mathcal{F}(\mathbb{Z}^d)$ is not isomorphic to ℓ_1 . Taking envelopes yields $\mathcal{F}_p(\mathbb{Z}^d) \not\simeq \ell_p$ for any $0 < p \leq 1$ and $d \geq 2$ (see [5, Corollary 4.2]). On the other hand, it is known and easy to prove, that $\mathcal{F}(\mathbb{Z}) \simeq \ell_1$. More generally, we have $\mathcal{F}(\mathcal{M}) \simeq \ell_1$ whenever \mathcal{M} is

the closure of a zero-measure subset of \mathbb{R} (see [9]). This result was extended to $0 < p < 1$ by the authors replacing the Euclidean distance $|\cdot|$ on \mathbb{R} with its anti-snowflaking $|\cdot|^{1/p}$. That is, we have $\mathcal{F}_p(\mathbb{Z}, |\cdot|^{1/p}) \simeq \ell_p$ for every $0 < p \leq 1$. A question that implicitly arose from [5] is whether the same holds for the Euclidean distance, i.e., whether $\mathcal{F}_p(\mathbb{Z})$ is isomorphic to ℓ_p or not for $0 < p < 1$. Section 2 is devoted to providing a negative answer to this problem.

1.1. Terminology. Through this article we use standard facts and notation from quasi-Banach spaces and Lipschitz free p -spaces over quasimetric spaces as can be found in [5]. Nonetheless, we will record the notation that is most heavily used. A *quasi-norm* on a vector space X over the real field \mathbb{R} is a map $\|\cdot\|: X \rightarrow [0, \infty)$ satisfying $\|x\| > 0$ unless $x = 0$, $\|tx\| = |t|\|x\|$ for all $t \in \mathbb{R}$ and all $x \in X$, and

$$\|x + y\| \leq \kappa(\|x\| + \|y\|), \quad x, y \in X. \quad (1.1)$$

for some constant $\kappa \geq 1$. The optimal constant such that (1.1) will be called the *modulus of concavity* of X .

Let $0 < p \leq 1$. If $\|\cdot\|$ fulfils the condition

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p, \quad x, y \in X,$$

we say that $\|\cdot\|$ is a *p-norm*. Any p -norm is a quasi-norm with modulus of concavity at most $2^{1/p-1}$. A quasi-norm induces a Hausdorff vector topology on X . If X is a complete topological vector space, we say that $(X, \|\cdot\|)$ is a *quasi-Banach space*. A p -Banach space will be a quasi-Banach space equipped with a p -norm.

A quasi-Banach space X is said to have the *bounded approximation property* (BAP for short) if there exists a net $(T_\alpha)_{\alpha \in \mathcal{A}}$ consisting of finite-rank linear operators with

$$\sup_{\alpha \in \mathcal{A}} \|T_\alpha\| < \infty$$

that converges to Id_X uniformly on compact sets. If, moreover, each operator T_α is a projection we say that X has the *π -property*.

A *Schauder basis* of a quasi-Banach space X is a sequence $(x_n)_{n=1}^\infty$ in X such that for every $x \in X$ there is a unique sequence $(a_n)_{n=1}^\infty$ with $x = \sum_{n=1}^\infty a_n x_n$. Associated to the Schauder basis $(x_n)_{n=1}^\infty$ the partial-sum projections $P_m: X \rightarrow X$, $m \in \mathbb{N}$, given by

$$x = \sum_{n=1}^\infty a_n x_n \mapsto P_m(x) = \sum_{n=1}^m a_n x_n,$$

are uniformly bounded. Therefore, if a quasi-Banach space X has a Schauder basis, then it has both the BAP and the π -property. Conversely, given a sequence $(P_m)_{m=1}^\infty$ of linear maps from X into X such that $\sup_m \|P_m\| < \infty$, $\cup_{m=1}^\infty P_m(X)$ is dense in X , $\dim(P_m(X)) = m$, and $P_m \circ P_n = P_{\min\{n,m\}}$ for all $n, m \in \mathbb{N}$, there is a Schauder basis whose associated projections are $(P_m)_{m=1}^\infty$. Namely, if for each $n \in \mathbb{N}$ we pick an arbitrary non-zero vector x_n in the one-dimensional space $P_n(X) \cap \text{Ker}(P_{n-1})$, then $(x_n)_{n=1}^\infty$ is such a Schauder basis.

We say that a quasi-Banach space X is K -complemented in Y if there are bounded linear maps $S: X \rightarrow Y$ and $P: Y \rightarrow X$ with $P \circ S = \text{Id}_X$ and $\|S\| \|P\| \leq K$. If, moreover, $S \circ P = \text{Id}_Y$, the spaces X and Y are said to be K -isomorphic. In the case when S is the inclusion map, that is, P is a projection onto X , we say that X is a K -complemented subspace of Y . If the constant K is irrelevant, we simply drop it from the notation. We can define similar notions replacing the quasi-Banach spaces X and Y with metric, or quasi-metric, spaces \mathcal{M} and \mathcal{N} , and replacing bounded linear maps with Lipschitz maps. In the ‘‘metric case’’, we will say that \mathcal{M} is a Lipschitz retract of \mathcal{N} with constant K , or that \mathcal{M} and \mathcal{N} are Lipschitz isomorphic with distortion at most K , respectively.

A subset \mathcal{N} of a metric space (\mathcal{M}, d) is said to be *uniformly separated* if

$$\inf\{d(x, y) : x, y \in \mathcal{N}, x \neq y\} > 0.$$

A *net* is a uniformly separated set \mathcal{N} with $\sup_{x \in \mathcal{M}} d(x, \mathcal{N}) < \infty$.

We use the symbol \mathbb{N}_* to denote the set consisting of all non-negative integers, i.e., $\mathbb{N}_* = \mathbb{N} \cup \{0\}$. Given $n \in \mathbb{N}$ we will put $\mathbb{N}_n = \mathbb{Z} \cap [0, n]$.

2. THE p -BANACH SPACE $\mathcal{F}_p(\mathbb{Z}^d)$ IS NOT ISOMORPHIC TO ℓ_p

Once we conjecture that two quasi-Banach spaces are not isomorphic, the best strategy for substantiating our guess is to come up with a feature that tells apart them. We find this wished-for property within the theory of locally complemented subspaces and \mathcal{L}_p -spaces developed by Kalton in [20]. A subspace X of a quasi-Banach space Y is said to be *locally K -complemented* in Y for some $K > 0$, if for every finite-dimensional space $V \subset Y$ and every $\varepsilon > 0$ there is $P: V \rightarrow X$ with

$$\|P\| \leq K \text{ and } \|P(x) - x\| \leq \varepsilon \|x\|, \quad x \in V \cap X.$$

If the constant K is irrelevant, we say that X is locally complemented in Y .

A quasi-Banach space X is a \mathcal{L}_p -space, $0 < p < \infty$, if it is isomorphic to a locally complemented subspace of $L_p(\mu)$ for some measure μ .

This notion was introduced in [20] with the aim of developing an attractive definition of a \mathcal{L}_p -space for $0 < p < 1$ as it is not clear whether $L_p(0 < p < 1)$ would satisfy the analogue of the classical Lindenstrass-Pełczyński definition. For $p = 1$, the above definition is equivalent to the classical one, while for $1 < p < \infty$ the only exception to the equivalence is that, with the definition used in this paper, Hilbert spaces become \mathcal{L}_p -spaces.

Lemma 2.1. *Let $0 < p < \infty$, let X be a quasi-Banach space and let $(V_\alpha)_{\alpha \in A}$ be an increasing net consisting of finite-dimensional subspaces of X with $\overline{\cup_{\alpha \in A} V_\alpha} = X$. Suppose that there is $K \in (0, \infty)$ such that for every $\alpha \in A$ and every $\varepsilon > 0$ there is $S: V_\alpha \rightarrow \ell_p$ and $T: \ell_p \rightarrow X$ with*

$$\|T\| \|S\| \leq K \text{ and } \|T(S(x)) - x\| \leq \varepsilon \|x\|, \quad x \in V_\alpha.$$

Then X is an \mathcal{L}_p -space.

Proof. Let V be a finite-dimensional subspace of X and let $\varepsilon > 0$. Set

$$\varepsilon_0 := \min \left\{ 1, \frac{\varepsilon}{\kappa + 2\kappa^2} \right\}.$$

A standard argument yields $\alpha \in A$ and $J: V \rightarrow V_\alpha$ with $\|J(x) - x\| \leq \varepsilon_0 \|x\|$ for all $x \in V$. By hypothesis, there are $S: V_\alpha \rightarrow \ell_p$ and $T: \ell_p \rightarrow X$ such that $\|T\| \|S\| \leq K$ and $\|T(S(x)) - x\| \leq \varepsilon_0 \|x\|$ for all $x \in V_\alpha$. We have

$$\|J\| = \|J - \text{Id}_V + \text{Id}_V\| \leq \kappa(\varepsilon_0 + 1) \leq 2\kappa.$$

Hence $\|T\| \|S \circ J\| \leq \|T\| \|S\| \|J\| \leq 2\kappa K$. If $x \in V$,

$$\begin{aligned} \|T(S(J(x))) - x\| &\leq \kappa(\|T(S(J(x))) - J(x)\| + \|J(x) - x\|) \\ &\leq \kappa(\varepsilon_0 \|J(x)\| + \varepsilon_0 \|x\|) \\ &\leq \kappa \varepsilon_0 (2\kappa \|x\| + \|x\|) = \varepsilon \|x\|. \end{aligned}$$

Appealing to [20, Theorem 6.1] finishes the proof. \square

It is known that complemented subspaces inherit the property of being \mathcal{L}_p -spaces (see [20, Proposition 3.3]). For the sake of clarity, we state and prove the quantitative version of this result, which is the one we will need.

