TILTING APPROXIMATIONS AND COTORSION PAIRS

PRINCIPAL POINTS OF THE DISSERTATION

Classical tilting theory generalizes Morita theory of equivalence of module categories. The key property – existence of category equivalences between large full subcategories of the module categories – forces the representing tilting module to be finitely generated, [1].

However, some aspects of the classical theory can be extended to infinitely generated modules over arbitrary rings. The main part of the Dissertation deals with such an aspect: the relation of tilting to approximations (preenvelopes and precovers) of modules, [3], [4], [5]. As an application, connections between tilting theory of infinitely generated modules and the finitistic dimension conjectures are presented in [6] and [7].

General existence theorems provide a big supply of approximations in the category Mod-R of all modules over an arbitrary ring R. However, the corresponding approximations may not be available in the subcategory of all finitely generated modules. So the usual sharp distinction between finitely and infinitely generated modules becomes unnatural, and even misleading.

Cotorsion pairs give a convenient tool for the study of module approximations. Tilting cotorsion pairs are defined as the cotorsion pairs induced by tilting modules. In [5] and [33], their characterization among all cotorsion pairs was given. This has recently been applied to a classification of tilting classes in particular cases - e.g., over Prüfer and Dedekind domains [45]. The point of the classification is that in the particular cases, the tilting classes are of finite type in the sense of [8]. This means that we can replace the single infinitely generated tilting module by a set of finitely presented modules; the tilting class is then axiomatizable in the language of the first order theory of modules.

In the following, we will present the main results of the Dissertation in the context of recent developments in the area. Our presentation will be based on a survey written by the author for the Handbook of Tilting Theory, [30]. To make our account more self-contained, we will give complete definitions and statements of the main results. Of course, proofs will be omitted; for full details, we will refer to the corresponding parts of the Dissertation, and to the recent papers and preprints listed in the references.

In §1, we will introduce cotorsion pairs and their relations to approximation theory of infinitely generated modules over arbitrary rings. In §2 and §3, we will discuss infinitely generated tilting and cotilting modules, and characterize the induced tilting and cotilting cotorsion pairs. §4 will deal with tilting classes of finite type and cotilting classes of cofinite type, and with their classification over particular rings. Finally, §5 will relate tilting approximations to the first and second finitistic dimension conjectures.

We start by fixing our notation. For an (associative, unital) ring R, Mod-R denotes the category of all (right R-) modules. mod-R denotes the subcategory of Mod-R formed by all modules possessing a projective resolution consisting of finitely generated modules.

Let $\mathcal{C}$ be a class of modules. For a cardinal κ, we denote by $\mathcal{C}^{<\kappa}$, and $\mathcal{C}^{\leq\kappa}$, the subclass of $\mathcal{C}$ consisting of the modules possessing a projective resolution containing only $<\kappa$-generated, and $\leq\kappa$-generated, modules, respectively. Further, $\varinjlim \mathcal{C}$ denotes the class of all modules that are direct limits of modules from $\mathcal{C}$.

Let $n < \omega$. We denote by $\mathcal{P}_n$ ($\mathcal{I}_n$, $\mathcal{F}_n$) the class of all modules of projective (injective, flat) dimension $\leq n$. Further, $\mathcal{P}$ ($\mathcal{I}$, $\mathcal{F}$) denotes the class of all modules of finite projective (injective, flat) dimension. The injective hull of a module M is denoted by $E(M)$.

We denote by $\mathbb{Z}$ the ring of all integers, and by $\mathbb{Q}$ the field of all rational numbers. For a commutative domain R, Q denotes the quotient field of R.

For a left R-module N, we denote by $N^* = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})$ the *character module* of N. Note that N^* is a (right R-) module.

Let M be a module. Then M is a *dual module* provided that $M = N^*$ for a left R-module N. M is *pure-injective* provided that M is a direct summand in a dual module. M is (Enochs) *cotorsion* provided that $\mathrm{Ext}^1_R(M, F) = 0$ for each $F \in \mathcal{F}_0$. Notice that any dual module is pure-injective, and any pure-injective module is cotorsion The class of all pure-injective, and cotorsion, modules is denoted by $\mathcal{PI}$, and $\mathcal{EC}$, respectively.

A module M is *divisible* if $\mathrm{Ext}^1_R(R/rR, M) = 0$ for each $r \in R$, and *torsion-free* if $\mathrm{Tor}_1^R(M, R/Rr) = 0$ for each $r \in R$. The class of all divisible and torsion-free modules is denoted by $\mathcal{DI}$ and $\mathcal{TF}$, respectively.

1. Cotorsion pairs and approximations

Cotorsion pairs are analogs of (non-hereditary) torsion pairs, with Hom replaced by Ext. They were introduced by Salce (under the name "cotorsion theories") in [83]. The analogy with the well-known torsion pairs makes it possible to derive easily some basic notions and facts about cotorsion pairs. However, the main point concerning cotorsion pairs is their close relation to special approximations of modules: cotorsion pairs provide a homological tie between the dual notions of a special preenvelope and a special precover. This tie (discovered in [83], cf. 1.8.3) is a sort of remedy for the non-existence of a duality in Mod-R.

Before introducing cotorsion pairs, we define various Ext-orthogonal classes.