Lemma 2.2. *Let X be an \mathcal{L}_p -space and let $\lambda \in [1, \infty)$. There is a constant $K = K(X, \lambda)$ such that for every quasi-Banach space Y which is λ -complemented in X , every finite-dimensional subspace V of Y , and every $\varepsilon > 0$ there are bounded linear maps $S: V \rightarrow \ell_p$ and $T: \ell_p \rightarrow Y$ with $\|S\| \|T\| \leq K$ and $\|T \circ S - \text{Id}_V\| \leq \varepsilon$.*

Proof. Let $J: Y \rightarrow X$ and $P: X \rightarrow Y$ be such that $\|J\| \|P\| \leq \lambda$ and $P \circ J = \text{Id}_Y$. By [20, Theorem 6.1] there are $S: J(V) \rightarrow \ell_p$ and $T: \ell_p \rightarrow X$ with $\|S\| \|T\| \leq K_0$ and $\|T \circ S - \text{Id}_{J(V)}\| \leq \varepsilon/\lambda$, where $K_0 \in [1, \infty)$ depends only on X . We have

$$\|S \circ J|_V\| \|P \circ T\| \leq \|S\| \|J\| \|P\| \|T\| \leq \lambda K_0$$

and

$$\|P \circ T \circ S \circ J|_V - \text{Id}_V\| = \|P \circ (T \circ S - \text{Id}_{J(V)}) \circ J\| \leq \varepsilon. \quad \square$$

Given a quasi-Banach space X and $0 < q \leq 1$ the q -Banach envelope of X is a pair $(X_{c,q}, E_{X,q})$, where $X_{c,q}$ is a q -Banach space and $E_{X,q}: X \rightarrow X_{c,q}$ is a linear contraction, defined by the following universal property: for every bounded linear map $T: X \rightarrow Y$, where Y is a q -Banach space, there is a unique linear map $T': X_{c,q} \rightarrow Y$ with $\|T'\| \leq \|T\|$ and $T = T' \circ E_{X,q}$. If X and Y are quasi-Banach spaces and $T: X \rightarrow Y$ is linear and bounded, there is a unique bounded linear map $T_{c,q}: X_{c,q} \rightarrow Y_{c,q}$ such that $T_{c,q} \circ E_{X,q} = E_{Y,q} \circ T$. Moreover, $\|T_{c,q}\| \leq \|T\|$. For background on envelopes, see e.g. [1, §9].

Banach envelopes inherit BAP. For further reference, we record this result.

Lemma 2.3. *Let X be a quasi-Banach space with the BAP. Then $X_{c,q}$ has the BAP for $0 < q \leq 1$.*

Given $0 < p \leq 1$, a subset \mathcal{C} of a vector space V is said to be *absolutely p -convex* if for any x and $y \in \mathcal{C}$ and any scalars λ and μ with $|\lambda|^p + |\mu|^p \leq 1$ we have $\lambda x + \mu y \in \mathcal{C}$. We will denote by $\text{co}_p(Z)$ the p -convex hull of $Z \subset V$, i.e., the smallest absolutely p -convex set containing Z .

Lemma 2.4. *Let Z be a q -Banach space and $K \subset Z$ be relatively compact. Then the absolutely q -convex hull $\text{co}_q(K)$ of K is relatively compact.*

Proof. Since the map $(t, x) \mapsto tx$ is continuous and the unit sphere of the scalar field is compact, we can suppose that $tx \in K$ whenever $x \in K$ and $|t| = 1$. It suffices to prove that $\text{co}_q(K)$ possesses a finite ε -net for every $\varepsilon > 0$. Let \mathcal{N}_0 be a finite $(2^{-1/q}\varepsilon)$ -net for K . Then

$$\mathcal{N}_1 = \left\{ \sum_{k=1}^m a_k x_k : x_k \in \mathcal{N}_0, a_k \geq 0, \sum_{k=1}^m a_k^q \leq 1 \right\}$$

is a $(2^{-1/q}\varepsilon)$ -net for $\text{co}_q(K)$

Enumerate $\mathcal{N}_0 = \{y_j: 1 \leq j \leq n\}$, and let $s \in \mathbb{N}$ be such that $2n \sup_j \|y_j\|^q \leq \varepsilon^q 2^{sq}$. We will conclude the proof by showing that

$$\left\{ 2^{-s} \sum_{j=1}^n b_j y_j : b_j \in \mathbb{N}_*, \sum_{j=1}^n b_j^q \leq 2^{sq} \right\}$$

is a (finite) $(2^{-1/q}\varepsilon)$ -net for \mathcal{N}_1 . Let $x \in \mathcal{N}_1$. There is $(a_j)_{j=1}^n$ in $[0, \infty)$ such that $x = \sum_{j=1}^n a_j y_j$ and $\sum_{j=1}^n a_j^q \leq 1$. For each $j = 1, \dots, n$, let $b_j \in \mathbb{N}_*$ be such that $b_j \leq 2^s a_j < b_j + 1$. Then,

$$\left\| x - 2^{-s} \sum_{j=1}^n b_j y_j \right\|^q \leq n 2^{-sq} \sup_j \|y_j\|^q \leq \frac{\varepsilon^q}{2}. \quad \square$$

Theorem 2.5. *Let $T: X \rightarrow Y$ be a compact linear operator between quasi-Banach spaces X and Y . Then the operator $T_{c,q}$ is compact for any $0 < q \leq 1$.*

Proof. Since the space of compact operators forms an ideal, we can assume that Y is q -Banach, so that $T_{c,q}$ is a map from $X_{c,q}$ into Y with $T_{c,q} \circ E_{q,X} = T$. Then, by construction,

$$\begin{aligned} \overline{T_{c,q}(B_{X_{c,q}})} &= \overline{T_{c,q}(\text{co}_q(E_{q,X}(B_X)))} \\ &= \overline{\text{co}_q(T_{c,q}(E_{q,X}(B_X)))} \\ &= \overline{\text{co}_q(T(B_X))}, \end{aligned}$$

which is compact by Lemma 2.4. \square

Let X be a subspace of a quasi-Banach space Y . We say that X has the *compact extension property* (CEP for short) in Y if every compact operator $T: X \rightarrow Z$, where Z is a quasi-Banach space, extends to a compact operator $\tilde{T}: Y \rightarrow Z$. Let $0 < q \leq 1$. We say that X has the *compact extension property for q -Banach spaces* (q -CEP for short) in Y if the compact extension property holds when Z is a q -Banach space. If, moreover, we can ensure that $\|\tilde{T}\| \leq K\|T\|$ for some constant $K \in [1, \infty)$ (depending on X, Y and q), we say that X has the q -CEP in Y with constant K . Notice that, by the Aoki-Rolewicz theorem, X has the CEP in Y if and only if X has the q -CEP in Y for every $0 < q \leq 1$. We have the following results in this respect.

Theorem 2.6. *Let X be a subspace of a quasi-Banach space Y . Suppose that X has the q -CEP in Y for some $0 < q \leq 1$. Then there is a constant $K \in [1, \infty)$ for which X has the q -CEP in Y with constant K .*

Proof. Although the proof of [23, Theorem 2.2] was done for Banach spaces, the arguments therein apply to our case without any important modifications. \square

Theorem 2.7. *Let $0 < q \leq 1$ and Y be a q -Banach space with the BAP. Let X be a closed subspace of Y . The following are equivalent.*

- (i) X is locally complemented in Y .
- (ii) X has the BAP and the CEP in Y .
- (iii) X has the BAP and the q -CEP in Y .