Let $\mathcal{C} \subseteq$ Mod-R. Define $\mathcal{C}^\perp = \bigcap_{n<\omega} \mathcal{C}^{\perp_n}$ where $\mathcal{C}^{\perp_n} = \{M \in \text{Mod-}R \mid \mathrm{Ext}^n_R(C, M) = 0 \text{ for all } C \in \mathcal{C}\}$ for each $n < \omega$. Dually, let ${}^\perp\mathcal{C} = \bigcap_{n<\omega} {}^{\perp_n}\mathcal{C}$ where ${}^{\perp_n}\mathcal{C} = \{M \in \text{Mod-}R \mid \mathrm{Ext}^n_R(M, C) = 0 \text{ for all } C \in \mathcal{C}\}$ for each $n < \omega$.

1.1. Cotorsion pairs. Let R be a ring. A *cotorsion pair* is a pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ of classes of modules such that $\mathcal{A} = {}^{\perp_1}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp_1}$. The class $\mathcal{A} \cap \mathcal{B}$ is called the *kernel* of $\mathfrak{C}$. The cotorsion pair $\mathfrak{C}$ is *hereditary* provided that $\mathrm{Ext}^i_R(A, B) = 0$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $i \geq 2$.

Each module M in the kernel of a cotorsion pair $\mathfrak{C}$ is a *splitter*, that is, M satisfies $\mathrm{Ext}^1_R(M, M) = 0$. We will see that the kernel of $\mathfrak{C}$ in the tilting and cotilting cases plays an important role: it determines completely the classes $\mathcal{A}$ and $\mathcal{B}$. (This contrasts with what happens for torsion pairs: since $\mathrm{id}_M \in \mathrm{Hom}_R(M, M)$ for each module M, the "kernel" of any torsion pair is trivial.)

1.2. By changing the category, we could take a complementary point of view, working modulo the kernel rather than stressing its role. By a result of Beligiannis and Reiten [49], each complete hereditary cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ in Mod-R determines a torsion pair, $(\underline{\mathcal{A}}, \underline{\mathcal{B}})$, in the stable module category $\underline{\text{Mod-}R}$ (of Mod-R

modulo the kernel of $\mathfrak{C}$), cf. 1.8.3. Consequently, special $\mathcal{A}$-precovers and special $\mathcal{B}$-preenvelopes are functorial modulo maps factoring through the kernel, cf. [76].

The class of all cotorsion pairs is partially ordered by inclusion in the first component: $(\mathcal{A},\mathcal{B}) \leq (\mathcal{A}',\mathcal{B}')$ iff $\mathcal{A} \subseteq \mathcal{A}'$. The $\leq$-least cotorsion pair is $(\mathcal{P}_0, \text{Mod-}R)$, the $\leq$-greatest $(\text{Mod-}R, \mathcal{I}_0)$; these are the *trivial* cotorsion pairs.

The cotorsion pairs over a ring R form a complete lattice, $\mathfrak{L}_R$: given a sequence of cotorsion pairs $\mathcal{S} = ((\mathcal{A}_i,\mathcal{B}_i) \mid i \in I)$, the infimum of $\mathcal{S}$ in $\mathfrak{L}_R$ is $(\bigcap_{i\in I}\mathcal{A}_i, (\bigcap_{i\in I}\mathcal{A}_i)^{\perp_1})$, the supremum being $({}^{\perp_1}(\bigcap_{i\in I}\mathcal{B}_i), \bigcap_{i\in I}\mathcal{B}_i)$.

For any class of modules $\mathcal{C}$, there are two cotorsion pairs associated with $\mathcal{C}$: $({}^{\perp_1}\mathcal{C}, ({}^{\perp_1}\mathcal{C})^{\perp_1})$, called the cotorsion pair *generated* by $\mathcal{C}$, and $({}^{\perp_1}(\mathcal{C}^{\perp_1}), \mathcal{C}^{\perp_1})$, the cotorsion pair *cogenerated* by $\mathcal{C}$. If $\mathcal{C}$ has a representative set of elements $\mathcal{S}$, then the first cotorsion pair is generated by the single module $\prod_{S\in\mathcal{S}} S$, while the second is cogenerated by the single module $\bigoplus_{S\in\mathcal{S}} S$.

The existence of cotorsion pairs generated and cogenerated by any class of modules indicates that $\mathfrak{L}_R$ is a large class in general.

For example, the condition of all cotorsion pairs being trivial is extremely restrictive: by [2] and [3], for a right hereditary ring R, this condition holds iff $R = S$ or $R = T$ or R is the ring direct sum $S \boxplus T$, where S is semisimple artinian and T is Morita equivalent to a 2×2-matrix ring over a skew-field. As another example, consider the case of $R = \mathbb{Z}$: by [67], any partially ordered set embeds in $\mathfrak{L}_\mathbb{Z}$; in particular, $\mathfrak{L}_\mathbb{Z}$ is a proper class.

1.3. Replacing Ext by Tor in 1.1, we can define a *Tor-torsion pair* as the pair $(\mathcal{A},\mathcal{B})$ where $\mathcal{A} = \{A \in \text{Mod-}R \mid \text{Tor}_1^R(A,B) = 0 \text{ for all } B \in \mathcal{B}\}$ and $\mathcal{B} = \{B \in R\text{-Mod} \mid \text{Tor}_1^R(A,B) = 0 \text{ for all } A \in \mathcal{A}\}$. Similarly to the case of cotorsion pairs, we can define Tor-torsion pairs generated (cogenerated) by a class of left (right) R-modules. Tor-torsion pairs over a ring R form a complete lattice; by 1.4.3 below, the cardinality of this lattice is $\leq 2^{2^\kappa}$ where $\kappa = \text{card}(R) + \aleph_0$.

The well-known Ext-Tor relations yield an embedding of the lattice of Tor-torsion pairs into $\mathfrak{L}_R$ as follows: a Tor-torsion pair $(\mathcal{A},\mathcal{B})$ is mapped to the cotorsion pair $(\mathcal{A}, \mathcal{A}^{\perp_1})$. The latter cotorsion pair is easily seen to be generated by the class $\{B^* \mid B \in \mathcal{B}\}$. In this way, Tor-torsion pairs are identified with particular cotorsion pairs generated by classes of pure-injective modules.

Most of the classes of modules defined above occur as first or second components of cotorsion pairs cogenerated by sets:

Lemma 1.4. [32], [4]. *Let R be a ring and $n < \omega$. Let $\kappa = card(R) + \aleph_0$.*

1. *$\mathfrak{C} = (\mathcal{P}_n, (\mathcal{P}_n)^\perp)$ is a hereditary cotorsion pair cogenerated by $\mathcal{P}_n^{\leq\kappa}$. If R is right noetherian then $\mathfrak{C}$ is cogenerated by $\mathcal{P}_n^{\leq\omega}$.*
2. *Let $\mathfrak{C} = (\mathcal{A},\mathcal{B})$ be a cotorsion pair generated by a class of pure-injective modules. Then $\mathfrak{C}$ is cogenerated by $\mathcal{A}^{\leq\kappa}$.*
3. *Let $(\mathcal{A},\mathcal{B})$ be a Tor-torsion pair. Then $(\mathcal{A},\mathcal{A}^{\perp_1})$ is a cotorsion pair cogenerated by $\mathcal{A}^{\leq\kappa}$, and generated by $\{B^* \mid B \in \mathcal{B}\}$. In particular, $(\mathcal{F}_n, (\mathcal{F}_n)^\perp)$ is a hereditary cotorsion pair cogenerated by $\mathcal{F}_n^{\leq\kappa}$.*
4. *$({}^\perp\mathcal{I}_n, \mathcal{I}_n)$ is a hereditary cotorsion pair cogenerated by $({}^\perp\mathcal{I}_n)^{\leq\lambda}$ where λ is the least infinite cardinal such that each right ideal of R is λ-generated.*
5. *Let R be a right noetherian ring. Then the cotorsion pair cogenerated by $\mathcal{I}_n$ is cogenerated by a set.*
6. *$({}^{\perp_1}\mathcal{DI}, \mathcal{DI})$ and $(\mathcal{TF}, \mathcal{TF}^{\perp_1})$ are cogenerated by sets of cardinality $\leq\kappa$.*

The key property of cotorsion pairs is their relation to approximations of modules. The connection is through the notion of a special approximation, [89]:

1.5. Special approximations. Let R be a ring, M a module and $\mathcal{C}$ a class of modules. An R-homomorphism $f : M \to C$ is a *special $\mathcal{C}$-preenvelope* of M provided that f induces a short exact sequence $0 \to M \overset{f}{\to} C \to D \to 0$ with $C \in \mathcal{C}$ and $D \in {}^{\perp_1}\mathcal{C}$. $\mathcal{C}$ is a *special preenveloping* class if each module $M \in$ Mod-R has a special $\mathcal{C}$-preenvelope.

Dually, an R-homomorphism $g : C \to M$ is a *special $\mathcal{C}$-precover* of M provided that g induces a short exact sequence $0 \to B \to C \overset{g}{\to} M \to 0$ with $C \in \mathcal{C}$ and $B \in \mathcal{C}^{\perp_1}$. $\mathcal{C}$ is a *special precovering* class if each module $M \in$ Mod-R has a special $\mathcal{C}$-precover.

The terminology of 1.5 comes from the fact that special preenvelopes and precovers are special instances of the following more general notions, [60], [89]:

1.6. Let R be a ring, M a module, and $\mathcal{C}$ a class of modules. An R-homomorphism $f : M \to C$ with $C \in \mathcal{C}$ is a *$\mathcal{C}$-preenvelope* of M provided that for each $C' \in \mathcal{C}$ and each R-homomorphism $f' : M \to C'$ there is an R-homomorphism $g : C \to C'$ such that $f' = gf$.

The $\mathcal{C}$-preenvelope f is a *$\mathcal{C}$-envelope* of M if f has the following minimality property: if g is an endomorphism of C such that $gf = f$ then g is an automorphism.

$\mathcal{C}$ is a *preenveloping (enveloping)* class provided that each module $M \in$ Mod-R has a $\mathcal{C}$-preenvelope (envelope).

The notions of a *$\mathcal{C}$-precover*, *$\mathcal{C}$-cover*, *precovering* class, and *covering* class are defined dually.

A preenvelope (precover) may be viewed as a kind of weak (co-) reflection [62]; however, we do not require the assignment $M \mapsto C$ ($C \mapsto M$) to be functorial or unique, cf. 1.2.

However, if a module M has a $\mathcal{C}$-envelope (cover) then the $\mathcal{C}$-envelope (cover) is easily seen to be uniquely determined up to isomorphism; morever the $\mathcal{C}$-envelope (cover) of M is isomorphic to a direct summand in any $\mathcal{C}$-preenvelope ($\mathcal{C}$-precover) of M, [89].

Classical examples of enveloping classes include $\mathcal{I}_0$ and $\mathcal{PI}$, see [56] and [88], and of covering classes, $\mathcal{P}_0$ in case R is a right perfect ring, and $\mathcal{TF}$ in case R is a domain, see [39] and [59].

1.7. The definitions above can be extended to the setting of an abitrary category $\mathcal{K}$ (in place of Mod-R) and its subcategory $\mathcal{C} \subseteq \mathcal{K}$. In the particular case when $\mathcal{K} =$ mod-R, we will say $\mathcal{C}$ is *covariantly finite* (*contravariantly finite*) provided that $\mathcal{C}$ is preenveloping (precovering) in mod-R, cf. [38].