Proof. Although [20, Theorem 5.1] only establishes the equivalence between (i) and (ii), the very same proof gives that (iii) implies (i). So, taking into account Theorem 2.6, (iii) is equivalent to (i) and (ii). \square

Theorem 2.8. *Let X be a subspace of a quasi-Banach space Y . Denote by J the inclusion of X into Y . Suppose that X has the BAP and that X has the q -CEP in Y for some $0 < q \leq 1$. Then $J_{c,q}$ is an isomorphic embedding and $J_{c,q}(X_{c,q})$ has the q -CEP in $Y_{c,q}$.*

Proof. Use Theorem 2.6 to pick K such that X has the q -CEP in Y with constant K . Let Z be a q -Banach space and let $T: X_{c,q} \rightarrow Z$ be a compact operator. Since $T \circ E_{X,q}$ is also compact, the compact extension property and Theorem 2.5 yield a compact operator $S: Y_{c,q} \rightarrow Z$ with $\|S\| \leq K\|T\|$ and such that the diagram

$$\begin{array}{ccc} Y & \xrightarrow{E_{Y,q}} & Y_{c,q} \\ J \uparrow & & \searrow S \\ X & \xrightarrow{E_{X,q}} & X_{c,q} \xrightarrow{T} Z \end{array}$$

commutes. Since $E_{X,q}(X)$ is dense in $X_{c,q}$, merging this commutative diagram with

$$\begin{array}{ccc} Y & \xrightarrow{E_{Y,q}} & Y_{c,q} \\ J \uparrow & & \uparrow J_{c,q} \\ X & \xrightarrow{E_{X,q}} & X_{c,q} \end{array}$$

yields that the diagram

$$\begin{array}{ccc} Y_{c,q} & & \\ J_{c,q} \uparrow & \searrow S & \\ X_{c,q} & \xrightarrow{T} & Z \end{array}$$

commutes. It remains to show that $J_{c,q}$ is an isomorphic embedding. To that end, use Lemma 2.3 to pick C such that $X_{c,q}$ has that BAP with constant C . Let $x \in X_{c,q}$ and $\varepsilon > 0$. There is a linear operator $T: X_{c,q} \rightarrow X_{c,q}$ with finite-dimensional range such that $\|x - T(x)\| \leq \varepsilon$ and $\|T\| \leq C$. Since T is compact, there is $S: Y_{c,q} \rightarrow X_{c,q}$ such $S \circ J_{c,q} = T$ and $\|S\| \leq CK$. We have

$$\|x\|^q \leq \|x - T(x)\|^q + \|S(J_{c,q}(x))\|^q \leq \varepsilon^q + C^q K^q \|J_{c,q}(x)\|^q.$$

Letting ε tend to zero we obtain $\|x\| \leq CK \|J_{c,q}(x)\|$. \square

Theorem 2.9. *Let $0 < p \leq q \leq 1$ and X be a separable \mathcal{L}_p -space with the BAP. Then $X_{c,q}$ is isomorphic to a locally complemented subspace of ℓ_q and has a Schauder basis.*

Proof. By [20, Theorem 6.4], X is isomorphic to a locally complemented subspace Y of ℓ_p , and Y has a Schauder basis. By Theorem 2.7, Y has CEP in ℓ_p . Hence, by Theorem 2.8, $X_{c,q}$ is isomorphic to a subspace Z of ℓ_q and has the q -CEP in ℓ_q . As it is clear that $Y_{c,q}$ (and so $X_{c,q}$) has a Schauder basis, applying once again Theorem 2.7 completes the proof. \square

The following straightforward consequence of Theorem 2.9 partially solves [2, Question 4.18].

Corollary 2.10. *Let $0 < p \leq q \leq 1$ and X be a separable \mathcal{L}_p -space with the BAP. Then $X_{c,q}$ is a \mathcal{L}_q -space.*

We are ready to prove the main results of this section.

Theorem 2.11. *Let \mathcal{M} be a p -metric space, $0 < p < 1$. Suppose that there is a constant C such that \mathbb{N}_n is a Lipschitz retract of \mathcal{M} with constant C for all $n \in \mathbb{N}$. Then $\mathcal{F}_p(\mathcal{M})$ is not an \mathcal{L}_p -space. In particular, $\mathcal{F}_p(\mathcal{M}) \not\cong \ell_p$.*

Theorem 2.12. *Let \mathcal{M} be a metric space. Suppose that there is a constant K such that, for all $n \in \mathbb{N}$, \mathbb{N}_n Lipschitz-isomorphically embeds in \mathcal{M} with distortion at most K . Then $\mathcal{F}_p(\mathcal{M})$ is not an \mathcal{L}_p -space. In particular, $\mathcal{F}_p(\mathcal{M}) \not\cong \ell_p$.*

Proof of Theorems 2.11 and 2.12. Since \mathbb{R} is a doubling metric space, there is $D \geq 1$ such that every subset of \mathbb{R} , in particular \mathbb{N}_n for all $n \in \mathbb{N}$, is a doubling metric space with doubling constant D . Thus, under the assumptions in Theorem 2.12, applying [7, Theorem 5.1] yields a constant K such that $\mathcal{F}_p(\mathbb{N}_n)$ is K -complemented in $\mathcal{F}_p(\mathcal{M})$ for all $n \in \mathbb{N}$. By [5, Lemma 4.19], this holds under the assumptions in Theorem 2.11 as well. Since

$$\mathcal{D}_k = \{x \in [-k, k]: 2^k x \in \mathbb{Z}\}$$

is a doubling metric space with constant D for all $k \in \mathbb{N}$, applying again [7, Theorem 5.1] yields a constant C such that the linearization of the inclusion of \mathcal{D}_k into \mathbb{R} is a C -isomorphic embedding for all $k \in \mathbb{N}$. Taking into account that \mathcal{D}_k is Lipschitz isomorphic to $\mathbb{N}_{k2^{k+1}}$ with distortion 1 we infer the existence of a constant K_1 such that the finite-dimensional subspace

$$V_k := \text{span}(\delta_{\mathbb{R}}(x) : x \in \mathcal{D}_k)$$

of $\mathcal{F}_p(\mathbb{R})$ is K_1 -complemented in $\mathcal{F}_p(\mathcal{M})$ for all $k \in \mathbb{N}$.

Suppose by contradiction that $\mathcal{F}_p(\mathcal{M})$ is an \mathcal{L}_p -space. Then, by Lemma 2.2, there is a constant K_2 such that for all $k \in \mathbb{N}$ and $\varepsilon > 0$ there are linear bounded maps $S: V_k \rightarrow \ell_p$ and $T: \ell_p \rightarrow V_k \subset \mathcal{F}_p(\mathbb{R})$ with $\|S\| \|T\| \leq K_2$ and $\|T \circ S - \text{Id}_{V_k}\| \leq \varepsilon$. Since the set \mathcal{D} consisting of all dyadic rationals is dense in \mathbb{R} ,

$$\cup_{k=1}^{\infty} V_k = \text{span}(\delta_{\mathbb{R}}(x) : x \in \mathcal{D})$$

is a dense subspace of $\mathcal{F}_p(\mathbb{R})$. Applying Lemma 2.1 yields that $\mathcal{F}_p(\mathbb{R})$ is an \mathcal{L}_p -space. Since $\mathcal{F}_p(\mathbb{R})$ has the BAP (see (A.8)) combining Theorem 2.9 with [5, Proposition 4.20] yields that $\mathcal{F}(\mathbb{R})$ is isomorphic to a subspace of ℓ_1 . Using that $\mathcal{F}(\mathbb{R})$ is isometric to $L_1(\mathbb{R})$ and that ℓ_2 is a subspace of $L_1(\mathbb{R})$, we obtain that ℓ_2 is isomorphic to a subspace of ℓ_1 , an absurdity.

For the last part of the statements, we just need to note that ℓ_p is an \mathcal{L}_p -space. \square

Corollary 2.13. *Let \mathcal{M} be a metric space containing a subset which is Lipschitz isomorphic either to $[0, 1]$ or to \mathbb{N} . Then $\mathcal{F}_p(\mathcal{M})$ is not a \mathcal{L}_p -space for any $0 < p < 1$.*

Proof. Just notice that $\{k/n : k \in \mathbb{N}_n\} \subset [0, 1]$ is Lipschitz isomorphic to \mathbb{N}_n with distortion 1, and apply Theorem 2.12. \square

Corollary 2.14. *Let X be a p -Banach space ($0 < p < 1$) with non-trivial dual. Then $\mathcal{F}_p(X)$ is not a \mathcal{L}_p -space.*

Proof. By Corollary 2.13, $\mathcal{F}_p(\mathbb{R})$ is not a \mathcal{L}_p -space. Since, by assumption, \mathbb{R} is a complemented subspace of X , by [5, Lemma 4.19] it follows that $\mathcal{F}_p(\mathbb{R})$ is a complemented subspace of $\mathcal{F}_p(X)$. Consequently, $\mathcal{F}_p(X)$ is not an \mathcal{L}_p -space either. \square

3. SCHAUDER BASES IN $\mathcal{F}_p([0, 1]^d)$ AND $\mathcal{F}_p(\mathbb{R}^d)$

The basic idea for building Schauder bases for $\mathcal{F}_p([0, 1]^d)$ and $\mathcal{F}_p(\mathbb{R}^d)$ comes, on one hand, from [22], where the authors present a canonical

way of extending linearly Lipschitz functions on d -dimensional hypercubes, and on the other hand from [6], where a method for building linear projections on $\mathcal{F}_p([0, 1])$ is given.