The following classical lemma connects cotorsion pairs to approximations of modules:

Lemma 1.8. *Let R be a ring, M a module, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ a cotorsion pair.*

(1) [87] *Assume M has a $\mathcal{B}$-envelope f. Then f is a special $\mathcal{B}$-preenvelope. So if $\mathcal{B}$ is enveloping then $\mathcal{B}$ is special preenveloping.*
(2) [87] *Assume M has a $\mathcal{A}$-cover f. Then f is a special $\mathcal{A}$-precover. So if $\mathcal{A}$ is covering then $\mathcal{A}$ is special precovering.*
(3) [83] *$\mathcal{A}$ is special precovering iff $\mathcal{B}$ is special preenveloping. In this case $\mathfrak{C}$ is called a* complete *cotorsion pair.*

The next example shows that in 1.8.3, we cannot claim that $\mathcal{A}$ is a covering class iff $\mathcal{B}$ is an enveloping one (however, by 1.10 below, the equivalence holds in case $\mathcal{A}$ is closed under direct limits):

Example 1.9. [46], [47], [48] Let R be a commutative domain and $\mathfrak{C}$ be the cotorsion pair cogenerated by the quotient field Q. Matlis proved that $\mathfrak{C}$ is hereditary iff proj.dim$(Q) \leq 1$ (that is, R is a *Matlis domain*).

The class $\mathcal{B} = \{Q\}^{\perp_1}$ is the class of all *Matlis cotorsion* modules. Since $\mathcal{B} = (\text{Mod-}Q)^{\perp_1}$, $\mathcal{B}$ is an enveloping class, [89]. For example, the $\mathcal{B}$-envelope of a torsion-free reduced module M coincides with the R-completion of M, cf. [64].

On the other hand, $\mathcal{A}$ (called the class of all *strongly flat* modules) is a covering class iff all proper factor-rings of R are perfect. For example, if R is a Prüfer domain then $\mathcal{A}$ is a covering class iff R is a Dedekind domain.

Cotorsion pairs $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ such that $\mathcal{A}$ is a covering class and $\mathcal{B}$ is an enveloping class are called *perfect*. By 1.8, each perfect cotorsion pair is complete. There is an important sufficient condition for perfectness of complete cotorsion pairs due to Enochs:

Theorem 1.10. [60], [89] *Let R be a ring, M a module, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ a cotorsion pair. Assume that $\mathcal{A}$ is closed under direct limits.*

1. *If M has a $\mathcal{B}$-preenvelope then M has a $\mathcal{B}$-envelope.*
2. *If M has an $\mathcal{A}$-precover then M has an $\mathcal{A}$-cover.*

In particular, $\mathfrak{C}$ is perfect iff $\mathfrak{C}$ is complete iff $\mathcal{A}$ is covering iff $\mathcal{B}$ is enveloping.

Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a complete cotorsion pair. It is an open problem whether $\mathcal{A}$ is a covering class iff $\mathcal{A}$ is closed under direct limits (The 'if' part is true by 1.10). 1.9 shows that $\mathcal{B}$ may be enveloping even if $\mathcal{A}$ is not closed under direct limits.

1.11. Invariants of modules. Assume $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is a perfect cotorsion pair. Then often the modules in the kernel, $\mathcal{K}$, of $\mathfrak{C}$ can be classified up to isomorphism by cardinal invariants. There are two ways to extend this classification:

a) Any module $A \in \mathcal{A}$ determines – by an iteration of $\mathcal{B}$-envelopes (of A, of the cokernel of the $\mathcal{B}$-envelope of A, etc.) – a long exact sequence all of whose members (except for A) belong to $\mathcal{K}$. This sequence is called the *minimal $\mathcal{K}$-coresolution* of A. The sequence of the cardinal invariants of the modules from $\mathcal{K}$ occuring in the coresolution provides for an invariant of A. In this way, the structure theory of the modules in $\mathcal{K}$ is extended to a structure theory of the modules in $\mathcal{A}$.

b) Dually, any module $B \in \mathcal{B}$ determines – by an iteration of $\mathcal{A}$-covers – a long exact sequence all of whose members (except for B) belong to $\mathcal{K}$, the *minimal $\mathcal{K}$-resolution* of B. This yields a sequence of cardinal invariants for any module $B \in \mathcal{B}$.

For specific examples to a) and b), we consider the case when R is a commutative noetherian ring:

If $\mathfrak{C} = (\text{Mod-}R, \mathcal{I}_0)$, then $\mathcal{K} = \mathcal{I}_0$, and by the classical theory of Matlis, each $M \in \mathcal{K}$ is determined up to isomorphism by the multiplicities of indecomposable injectives $E(R/p)$ (p a prime ideal of R) occuring in an indecomposable decomposition of M. The cardinal invariants of arbitary modules (in $\mathcal{A} = \text{Mod-}R$) constructed in a) are called the *Bass numbers*. A formula for their computation goes back to Bass: the multiplicity of $E(R/p)$ in the i-th term of the minimal injective coresolution of a module N is $\mu_i(p, N) = \dim_{k(p)} \text{Ext}^i_{R_p}(k(p), N_p)$ where $k(p) = R_p/\text{Rad}(R_p)$, and R_p and N_p is the localization of R and N at p, respectively, cf. [77].

If $\mathfrak{C} = (\mathcal{F}_0, \mathcal{EC})$, then $\mathcal{K}$ consists of the flat pure-injective modules: these are described by the ranks of the completions, T_p, of free modules over localizations R_p (p a prime ideal of R) occuring in their decomposition, [60]. The construction b) yields a sequence of invariants for any cotorsion module N. These invariants are called the *dual Bass numbers*. A formula for their computation is due to Xu [89]:

the rank of T_p in the i-th term of the minimal flat resolution of N is $\pi_i(p, N) = \dim_{k(p)} \operatorname{Tor}_i^{R_p}(k(p), \operatorname{Hom}_R(R_p, N))$.

In view of 1.4, the following result — first proved in [3] – says that most cotorsion pairs are complete, hence provide for approximations of modules.

For a module M and a class of modules $\mathcal{C}$, a *$\mathcal{C}$-filtration* of M is an increasing sequence of submodules of M, $(M_\alpha \mid \alpha < \sigma)$, such that $M = \bigcup_{\alpha<\sigma} M_\alpha$, $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for all limit ordinals $\alpha < \sigma$, and $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathcal{C}$ for each $\alpha < \sigma$. A module possessing a $\mathcal{C}$-filtration is called *$\mathcal{C}$-filtered*.

Theorem 1.12. [3] *Let R be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ a cotorsion pair cogenerated by a set of modules $\mathcal{S}$. Then $\mathfrak{C}$ is complete, and $\mathcal{A}$ is the class of all direct summands of all $\mathcal{S} \cup \{R\}$-filtered modules.*

1.12 was applied by Enochs to prove the flat cover conjecture: each module has a flat cover and a cotorsion envelope, [50]. This was generalized in [4] as follows:

Theorem 1.13. [4] *Let R be a ring and $\mathfrak{C}$ be a cotorsion pair generated by a class of pure-injective modules. Then $\mathfrak{C}$ is perfect.*

The flat cover conjecture is the particular case of 1.13 for $\mathfrak{C}$ generated by $\mathcal{PI}$.

For Dedekind domains, we can extend 1.13 further:

Theorem 1.14. [4] *Let R be a Dedekind domain and $\mathfrak{C}$ be a cotorsion pair generated by a class of cotorsion modules. Then $\mathfrak{C}$ is perfect.*

Recently, it turned out that the possibility of extending 1.13 to larger classes of modules depends on the extension of set theory (ZFC) that we work in.

In the positive direction, Gödel's axiom of constructibility (V = L) is useful, or rather its combinatorial consequence called Jensen's diamond principle $\diamondsuit$, see [57]. The following result is proved in [4] by induction, applying $\diamondsuit$ in regular cardinals, and Shelah's Singular Compactness Theorem in the singular ones:

Theorem 1.15. [4] *Assume $\diamondsuit$. Let R be a right hereditary ring and $\mathfrak{C}$ a cotorsion pair generated by a set of modules. Then $\mathfrak{C}$ is complete.*

In the negative direction, Shelah's uniformization principle UP^+ is useful, cf. [57]. Like Gödel's axiom of constructibility, UP^+ is relatively consistent with ZFC + GCH, but UP^+ and $\diamondsuit$ are mutually inconsistent.

In [2], UP^+ was used, for any non-right perfect ring R, to construct particular free modules $G \subseteq F$ such that $M = F/G$ is a non-projective module satisfying $\operatorname{Ext}_R^1(M, N) = 0$ for each module N with $\operatorname{card}(N) < \lambda$.

The following is proved in [14] (cf. with 1.14):

Theorem 1.16. *Assume UP^+. Let R be a Dedekind domain with a countable spectrum, and $\mathfrak{C}$ a cotorsion pair generated by a set containing at least one non-cotorsion module. Then $\mathfrak{C}$ is not cogenerated by a set of modules.*

Let us note that in the particular case of $R = \mathbb{Z}$, there is a stronger recent result by Eklof and Shelah [58], using a much stronger version of UP^+ denoted by SUP (SUP is also relatively consistent with ZFC + GCH, cf. [58]):

Theorem 1.17. *Assume SUP. Denote by $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ the cotorsion pair generated by $\mathbb{Z}$. Then $\mathbb{Q}$ does not have an $\mathcal{A}$-precover; in particular, $\mathfrak{C}$ is not complete.*

The class $\mathcal{A}$ in 1.17 is the well-known class of all *Whitehead groups*.

2. Tilting modules and approximations

In this section, we will deal with relations between tilting and approximation theory of modules. As observed in [5] and [33], for this purpose, it is convenient to work with a rather general definition of a tilting module. Our definition will allow for infinitely generated modules, and also modules of finite projective dimension > 1.

2.1. Tilting modules. Let R be a ring. A module T is *tilting* provided that

(1) $\text{proj.dim}(T) < \infty$;
(2) $\text{Ext}^i_R(T, T^{(\kappa)}) = 0$ for any cardinal κ and any $i \geq 1$;
(3) There are $k < \omega$, $T_i \in \text{Add}(T)$ $(i \leq k)$, and an exact sequence

$$0 \to R \to T_0 \to T_1 \to \dots \to T_k \to 0.$$

Here, $\text{Add}(T)$ denotes the class of all direct summands of arbitrary direct sums of copies of the module T.

Let $n < \omega$. Tilting modules of projective dimension $\leq n$ are called *n-tilting*. A class of modules $\mathcal{C}$ is *n-tilting* if there is an n-tilting module T such that $\mathcal{C} = \{T\}^\perp$.

A cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is *n-tilting* provided that $\mathcal{B}$ is an n-tilting class.