Fix $d \in \mathbb{N}$. Given $R > 0$ and $w \in \mathbb{Z}^d$, we denote by $Q_{w,R}^d$, the hypercube having edge-length R and with vertices in the points

$$V_{w,R}^d = \{Rw + R\varepsilon : \varepsilon \in \{0, 1\}^d\},$$

That is, if $w = (w_i)_{i=1}^d$,

$$Q_{w,R}^d = \text{co}[V_{w,R}^d] = \prod_{i=1}^d [Rw_i, Rw_i + R].$$

For $R > 0$ fixed, the set of hypercubes

$$\mathcal{Q}_R^d = \{Q_{w,R}^d : w \in \mathbb{Z}^d\}$$

tessellates the space \mathbb{R}^d . If $Q \in \mathcal{Q}_R^d$, we denote by $\mathcal{V}(Q)$ its set of vertices, that is $\mathcal{V}(Q_{w,R}^d) = V_{w,R}^d$ for every $w \in \mathbb{Z}^d$. We have

$$\bigcup_{Q \in \mathcal{Q}_R^d} \mathcal{V}(Q) = \mathcal{V}_R^d := \{Rw : w \in \mathbb{Z}^d\}.$$

We shall define a fuzzy pull back of \mathbb{R}^d into the set of vertices \mathcal{V}_R^d . Given $x \in [0, 1]$ and $w \in \mathbb{Z}$ we set

$$x^{(w)} = \begin{cases} x & \text{if } w = 1, \\ 1 - x & \text{if } w = 0, \\ 0 & \text{if } w \in \mathbb{Z} \setminus \{0, 1\}. \end{cases}$$

Given $x = (x_i)_{i=1}^d \in [0, 1]^d$ and $w = (w_i)_{i=1}^d \in \mathbb{Z}^d$ we put

$$x^{(w)} = \prod_{i=1}^d x_i^{(w_i)}.$$

Lemma 3.1. *Let $d \in \mathbb{N}$ and $R > 0$. There is a mapping*

$$\Lambda = (\Lambda(v, \cdot))_{v \in \mathcal{V}_R^d} : \mathbb{R}^d \rightarrow [0, 1]^{\mathcal{V}_R^d}$$

such that $\Lambda(Ru, Rw + Rx) = x^{(u-w)}$ for every $x \in [0, 1]^d$ and $u, w \in \mathbb{Z}^d$.

Proof. Since the function $x \mapsto Rw + Rx$ maps $[0, 1]^d$ onto $Q_{w,R}^d$, if such a function Λ exists, it is unique. By dilation, it suffices to consider the case $R = 1$. If Λ is as desired in the one-dimensional case, then

$$\Lambda(u, \dots) = \Lambda(u_1, \cdot) \otimes \dots \otimes \Lambda(u_i, \cdot) \otimes \dots \otimes \Lambda(u_d, \cdot), \quad u = (u_i)_{i=1}^d \in \mathbb{Z}^d,$$

is as desired in the d -dimensional case. Hence, we can also assume that $d = 1$. To prove the result in this particular case, we must check that, given $w \in \mathbb{Z}$, the function given for $x \in \mathbb{R}$ by

$$x \mapsto (x - w)^{(u-w)} \text{ if } u \in \mathbb{Z} \text{ and } w \leq x \leq w + 1$$

is well-defined. Suppose that $w \leq x \leq w + 1$ and $v \leq x \leq v + 1$ with $v, w \in \mathbb{Z}$. Assume without loss of generality that $v < w$. Then $x = w = v + 1$. Since $x - w = 0$, we have $(x - w)^{(u-w)} = 0$ unless $u - w = 0$, in which case $(x - w)^{(u-w)} = 1$. Since $x - v = 1$, we have $(x - v)^{(u-v)} = 0$ unless $u - v = 1$, in which case $(x - v)^{(u-v)} = 1$. Since $u - w = u - v - 1$ for every $u \in \mathbb{Z}$, we are done. \square

Definition 3.2. Given $R > 0$ and $d \in \mathbb{N}$, we define

$$\Lambda_R^d = (\Lambda_R^d(v, \cdot))_{v \in \mathcal{V}_R^d}$$

as the function provided by Lemma 3.1.

If $d = 1$ we simply put $\Lambda_R = \Lambda_R^1$ and $\mathcal{V}_R = \mathcal{V}_R^1$. Given a finite set $A \subset \mathbb{N}$ we can carry out the above construction replacing the set $\{1, \dots, d\}$ with the set A . We will denote by \mathcal{V}_R^A the corresponding set of vertices and by Λ_R^A the corresponding function defined as in Definition 3.2.

Let us give an auxiliary lemma followed by some properties of the function Λ_R^d .

Lemma 3.3. *Let $x = (x_i)_{i=1}^d, y = (y_i)_{i=1}^d \in [0, 1]^d$. Then*

$$\left| \prod_{i=1}^d x_i - \prod_{i=1}^d y_i \right| \leq |x - y|_1.$$

Proof. We proceed by induction on d . For $d = 1$ the result is obvious. Assume that $d \in \mathbb{N}$ and that the result holds for $d - 1$. Then

$$\begin{aligned} \left| \prod_{i=1}^d x_i - \prod_{i=1}^d y_i \right| &\leq \left| \prod_{i=1}^d x_i - y_d \prod_{i=1}^{d-1} x_i \right| + \left| y_d \prod_{i=1}^{d-1} x_i - \prod_{i=1}^d y_i \right| \\ &= |x_d - y_d| \prod_{i=1}^{d-1} x_i + y_d \left| \prod_{i=1}^{d-1} x_i - \prod_{i=1}^{d-1} y_i \right| \\ &\leq |x_d - y_d| + \sum_{i=1}^{d-1} |x_i - y_i| = |x - y|_1. \quad \square \end{aligned}$$

Lemma 3.4. *Let $d \in \mathbb{N}$ and $S \geq R > 0$ with $S/R \in \mathbb{Z}$. We have:*

- (i) $\Lambda_R^d(v, x) = 0$ if $x \in Q \in \mathcal{Q}_R^d$ and $v \notin \mathcal{V}(Q)$.
- (ii) $\Lambda_R^d(v, u) = \delta_{u,v}$ for every $u, v \in \mathcal{V}_R^d$.

(iii) $\sum_{v \in \mathcal{V}_R} \Lambda_R^d(v, x) = 1$ for every $x \in \mathbb{R}^d$.

(iv) If there is $Q \in \mathcal{Q}_R^d$ such that $x, y \in Q$, then

$$|\Lambda_R^d(v, x) - \Lambda_R^d(v, y)| \leq R^{-1}|x - y|_1$$

for every $v \in \mathcal{V}_R^d$.

(v) Let (A, B) be a partition of $\{1, \dots, d\}$. Then

$$\Lambda_R^d((u, v), (x, y)) = \Lambda_R^A(u, x) \Lambda_R^B(v, y)$$

for every $u \in \mathcal{V}_R^A$, $v \in \mathcal{V}_R^B$, $x \in \mathbb{R}^A$ and $y \in \mathbb{R}^B$.

(vi) $\Lambda_S^d(v, x) = \sum_{u \in \mathcal{V}_R^d} \Lambda_S^d(v, u) \Lambda_R^d(u, x)$ for every $x \in \mathbb{R}^d$ and $v \in \mathcal{V}_S^d$.

Proof. (i) is clear from the definition. (ii) follows from the equality $0^{(0)} = 1$. A straightforward induction on d yields

$$\sum_{w \in \mathbb{Z}^d} x^{(w)} = 1, \quad x \in [0, 1]^d,$$

and (iii) is clear from this identity. (iv) is a consequence of Lemma 3.3. (v) is clear from the definition. In light of (v), in order to prove (vi) it suffices to consider the case $d = 1$. Given $x \in \mathbb{R}$ there are $u_0, u_1 \in \mathcal{V}_S$ and $v_0, v_1 \in \mathcal{V}_R$ with $u_0 \leq v_0 \leq x < v_1 \leq u_1$, and we have $v_1 = v_0 + R$ and $u_1 = u_0 + S$. Suppose that $u \in \mathcal{V}_S \setminus \{u_0, u_1\}$. Then $\Lambda_S(u, v) = 0$ for $v \in \{v_0, v_1\}$. Since $\Lambda_R(v, x) = 0$ for $v \in \mathcal{V}_R \setminus \{v_0, v_1\}$ we have

$$\Gamma(u, x) := \sum_{v \in \mathcal{V}_R} \Lambda_S(u, v) \Lambda_R(v, x) = 0 = \Lambda_S(u, x).$$

Hence, considering also the symmetry $x \mapsto -x$, it suffices to prove the result in the case when $u = u_1$. We have

$$\begin{aligned} \Gamma(u_1, x) &= \Lambda_S(u_1, v_0) \Lambda_R(v_0, x) + \Lambda_S(u_1, v_1) \Lambda_R(v_1, x) \\ &= \eta(x) := \frac{v_0 - u_0}{S} \frac{v_1 - x}{R} + \frac{v_1 - u_0}{S} \frac{x - v_0}{R}. \end{aligned}$$

Since $\eta(u_0) = 0$ we have $\eta(y) = \gamma(y - u_0)$ for all $y \in \mathbb{R}$, where

$$\gamma = \left(-\frac{v_0 - u_0}{SR} + \frac{v_1 - u_0}{SR} \right) = \frac{1}{S}.$$

Since $\Lambda_S(u_1, x) = (x - u_0)/S$ we are done. \square

Although the previous auxiliary results are stated in terms of the ℓ_1 -norm, in this section we will consider \mathbb{R}^d and its subsets equipped with the supremum norm $\|\cdot\|_\infty$.

Theorem 3.5. *Let $d \in \mathbb{N}$ and $0 < p \leq 1$. There is a constant $C = C(p, d)$ such that for every $R > 0$ and every $\mathcal{R} \subset \mathcal{Q}_R^d$, if we denote $K = \cup_{Q \in \mathcal{R}} Q$ and $V = \cup_{Q \in \mathcal{R}} \mathcal{V}(Q)$, and we choose an arbitrary point*

of V as base point of both metric spaces, there is a C -Lipschitz map $r = r_{K,V}: K \rightarrow \mathcal{F}_p(V)$ such that

$$r(x) = \sum_{v \in V} \Lambda_R^d(v, x) \delta_V(v), \quad x \in K.$$

Proof. Let $Q \in \mathcal{R}$ and $x, y \in Q$. Pick $u \in \mathcal{V}(Q)$. By Lemma 3.4,

$$\begin{aligned} \|r(x) - r(y)\|^p &= R \left\| \sum_{v \in \mathcal{V}(Q) \setminus \{u\}} (\Lambda_R^d(v, x) - \Lambda_R^d(v, y)) \frac{\delta_V(v) - \delta_V(u)}{|v - u|_\infty} \right\|^p \\ &\leq R \sum_{v \in \mathcal{V}(Q) \setminus \{u\}} |\Lambda_R^d(v, x) - \Lambda_R^d(v, y)|^p \\ &\leq (2^d - 1) |x - y|_1^p. \end{aligned}$$

Let $x = (x_i)_{i=1}^d \in K$ and $y = (y_i)_{i=1}^d \in K$. Pick $u = (u_i)_{i=1}^d$ and $w = (w_i)_{i=1}^d \in \mathbb{Z}^d$ such that $x \in Q_{R,u}^d$ and $y \in Q_{R,w}^d$. Define

$$F = \{i \in \{1, \dots, d\} : u_i = w_i\}.$$

For each $i \in G = \{1, \dots, d\} \setminus F$ there is $m_i \in \{u_i, u_i + 1\}$ and $n_i \in \{w_i, w_i + 1\}$ such that

$$|y_i - x_i| = |y_i - Rn_i| + |Rn_i - Rm_i| + |Rm_i - x_i|.$$

Suppose that $n_i = m_i$ for every $i \in G$. Define $z = (z_i)_{i=1}^d$ by

$$z_i = \begin{cases} x_i & \text{if } i \in F, \\ Rn_i = Rm_i & \text{if } i \in G. \end{cases}$$

We have $z \in Q_{R,u}^d \cap Q_{R,w}^d$. Consequently,

$$\begin{aligned} \|r(x) - r(y)\|^p &\leq \|r(x) - r(z)\|^p + \|r(z) - r(y)\|^p \\ &\leq (2^d - 1) (|x - z|_1^p + |z - y|_1^p) \\ &\leq 2^{1-p} (2^d - 1) (|x - z|_1 + |z - y|_1)^p \\ &= 2^{1-p} (2^d - 1) |x - y|_1^p. \end{aligned}$$

Suppose that $m_i \neq n_i$ for some i . Define $x' = (x'_i)_{i=1}^d$ and $y' = (y'_i)_{i=1}^d$ by

$$x'_i = \begin{cases} Ru_i = Rw_i & \text{if } i \in F, \\ Rm_i & \text{if } i \in G, \end{cases} \quad y'_i = \begin{cases} Ru_i = Rw_i & \text{if } i \in F, \\ Rn_i & \text{if } i \in G. \end{cases}$$

We have $x' \in V_{R,u}^d$, $y' \in V_{R,w}^d$, and $1 \leq \mu := \max_{i \in G} |m_i - n_i|$. Hence

$$\begin{aligned} \|r(x) - r(y)\|^p &\leq \|r(x) - r(x')\|^p + \|r(x') - r(y')\|^p + \|r(y') - r(y)\|^p \end{aligned}$$

$$\begin{aligned}
&\leq (2^d - 1)|x - x'|_1^p + |x' - y'|_\infty^p + (2^d - 1)|y - y'|_1^p \\
&\leq 2(2^d - 1)dR^p + \mu^p R^p \\
&\leq (1 + 2d(2^d - 1))\mu^p R^p \\
&\leq (1 + 2d(2^d - 1))|x - y|_\infty.
\end{aligned}$$

This way the result holds with $C(p, d) = (1 + 2d(2^d - 1))^{1/p}$. \square

Next, we show that $\mathcal{F}_p(\mathbb{R}^d)$ has a Schauder basis for every $d \in \mathbb{N}$.

Lemma 3.6. *Let \mathcal{M} and \mathcal{N} be quasi-metric spaces. For $i = 1, 2$, let $f_i: \mathcal{M}_i \subset \mathcal{M} \rightarrow \mathcal{N}$ be a Lipschitz function. Assume that*

- (i) $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$,
- (ii) $f_1|_{\mathcal{M}_1 \cap \mathcal{M}_2} = f_2|_{\mathcal{M}_1 \cap \mathcal{M}_2}$, and
- (iii) *There is a constant C such that $d(x_1, \mathcal{M}_1 \cap \mathcal{M}_2) \leq Cd(x_1, x_2)$ for every $(x_1, x_2) \in \mathcal{M}_1 \times \mathcal{M}_2$.*

Then, the map $f: \mathcal{M} \rightarrow \mathcal{N}$ defined by $f|_{\mathcal{M}_i} = f_i$ for $i = 1, 2$ is Lipschitz. Moreover, if $k_{\mathcal{N}}$ and $k_{\mathcal{M}}$ are the quasi-metric constants of \mathcal{N} and \mathcal{M} , respectively, we have

$$\text{Lip}(f) \leq k_{\mathcal{N}}(C + k_{\mathcal{M}} + Ck_{\mathcal{M}}) \max_{i=1,2} \text{Lip}(f_i).$$

Proof. Put $L = \max_{i=1,2} \text{Lip}(f_i)$. Let $(x_1, x_2) \in \mathcal{M}_1 \times \mathcal{M}_2$ and pick $x \in \mathcal{M}_1 \cap \mathcal{M}_2$. Since $f_1(x) = f_2(x)$ we have

$$\begin{aligned}
d_{\mathcal{N}}(f_1(x_1), f_2(x_2)) &\leq k_{\mathcal{N}}(d_{\mathcal{N}}(f_1(x_1), f_1(x)) + d_{\mathcal{N}}(f_2(x), f_2(x_2))) \\
&\leq k_{\mathcal{N}}L(d_{\mathcal{M}}(x_1, x) + d_{\mathcal{M}}(x, x_2)) \\
&\leq k_{\mathcal{N}}L((1 + k_{\mathcal{M}})d_{\mathcal{M}}(x_1, x) + k_{\mathcal{M}}d_{\mathcal{M}}(x_1, x_2)).
\end{aligned}$$

Since the element x can be chosen so that $d_{\mathcal{M}}(x_1, x)$ is arbitrarily close to $d_{\mathcal{M}}(x_1, \mathcal{M}_1 \cap \mathcal{M}_2)$, using the assumption (iii) yields the desired result. \square

The following lemma exhibits a situation in which Lemma 3.6 is useful that will occur several times throughout this section,.

Lemma 3.7. *Let \mathcal{M} and \mathcal{N} be quasi-metric spaces. For $i = 1, 2$, let $f_i: \mathcal{M}_i \subset \mathcal{M} \rightarrow \mathcal{N}$ be a Lipschitz function. Assume that $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$ and that $f_1|_{\mathcal{M}_1 \cap \mathcal{M}_2} = f_2|_{\mathcal{M}_1 \cap \mathcal{M}_2}$. Suppose that there are constants $\lambda > 0$ and $C \geq 1$ such that*

- (i) \mathcal{M} is λ -separated, and
- (ii) $d(x_1, \mathcal{M}_1 \cap \mathcal{M}_2) \leq C$ for every $x_1 \in \mathcal{M}_1$.

Then, the map $f: \mathcal{M} \rightarrow \mathcal{N}$ defined by $f|_{\mathcal{M}_i} = f_i$ for $i = 1, 2$ is Lipschitz. Moreover, if $k_{\mathcal{N}}$ and $k_{\mathcal{M}}$ are the quasi-metric constant of \mathcal{N} and \mathcal{M} , respectively,

$$\text{Lip}(f) \leq k_{\mathcal{N}} \left(\frac{C}{\lambda} (1 + k_{\mathcal{M}}) + k_{\mathcal{M}} \right) \max_{i=1,2} \text{Lip}(f_i).$$

Proof. Let $(x_1, x_2) \in \mathcal{M}_1 \times \mathcal{M}_2$ with $x_1 \neq x_2$. We have

$$d(x_1, \mathcal{M}_1 \cap \mathcal{M}_2) \leq \frac{C}{\lambda} d(x_1, x_2).$$

Hence, the result follows from Lemma 3.6. \square

Theorem 3.8 (cf. [6, Theorem 5.7]). *Let $0 < p \leq 1$ and $d \in \mathbb{N}$. Then $\mathcal{F}_p([0, 1]^d)$ has a Schauder basis. In fact, if $V_n = [0, 1]^d \cap 2^{-n}\mathbb{Z}^d$ for all $n \in \mathbb{N}_*$ and $V_{-1} = \{0\}$, and we put $\alpha(x) = n$ if $x \in V_n \setminus V_{n-1}$, then, any arrangement $(f(x_j))_{j=1}^\infty$ of the family*

$$f(x) = \delta_{[0,1]^d}(x) - \sum_{v \in V_{n-1}} \Lambda_{2^{-n+1}}^d(v, x) \delta_{[0,1]^d}(v) \quad n \in \mathbb{N}_*, x \in V_n \setminus V_{n-1}$$

such that $(\alpha(x_j))_{j=1}^\infty$ is non-decreasing is a Schauder basis of $\mathcal{F}_p([0, 1]^d)$.