Notice that the notions above do not change when replacing the tilting module T by the tilting module $T^{(\kappa)}$ $(\kappa > 1)$. It is convenient to define an equivalence of tilting modules as follows: T is *equivalent* to T' provided that the induced tilting classes coincide: $\{T\}^\perp = \{T'\}^\perp$

Clearly, 0-tilting modules coincide with the projective generators. Finite dimensional tilting modules over artin algebras have been studied in great detail - see [35], [79] and [86] for much more on this classical case. We will now give three examples of infinitely generated 1-tilting modules:

2.2. The Fuchs tilting module. [64], [65] Let R be a commutative domain, and S a multiplicative subset of R. Let $\delta_S = F/G$ where F is the free module with the basis given by all sequences $(s_0, \dots, s_n)$ where $n \geq 0$, and $s_i \in S$ for all $i \leq n$, and the empty sequence $w = ()$. The submodule G is generated by the elements of the form $(s_0, \dots, s_n)s_n - (s_0, \dots, s_{n-1})$ where $0 < n$ and $s_i \in S$ for all $1 \leq i \leq n$, and of the form $(s)s - w$ where $s \in S$.

The module $\delta = \delta_{R \setminus \{0\}}$ was introduced by Fuchs. Facchini [61] proved that δ is a 1-tilting module. The general case of δ_S comes from [65]: the module δ_S is a 1-tilting module, called the *Fuchs tilting module*. The corresponding 1-tilting class is $\{\delta_S\}^\perp = \{M \in \text{Mod-}R \mid Ms = M \text{ for all } s \in S\}$, the class of all *$S$-divisible modules*. If R is a Prüfer domain or a Matlis domain, then the 1-tilting cotorsion pair cogenerated by δ is $(\mathcal{P}_1, \mathcal{DI})$.

Example 2.3. [8] Let R be a commutative 1-Gorenstein ring. Let P_0 and P_1 denote the set of all prime ideals of height 0 and 1, respectively. By a classical result of Bass, the minimal injective coresolution of R has the form

$$0 \to R \to \bigoplus_{q \in P_0} E(R/q) \xrightarrow{\pi} \bigoplus_{p \in P_1} E(R/p) \to 0.$$

Consider a subset $P \subseteq P_1$. Put $R_P = \pi^{-1}(\bigoplus_{p \in P} E(R/p))$. Then $T_P = R_P \oplus \bigoplus_{p \in P} E(R/p)$ is a 1-tilting module, the corresponding 1-tilting class being $\{T_P\}^\perp = \{M \mid \text{Ext}^1_R(E(R/p), M) = 0 \text{ for all } p \in P\}$. In particular, if R is a Dedekind domain then $\{T_P\}^\perp = \{M \mid \text{Ext}^1_R(R/p, M) = 0 \text{ for all } p \in P\} = \{M \mid pM = M \text{ for all } p \in P\}$.

In his classical work [81], Ringel discovered analogies between modules over Dedekind domains and tame hereditary algebras. The analogies extend to the setting of infinite dimensional tilting modules:

2.4. The Ringel tilting module. [81], [82] Let R be a tame hereditary algebra over an algebraically closed field k. Let G be the generic module. Then $S = \mathrm{End}(G)$ is a skew-field and $\dim_S G = n < \omega$. Denote by $\mathcal{T}$ the set of all tubes. If $\alpha \in \mathcal{T}$ is a homogenous tube, we denote by R_α the corresponding Prüfer module. If $\alpha \in \mathcal{T}$ is not homogenous, denote by R_α the direct sum of all Prüfer modules corresponding to the rays in α. Then there is an exact sequence

$$0 \to R \to G^{(n)} \xrightarrow{\pi} \bigoplus_{\alpha \in \mathcal{T}} R_\alpha^{(\lambda_\alpha)} \to 0$$

where $\lambda_\alpha > 0$ for all $\alpha \in \mathcal{T}$.

Let $P \subseteq \mathcal{T}$. Put $R_P = \pi^{-1}(\bigoplus_{\alpha \in P} R_\alpha^{(\lambda_\alpha)})$. Then $T_P = R_P \oplus \bigoplus_{\alpha \in P} R_\alpha$ is a 1-tilting module, called the *Ringel tilting module*. The corresponding 1-tilting class is the class of all modules M such that $\mathrm{Ext}^1_R(N, M) = 0$ for all (simple) regular modules $N \in P$.

Now, we will consider a simple example of an infinitely generated n-tilting module. In §5, we will see that this example is related to the validity of the first finitistic dimension conjecture for Iwanaga-Gorenstein rings.

A ring R is called *Iwanaga-Gorenstein* provided that R is left and right noetherian and the left and right injective dimensions of the regular module are finite, [60]. In this case, $\mathrm{inj.dim}(R_R) = \mathrm{inj.dim}({}_R R) = n$ for some $n < \omega$, and R is called *n-Gorenstein*. Notice that 0-Gorenstein rings coincide with the quasi-Frobenius rings.

Example 2.5. Let R be an n-Gorenstein ring. Let

$$0 \to R \to E_0 \to \cdots \to E_n \to 0$$

be the minimal injective coresolution of R. Then $T = \bigoplus_{i \le n} E_i$ is an n-tilting module. The only non-trivial fact needed for this is that $\mathcal{P} = \mathcal{P}_n = \mathcal{I}_n = \mathcal{I}$ $(= \mathcal{F}_n = \mathcal{F})$ for any n-Gorenstein ring, cf. [60, §9].