Proof. Set $\delta = \delta_{[0,1]^d}$ and $\delta_n = \delta_{V_n}$ for $n \in \mathbb{N}_*$. By Theorem 3.5, there exist a constant C and linear maps $T_n: \mathcal{F}_p([0, 1]^d) \rightarrow \mathcal{F}_p(V_n)$ such that $\|T_n\| \leq C$ and

$$T_n(\delta(x)) = \sum_{v \in V_n} \Lambda_{2^{-n}}^d(v, x) \delta_n(v), \quad x \in [0, 1]^d.$$

Let $m, n \in \mathbb{N}_*$ with $m \leq n$. Since $V_m \subset V_n$, we can consider the canonical map $L_{m,n}: \mathcal{F}_p(V_m) \rightarrow \mathcal{F}_p(V_n)$ associated to the inclusion. Consider also the canonical map $L_n: \mathcal{F}_p(V_n) \rightarrow \mathcal{F}_p([0, 1]^d)$ associated to the inclusion of V_n into $[0, 1]^d$. Applying Proposition 3.4 (vi) yields

$$\bullet T_m \circ L_n \circ T_n = T_m \text{ and } T_n \circ L_m \circ T_m = L_{m,n} \circ T_m.$$

Moreover, since $\cup_{n=0}^\infty V_n$ is dense in $[0, 1]^d$,

$$\bullet \text{ if } X_n = L_n(T_n(\mathcal{F}_p([0, 1]^d))), \cup_{n=0}^\infty X_n \text{ is dense in } \mathcal{F}_p([0, 1]^d).$$

Since, with the convention $L_{-1} \circ T_{-1} = 0$,

$$Y_n = \{x - L_{n-1}(T_{n-1}(x)): x \in X_n\},$$

and $X_n = \text{span}\{\delta(x): x \in V_n\}$, the family of nonzero vectors

$$\mathcal{B}_n := (f(x))_{x \in V_n \setminus V_{n-1}}$$

generates the space Y_n . We shall prove that \mathcal{B}_n is an unconditional basis of Y_n with uniformly bounded unconditional basis constant. Let

$F \subset V_n \setminus V_{n-1}$. We define $r_{n,F}: V_n \rightarrow \mathcal{F}_p([0, 1]^d)$ by

$$r_{n,F}(x) = \begin{cases} L_{n-1} \circ T_{n-1}(\delta(x)) & \text{if } x \in V_n \setminus F, \\ \delta(x) & \text{if } x \in F. \end{cases}$$

If $z \in V_{n-1}$, then $L_{n-1}(T_{n-1}(\delta(z))) = L_{n-1}(\delta_{n-1}(z)) = \delta(z)$. For every $x \in V_n$ there is $z \in V_{n-1}$ such that $|x - z|_\infty = 2^{-n}$, and V_n is 2^{-n} -separated. By Lemma 3.7, $r_{n,F}$ is C_1 -Lipschitz with $C_1 = (1 + 2^{1/p})2^{1/p-1}C$. We infer that there is $U_{n,F}: \mathcal{F}_p(V_n) \rightarrow \mathcal{F}_p([0, 1]^d)$ such that $\|U_{n,F}\| \leq C_1$ and $U_{n,F} \circ \delta_n = r_{n,F}$. Put $Q_{n,F} = U_{n,F} \circ T_n|_{Y_n}$. We have $\|Q_{n,F}\| \leq CC_1$ and

$$Q_{n,F}(f(x)) = \begin{cases} f(x) & \text{if } x \in F, \\ 0 & \text{if } x \in (V_n \setminus V_{n-1}) \setminus F. \end{cases}$$

Thus, the mappings $(Q_{n,F})_{F \subset V_n \setminus V_{n-1}}$ are commuting projections satisfying $Q_{n,F}(Y_n) = \text{span}\{f(x): x \in F\}$ and $Q_{n,F}^{-1}(0) = \text{span}\{f(x): x \in (V_n \setminus V_{n-1}) \setminus F\}$, which implies that \mathcal{B}_n is unconditional basis of Y_n with basis constant at most CC_1 . \square

We have explicitly constructed a Schauder basis of $\mathcal{F}_p([0, 1]^d)$. Using the isomorphism $\mathcal{F}_p([0, 1]^d) \simeq \mathcal{F}_p(\mathbb{R}^d)$ we infer that $\mathcal{F}_p(\mathbb{R}^d)$ has a Schauder basis, but now our proof is not constructive. So, it is worth to mention that an argument slightly more involved than the one we have used to prove Theorem 3.8 yields an explicit Schauder basis of $\mathcal{F}_p(\mathbb{R}^d)$.

Theorem 3.9. *Let $0 < p \leq 1$ and $d \in \mathbb{N}$. Given an increasing sequence of natural numbers $(k_n)_{n=-1}^\infty$, put*

$$V_n = \{x \in \mathbb{R}^d: |x|_\infty \leq k_n\} \cap 2^{-n}\mathbb{Z}^d, \text{ and} \\ W_n = \{x \in \mathbb{R}^d: |x|_\infty \leq k_{n-1}\} \cap 2^{-n}\mathbb{Z}^d$$

if $n \in \mathbb{N}_*$, and set $V_{-1} = \{0\}$. Define

$$s_n(x) = (\min\{-2^{-n} + |x|_\infty, |x_i|\})_{i=1}^d \text{sgn}(x_i), \quad x = (x_i)_{i=1}^d \in V_n \setminus W_n,$$

and put $\eta(x) = (n, |x|_\infty - k_{n-1})$ if $x \in V_n \setminus W_n$ and $\eta(x) = (n, 0)$ if $x \in W_n \setminus V_{n-1}$. Define $f(x)$ for $x \in \cup_{n=0}^\infty V_n$ by

$$f(x) = \begin{cases} \delta_{\mathbb{R}^d}(x) - \delta_{\mathbb{R}^d}(s_n(x)) & \text{if } x \in V_n \setminus W_n, \\ \delta_{\mathbb{R}^d}(x) - \sum_{v \in V_{n-1}} \Lambda_{2^{-n+1}}^d(v, x) \delta_{\mathbb{R}^d}(v) & \text{if } x \in W_n \setminus V_{n-1}. \end{cases}$$

Let $(x_j)_{j=1}^\infty$ be an arrangement of $\cup_{n=0}^\infty V_n$ such that $(\eta(x_j))_{j=1}^\infty$ is non-decreasing with respect to the lexicographical order. Then $(f(x_j))_{j=1}^\infty$ is a Schauder basis of $\mathcal{F}_p(\mathbb{R}^d)$.

Proof. Given $t \in (0, \infty)$, for $d = 1$ define $r_t: \mathbb{R} \rightarrow \mathbb{R}$ by

$$r_t(x) = \min \left\{ 1, \frac{t}{|x|} \right\} x, \quad x \in \mathbb{R}.$$

In general, we define $r_t: \mathbb{R}^d \rightarrow \mathbb{R}^d$ by

$$r_t((x_i)_{i=1}^d) = (r_t(x_i))_{i=1}^d.$$

The map r_t is 1-Lipschitz and its range is $B_t := \{x \in \mathbb{R}^d: \|x\|_\infty \leq t\}$. Let $S_t: \mathcal{F}_p(\mathbb{R}^d) \rightarrow \mathcal{F}_p(B_t)$ be obtained by linearization of r_t . Given $R > 0$ such that $t/R \in \mathbb{N}$, denote $V_{t,R} = \mathcal{V}_R^d \cap B_t$, and let $T_R: \mathcal{F}_p(\mathbb{R}^d) \rightarrow \mathcal{F}_p(\mathcal{V}_R^d)$ and $T_{t,R}: \mathcal{F}_p(B_t) \rightarrow \mathcal{F}_p(V_{t,R})$ be the linear maps provided by Theorem 3.5. Notice that Theorem 3.5 yields a constant C be such that $\|T_{t,R}\| \leq C$ for every $t > 0$ and R with $t/R \in \mathbb{N}$. Let $L_{\mathcal{M},\mathcal{N}}$ denote the linearization of the inclusion of \mathcal{M} into \mathcal{N} , and put $L_R = L_{\mathcal{V}_R^d, \mathbb{R}^d}$, $L_{t,R} = L_{V_{t,R}, B_t}$ and $L'_{t,R} = L_{V_{t,R}, \mathbb{R}^d}$. We shall prove that

$$L_{t,R} \circ T_{t,R} \circ S_t = S_t \circ L_R \circ T_R, \quad \frac{t}{R} \in \mathbb{N}. \quad (3.2)$$

Set $\delta = \delta_{\mathbb{R}^d}$, $\delta_t = \delta_{B_t}$, and $\delta_{t,R} = \delta_{V_{t,R}}$.