For any tilting cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$, there is a close relation among the classes $\mathcal{A}$, $\mathcal{B}$, and the kernel of $\mathfrak{C}$:

Lemma 2.6. [5], [33] *Let R be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ a tilting cotorsion pair. Let T be an n-tilting module with $\{T\}^\perp = \mathcal{B}$. Then*

(1) *$\mathfrak{C}$ is hereditary and complete. Moreover, $\mathfrak{C} \le (\mathcal{P}_n, \mathcal{P}_n^\perp)$, and the kernel of $\mathfrak{C}$ equals Add(T).*
(2) *$\mathcal{A}$ coincides with the class of all modules M such that there is an exact sequence*

$$0 \to M \to T_0 \to \cdots \to T_n \to 0$$

where $T_i \in$ Add(T) for all $i \le n$.
(3) *Let $0 \to F_n \to \cdots \to F_0 \to T \to 0$ be a free resolution of T and let $\mathcal{S} = \{S_i \mid i \le n\}$ be the corresponding set of syzygies of T. Then $\mathcal{A}$ coincides with the class of all direct summands of all $\mathcal{S}$-filtered modules.*
(4) *$\mathcal{B}$ coincides with the class of all modules N such that there is a long exact sequence*

$$\cdots \to T_{i+1} \to T_i \to \cdots \to T_0 \to N \to 0$$

where $T_i \in Add(T)$ for all $i < \omega$. In particular, $\mathcal{B}$ is closed under arbitrary direct sums.

We arrive at the characterization of tilting cotorsion pairs in terms of approximations. The case of $n = 1$ was treated in [5]:

Theorem 2.7. [5] *Let R be a ring.*

(1) *A class of modules $\mathcal{C}$ is 1-tilting iff $\mathcal{C}$ is a special preenveloping torsion class.*
(2) *Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Then $\mathfrak{C}$ is 1-tilting iff $\mathfrak{C}$ is complete, $\mathfrak{C} \leq (\mathcal{P}_1, \mathcal{P}_1^{\perp})$, and $\mathcal{B}$ is closed under arbitrary direct sums.*

We stress that the special approximations induced by 1-tilting modules may not have minimal versions in general (compare this with 3.5.1 below). For example, if R is a domain and δ is the Fuchs tilting module from 2.2 then the special $\{\delta\}^{\perp}$-preenvelopes coincide with the special divisible preenvelopes (and also with the special FP-injective preenvelopes). However, if R is a Prüfer domain of global dimension ≥ 2, then the regular module R does not have a divisible envelope (and so it does not have an FP-injective envelope), see [27].

The characterization in the general case is due to Angeleri Hügel and Coelho [33]:

Theorem 2.8. [33] *Let R be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Then $\mathfrak{C}$ is n-tilting iff $\mathfrak{C}$ is hereditary and complete, $\mathfrak{C} \leq (\mathcal{P}_n, \mathcal{P}_n^{\perp})$, and $\mathcal{B}$ is closed under arbitrary direct sums.*

3. Cotilting modules and approximations

In this section, we will consider the dual case of cotilting modules and cotilting cotorsion pairs.

Similarly as tilting modules, the cotilting ones have first appeared in the representation theory of finite dimensional k-algebras. There, the finite dimensional cotilting modules coincide with the k-duals of the finite dimensional tilting modules, so the theory is obtained by applying the k-duality.

1-cotilting modules over general rings are closely related to dualities (see [54] for more details). Also, in §4, we will see that restricting to tilting modules and classes of finite type, we actually have an explicit homological duality available producing the corresponding cotilting modules and classes of cofinite type.

However, there is no explicit duality available in the general case. The problem is that the dual of the key approximation construction of 1.12 does not work in ZFC: by 1.17, there is an extension of ZGC + GCH with a cotorsion pair $\mathfrak{C}$ generated by a set such that $\mathfrak{C}$ is not complete.

Fortunately, there is a remedy. First, for $n = 1$, a fundamental recent result of Bazzoni says that 1-cotilting modules are pure-injective (see 3.3 below), so we can apply 1.13 directly. As shown in [33], for $n > 1$, the classical work of Auslander and Buchweitz [36] makes it possible to overcome the problem.

3.1. Cotilting modules. Let R be a ring. A module C is *cotilting* provided that

(1) $\text{inj.dim}(C) < \infty$;
(2) $\text{Ext}^i_R(C^\kappa, C) = 0$ for any cardinal κ and any $i \geq 1$;

(3) There are $k < \omega$, $C_i \in \mathrm{Prod}(C)$ $(i \leq k)$, and an exact sequence

$$0 \to C_n \to \dots \to C_0 \to W \to 0,$$

where W is an injective cogenerator for Mod-R, and $\mathrm{Prod}(C)$ denotes the class of all direct summands of arbitrary direct products of copies of the module C.

Let $n < \omega$. Cotilting modules of injective dimension $\leq n$ are called *n-cotilting.* A class of modules $\mathcal{C}$ is *n-cotilting* if there is an n-cotilting module C such that $\mathcal{C} = {}^{\perp}\{C\}$. A cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is *n-cotilting* provided that $\mathcal{A}$ is an n-cotilting class.

The equivalence of cotilting modules is defined as follows: C is *equivalent* to C' provided that the induced cotilting classes coincide: ${}^{\perp}\{C\} = {}^{\perp}\{C'\}$.

0-cotilting modules coincide with the injective cogenerators. In 4.11 below, we will see that any resolving subclass of $\mathcal{P}_n^{<\omega}$ yields an n-cotilting class (of left R-modules), so there is a big supply of n-cotilting modules for $n \geq 1$ in general.

A class $\mathcal{C}$ of modules is *definable* provided that $\mathcal{C}$ is closed under arbitrary direct products, direct limits, and pure submodules, [55]. (Definability implies axiomatizability: definable classes are axiomatized by equality to 1 of certain of the Baur-Garavaglia-Monk invariants, cf. [78].)