By Proposition 3.4 (v) a similar argument works for the general case, hence for notational ease we will deal with the case $d = 1$. Let $x \in \mathbb{R}$. In the case when $|x| \leq t$, we have $r_t(x) = x$ and $\Lambda_R(v, x) = 0$ unless $|v| \leq t$, in which case $r_t(v) = v$. Consequently,

$$\begin{aligned} S_t(L_R(T_R(\delta(x)))) &= \sum_{v \in \mathcal{V}_R} \Lambda_R(v, x) \delta_t(r_t(v)) \\ &= \sum_{v \in \mathcal{V}_R} \Lambda_R(v, x) \delta_t(v) \\ &= L_{t,R}(T_{t,R}(\delta_t(x))) \\ &= L_{t,R}(T_{t,R}(\delta_t(r_t(x)))). \end{aligned}$$

In the case when $|x| > t$ we have $\Lambda_R(v, x) = 0$ unless $|v| \geq t$ and $\text{sgn}(v) = \text{sgn}(x)$, in which case $r_t(v) = r_t(x)$. Then we have

$$\begin{aligned} S_t(L_R(T_R(\delta(x)))) &= \sum_{v \in \mathcal{V}_R} \Lambda_R(v, x) \delta_t(r_t(v)) \\ &= \sum_{v \in \mathcal{V}_R} \Lambda_R(v, x) \delta_t(r_t(x)) \\ &= \delta_t(r_t(x)) \end{aligned}$$

$$= L_{t,R}(T_{t,R}(\delta_t(r_t(x))))).$$

Since $\delta_t(r_t(x)) = S_t(\delta(x))$ and $\{\delta(x) : x \in \mathbb{R}\}$ spans $\mathcal{F}_p(\mathbb{R})$, (3.2) holds. Note that the range of $U_{t,R} := T_{t,R} \circ S_t$ is $\text{span}\{\delta_{t,R}(x) : x \in V_{t,R}\}$. Put $P_{t,R} := L'_{t,R} \circ T_{t,R} \circ S_t$. By (3.2), for every $x \in \mathbb{R}^d$ we have

$$P_{t,R}(\delta(x)) = \sum_{v \in \mathcal{V}_R^d} \Lambda_R^d(v, x) \delta(r_t(v)), \quad \frac{t}{R} \in \mathbb{N}. \quad (3.3)$$

Thus, for every $v \in \mathcal{V}_R^d$ we have $P_{t,R}(\delta(v)) = \delta(r_t(v))$, which in combination with (3.3) implies that $P_{t,R}$ is a projection with range equal to $\text{span}\{\delta(x) : x \in V_{t,R}\}$. We infer that

$$P_{t,R} \circ P_{t',R'} = P_{t',R'} \circ P_{t,R} = P_{t,R}, \quad \frac{R}{R'}, \frac{t'}{R'}, \frac{t}{R} \in \mathbb{N}, t' \geq t > 0. \quad (3.4)$$

Indeed, $P_{t',R'} \circ P_{t,R} = P_{t,R}$ follows from the fact that the range of $P_{t,R}$ is contained in the range of $P_{t',R'}$ and we have

$$\begin{aligned} P_{t,R} \circ P_{t',R'} &= L'_{t,R} \circ T_{t,R} \circ S_t \circ L_{B_{t',R^d}} \circ S_{t'} \circ L_{R'} \circ T_{R'} \\ &= L'_{t,R} \circ T_{t,R} \circ S_t \circ L_{R'} \circ T_{R'} \\ &= P_{t,R}, \end{aligned}$$

where in the first equality we used (3.2), in the second we used the observation that $S_t \circ L_{B_{t',R^d}} \circ S_{t'} = S_t$ and the third equality follows from Proposition 3.4 (vi).

Hence, if for $n \in \mathbb{N}_*$ we put

$$P_{2n} = P_{k_n, 2^{-n}} \text{ and } P_{2n-1} = P_{k_{n-1}, 2^{-n}},$$

we have $P_j \circ P_{j'} = P_{j'} \circ P_j = P_j$ whenever $-1 \leq j \leq j'$. Therefore, there is Schauder decomposition $(Y_j)_{j=-1}^\infty$ of $\mathcal{F}_p(\mathbb{R}^d)$ whose associated projections are $(P_j)_{j=-1}^\infty$. Moreover, for all $n \in \mathbb{N}_*$ the range of P_{2n} is $\text{span}\{\delta(x) : x \in V_n\}$ and the range of P_{2n-1} is $\text{span}\{\delta(x) : x \in W_n\}$.

Similar arguments as in the proof of Theorem 3.8 show that there is a constant $C = C(p, d)$ such that

$$(f(x))_{x \in W_n \setminus V_{n-1}}$$

is a C -unconditional basis of this Y_{2n-1} for all $n \in \mathbb{N}_*$. Let $n \in \mathbb{N}_*$. Note that $r_t(x) = r_t(s_n(x))$ for all t with $2^n t \in \mathbb{N}$ and all $x \in V_n \setminus W_n$ with $\|x\|_\infty > t$. In particular,

$$r_{k_{n-1}}(x) = r_{k_{n-1}}(s_n(x)), \quad x \in V_n \setminus W_n.$$

Therefore, $P_{2n-1}(f(x)) = 0$ for every $x \in V_n \setminus W_n$, which in turn implies that $f(x) \in Y_{2n}$. An inductive argument yields that for every $x \in V_n \setminus W_n$,

$$g(x) := \delta(x) - \delta(r_{k_{n-1}}(x))$$

is a linear combination of the family (of nonzero vectors)

$$\mathcal{B}_n = (f(y))_{y \in V_n \setminus W_n}.$$

Since, by an argument similar to that used in the proof of Theorem 3.8, $(g(x))_{x \in V_n \setminus W_n}$ generates Y_{2n} , \mathcal{B}_n also generates Y_{2n} . Let $F \subset V_n \setminus W_n$ be such that

$$V_{t-2^{-n}, 2^{-n}} \setminus W_n \subset F \subset V_{t, 2^{-n}} \setminus W_n$$

for some $t \in (k_{n-1}, k_n] \cap 2^{-n}\mathbb{Z}$. Set $\alpha = (t, 2^{-n})$ and let $\hat{\alpha}$ be the ‘‘predecessor’’ of α given by $\hat{\alpha} = (t - 2^{-n}, 2^{-n})$. Define $r_{\alpha, F}: V_\alpha \rightarrow \mathbb{R}^d$ by

$$r_{\alpha, F}(x) = \begin{cases} r_{t-2^{-n}}(x) & \text{if } x \in V_\alpha \setminus F, \\ x & \text{if } x \in F. \end{cases}$$

If $x \in V_{\hat{\alpha}}$ then $r_{t-2^{-n}}(x) = x$. Given $x \in V_\alpha$ there is $z \in V_{\hat{\alpha}}$ with $|x - z|_\infty = 2^{-n}$, and V_α is 2^{-n} -separated. Hence, by Lemma 3.7, $r_{\alpha, F}$ is C_p -Lipschitz, where $C_p = (1 + 2^{1/p})$. Let $U_{\alpha, F}: \mathcal{F}_p(V_\alpha) \rightarrow \mathcal{F}_p(\mathbb{R}^d)$ be the linear map defined by $\delta_\alpha(x) \mapsto \delta(x)$ if $x \in F$ and $\delta_\alpha(x) \mapsto \delta(r_{t-2^{-n}}(x))$ if $x \in V_\alpha \setminus F$. Set $U_\alpha = T_\alpha \circ S_t$. We infer that, if

$$Q_{\alpha, F} = U_{\alpha, F} \circ U_\alpha|_{Y_{2n}},$$

then $\|Q_{\alpha, F}\| \leq CC_p$. Note that $U_\alpha(\delta(x)) = \delta_\alpha(x)$ for all $x \in V_\alpha$ and that $U_\alpha(\delta(x)) = U_\alpha(\delta(s_n(x)))$ for all $x \in V_n \setminus V_\alpha$. If $x \in V_\alpha$, then $s_n(x) \in V_{\hat{\alpha}}$, and so $Q_{\alpha, F}(f(x)) = f(x)$ for every $f \in F$. If $x \in V_\alpha \setminus V_{\hat{\alpha}}$, then $s_n(x) = r_{t-2^{-n}}(x)$. Consequently, $Q_{\alpha, F}(f(x)) = 0$ for every $x \in V_\alpha \setminus F$. Finally, we deduce that $Q_{\alpha, F}(f(x)) = 0$ for every $x \in V_n \setminus V_\alpha$. We infer that, if $(x_j)_{j=1}^{|V_n| - |W_n|}$ is an arrangement of $V_n \setminus W_n$ with $(|x_j|)_{j=1}^{|V_n| - |W_n|}$ non-decreasing, then $(f(x_j))_{j=1}^{|V_n| - |W_n|}$ is a Schauder basis of Y_{2n-1} with basis constant at most CC_p . \square

4. OPEN PROBLEMS

By [21, Theorem 5.2], there exists a subspace Z of ℓ_p whose Banach envelope is isomorphic to L_1 , so we would like to know whether $\mathcal{F}_p(\mathbb{R})$ is different from the subspace Z , whose existence is guaranteed by a general abstract construction.

Question 4.1. Let $p \in (0, 1)$. Is $\mathcal{F}_p(\mathbb{R})$ isomorphic to a subset of ℓ_p ?

Once we have told apart $\mathcal{F}_p(\mathbb{N})$ from ℓ_p for $p < 1$, it is natural to go on with the topic **(Q.b)** by trying to determine how many non-isomorphic Lipschitz free p -spaces one can obtain from subsets of \mathbb{N} . Note that if \mathcal{N} is a subset of \mathbb{N} then $\mathcal{F}_p(\mathcal{N})$ is a complemented subspace of $\mathcal{F}_p(\mathbb{N})$ by **(A.7)**. So, this problem connects with that of characterizing the complemented subspaces of $\mathcal{F}_p(\mathbb{N})$. Let us illustrate the question with

an example. Suppose that $\mathcal{N} \subset \mathbb{N}$ contains arbitrarily long chains of consecutive integers. Then, there is $(a_k)_{k=0}^\infty$ in \mathbb{N} such that $a_k + \mathbb{N}_{2^k-1} \subset \mathcal{N}$ and $2(a_k + 2^k - 1) \leq a_{k+1}$ for all $k \in \mathbb{N}_*$. Set

$$\mathcal{N}_0 = \bigcup_{k=0}^{\infty} a_k + \mathbb{N}_{2^k-1}.$$

Combining [6, Lemma 2.1] with [7, Theorem 5.8] yields $\mathcal{F}_p(\mathcal{N}_0) \simeq \mathcal{F}_p(\mathbb{N})$. Then, by **(A.7)**, $\mathcal{F}_p(\mathcal{N})$ is complemented in $\mathcal{F}_p(\mathbb{N})$ and, the other way around, $\mathcal{F}_p(\mathbb{N})$ is complemented in $\mathcal{F}_p(\mathcal{N})$. Taking into account **(A.5)**, Pełczyński's decomposition method yields $\mathcal{F}_p(\mathcal{N}) \simeq \mathcal{F}_p(\mathbb{N})$.

Question 4.2. Let $0 < p < 1$. Does there exist $\mathcal{N} \subset \mathbb{N}$ such that $\mathcal{F}_p(\mathcal{N})$ is neither isomorphic to ℓ_p nor to $\mathcal{F}_p(\mathbb{N})$?

A well-known problem in Geometric Functional Analysis is whether $\mathcal{F}(\mathbb{N}^2)$ is isomorphic to $\mathcal{F}(\mathbb{N}^3)$ or, more generally, whether $\mathcal{F}(\mathbb{N}^d)$ is isomorphic to $\mathcal{F}(\mathbb{N}^{d+1})$ for $d \geq 2$. As, by [5, Proposition 4.20], if $\mathcal{F}_p(\mathbb{N}^d) \simeq \mathcal{F}_p(\mathbb{N}^{d+1})$ for some $p < 1$ then $\mathcal{F}(\mathbb{N}^d) \simeq \mathcal{F}(\mathbb{N}^{d+1})$, investigating in more depth the geometry of $\mathcal{F}_p(\mathbb{N}^d)$ for $p < 1$ and $d \geq 2$ could shed some light into this important question. Similarly, telling apart $\mathcal{F}_p(\mathbb{R}^d)$ from $\mathcal{F}_p(\mathbb{R}^{d+1})$ for $p < 1$ could be easier than telling apart $\mathcal{F}(\mathbb{R}^d)$ from $\mathcal{F}(\mathbb{R}^{d+1})$. By [7, Theorem 4.21] the latter is equivalent to proving that Lipschitz p -free spaces over Euclidean spaces are not isomorphic to Lipschitz p -free spaces over its spheres.

Question 4.3. Let $0 < p \leq 1$ and $d \geq 2$. Is $\mathcal{F}_p(\mathbb{N}^d)$ isomorphic to $\mathcal{F}_p(\mathbb{N}^{d+1})$?

Question 4.4. Let $0 < p \leq 1$ and $d \geq 3$. Is $\mathcal{F}_p(\mathbb{R}^d)$ isomorphic to $\mathcal{F}_p(S_{\mathbb{R}^d})$?

REFERENCES

- [1] F. Albiac, J. L. Ansorena, P. M. Berná, and P. Wojtaszczyk, *Greedy approximation for biorthogonal systems in quasi-banach spaces*, arXiv e-prints (2019), available at [1903.11651](#).
- [2] F. Albiac, J. L. Ansorena, and P. Wojtaszczyk, *On certain subspaces of ℓ_p for $0 < p \leq 1$ and their applications to conditional quasi-greedy bases in p -banach spaces*, arXiv e-prints (2019), available at [1912.08449](#).
- [3] F. Albiac and N. J. Kalton, *Lipschitz structure of quasi-Banach spaces*, Israel J. Math. **170** (2009), 317–335.
- [4] ———, *Topics in Banach space theory*, Second, Graduate Texts in Mathematics, vol. 233, Springer, [Cham], 2016. With a foreword by Gilles Godefroy.
- [5] F. Albiac, J. L. Ansorena, M. Cuth, and M. Doucha, *Lipschitz free p -spaces for $0 < p < 1$* , arXiv e-prints (2018), available at [1811.01265](#). Accepted for publication in Isr. J. Math.

- [6] F. Albiac, J. L. Ansorena, M. Cúth, and M. Doucha, *Embeddability of ℓ_p and bases in Lipschitz free p -spaces for $0 < p \leq 1$* , J. Funct. Anal. **278** (2020), no. 4, 108354, 33.
- [7] F. Albiac, J. L. Ansorena, M. Cuth, and M. Doucha, *Lipschitz free spaces isomorphic to their infinite sums and geometric applications*, arXiv e-prints (2020), available at 2005.06555.
- [8] L. Ambrosio and D. Puglisi, *Linear extension operators between spaces of Lipschitz maps and optimal transport*, 2016.
- [9] M. Cúth and M. Doucha, *Lipschitz-free spaces over ultrametric spaces*, Mediterr. J. Math. **13** (2016), no. 4, 1893–1906.
- [10] A. Dalet, *Free spaces over countable compact metric spaces*, Proc. Amer. Math. Soc. **143** (2015), no. 8, 3537–3546.
- [11] ———, *Free spaces over some proper metric spaces*, Mediterr. J. Math. **12** (2015), no. 3, 973–986.
- [12] M. Doucha and P. L. Kaufmann, *Approximation properties in Lipschitz-free spaces over groups*, arXiv e-prints (2020), available at 2005.09785.
- [13] V. P. Fonf and P. Wojtaszczyk, *Properties of the Holmes space*, Topology Appl. **155** (2008), no. 14, 1627–1633.
- [14] G. Godefroy and N. J. Kalton, *Lipschitz-free Banach spaces*, Studia Math. **159** (2003), no. 1, 121–141. Dedicated to Professor Aleksander Pełczyński on the occasion of his 70th birthday.
- [15] G. Godefroy, *Extensions of Lipschitz functions and Grothendieck’s bounded approximation property*, North-West. Eur. J. Math. **1** (2015), 1–6.
- [16] ———, *A survey on Lipschitz-free Banach spaces*, Comment. Math. **55** (2015), no. 2, 89–118.
- [17] G. Godefroy and N. Ozawa, *Free Banach spaces and the approximation properties*, Proc. Amer. Math. Soc. **142** (2014), no. 5, 1681–1687.
- [18] P. Hájek and M. Novotný, *Some remarks on the structure of Lipschitz-free spaces*, Bull. Belg. Math. Soc. Simon Stevin **24** (2017), no. 2, 283–304.
- [19] P. Hájek and E. Pernecká, *On Schauder bases in Lipschitz-free spaces*, J. Math. Anal. Appl. **416** (2014), no. 2, 629–646.
- [20] N. J. Kalton, *Locally complemented subspaces and \mathcal{L}_p -spaces for $0 < p < 1$* , Math. Nachr. **115** (1984), 71–97.
- [21] ———, *Banach envelopes of nonlocally convex spaces*, Canad. J. Math. **38** (1986), no. 1, 65–86.
- [22] G. Lancien and E. Pernecká, *Approximation properties and Schauder decompositions in Lipschitz-free spaces*, J. Funct. Anal. **264** (2013), no. 10, 2323–2334.
- [23] J. Lindenstrauss, *Extension of compact operators*, Mem. Amer. Math. Soc. **48** (1964), 112.
- [24] A. Naor and G. Schechtman, *Planar earthmover is not in L_1* , SIAM J. Comput. **37** (2007), no. 3, 804–826.
- [25] E. Pernecká and R. J. Smith, *The metric approximation property and Lipschitz-free spaces over subsets of \mathbb{R}^N* , J. Approx. Theory **199** (2015), 29–44.

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