It is an open problem whether each cotilting module is pure-injective. There is a criterion of pure-injectivity of cotilting modules, [42]:

Lemma 3.2. *Let R be a ring and C a cotilting module. Then C is pure-injective iff ${}^{\perp}\{C\}$ is closed under direct limits iff ${}^{\perp}\{C\}$ is closed under pure submodules iff ${}^{\perp}\{C\}$ is definable.*

The criterion is satisfied for $n = 1$:

Theorem 3.3. [40] *Let R be a ring and C a 1-cotilting module. Then C is pure-injective. In particular, ${}^{\perp}\{C\}$ is a definable class.*

Now, we turn to relations between cotilting modules and approximations. Except for part 3., the dual of 2.6 holds true − a proof in case $n = 1$ appears in [5]; the general case makes use of [36], and is proved in [33]. (In view of 3.3, one can proceed more directly for $n = 1$, by dualizing the proof of 2.6 with help of 1.13):

Lemma 3.4. *Let R be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotilting cotorsion pair. Let C be an n-cotilting module with ${}^{\perp}\{C\} = \mathcal{A}$. Then*

(1) *$\mathfrak{C}$ is hereditary and complete. Moreover, $({}^{\perp}\mathcal{I}_n, \mathcal{I}_n) \leq \mathfrak{C}$, and the kernel of $\mathfrak{C}$ equals Prod(C).*

(2) *$\mathcal{A}$ coincides with the class of all modules M such that there is a long exact sequence*

$$0 \to M \to C_0 \to \dots \to C_i \to C_{i+1} \to \dots$$

where $C_i \in$ Prod(C) for all $i < \omega$. In particular, $\mathcal{A}$ is closed under arbitrary direct products.

(3) *$\mathcal{B}$ coincides with the class of all modules N such that there is an exact sequence*

$$0 \to C_n \to \dots \to C_0 \to N \to 0$$

where $C_i \in$ Prod(C) for all $i \leq n$.

Theorem 3.5. *Let R be a ring.*

(1) [5] *A class of modules* $\mathcal{C}$ *is* 1*-cotilting iff* $\mathcal{C}$ *is a covering torsion-free class.*
(2) [33] *Let* $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ *be a cotorsion pair. Then* $\mathfrak{C}$ *is* n*-cotilting iff* $\mathfrak{C}$ *is hereditary and complete,* $({}^{\perp}\mathcal{I}_n, \mathcal{I}_n) \leq \mathfrak{C}$*, and* $\mathcal{A}$ *is closed under arbitrary direct products.*

In particular, 1-cotilting classes coincide with those torsion-free classes $\mathcal{C}$ that are covering. If R is right noetherian, then $\mathcal{C}$ is completely determined by its subclass $\mathcal{C} \cap \text{mod-}R$, and the latter is characterized as a torsion-free class in mod-R containing R. More precisely, we have

Theorem 3.6. [51] *Let* R *be a right noetherian ring. There is a bijective correspondence between* 1*-cotilting classes of modules,* $\mathcal{C}$*, and torsion-free classes,* $\mathcal{E}$*, in mod-*R *containing* R*. The correspondence is given by the mutually inverse assignments* $\mathcal{C} \mapsto \mathcal{C} \cap \text{mod-}R$ *and* $\mathcal{E} \mapsto \varinjlim \mathcal{E}$*.*

4. Finite and cofinite type

The duality between the notions of a tilting and cotilting module can be made explicit in case the modules are of finite and cofinite type, respectively. These notions were introduced and studied in [8] and [45], as a continuation of [7] and [27].

4.1. Tilting modules of finite type. Let R be a ring.

(1) Let $\mathcal{C}$ be a class of modules. Then $\mathcal{C}$ is of *finite type* (*countable type*) provided there exist $n < \omega$ and a subset $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$ ($\mathcal{S} \subseteq \mathcal{P}_n^{\leq\omega}$) such that $\mathcal{C} = \mathcal{S}^{\perp}$.
(2) Let T be a tilting module. Then T is of *finite type* (*countable type*, *definable*) provided the class $\{T\}^{\perp}$ is of finite type (countable type, definable).

Lemma 4.2. [8] *Let* R *be a ring and* $\mathcal{C}$ *be a class of modules of finite type. Then* $\mathcal{C}$ *is tilting and definable.*

4.2 says that there is a rich supply of tilting classes in general: any subset $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$ (for some $n < \omega$) determines one (a more precise description appears in 4.11 below).

There is a criterion for tilting modules to be of finite type:

Lemma 4.3. [8] *Let* R *be a ring and* T *be a tilting module. Let* $\mathcal{B} = \{T\}^{\perp}$*, and* $(\mathcal{A}, \mathcal{B})$ *be the corresponding tilting cotorsion pair. Then* T *is of finite type iff* T *is definable and* $T \in \varinjlim \mathcal{A}^{<\omega}$*.*

The last condition of 4.3 is always satisfied for $n = 1$:

Lemma 4.4. [30] *Let* R *be a ring and* M *be a module of projective dimension* ≤ 1*. Let* $(\mathcal{A}, \mathcal{B})$ *be the cotorsion pair cogenerated by* M*. Then* $M \in \varinjlim \mathcal{A}^{<\omega}$*.*

For 1-tilting modules, 4.3 and 4.4 yield

Theorem 4.5. [45] *Let* R *be a ring and* T *be a* 1*-tilting module. Then* T *is definable iff* T *is of finite type iff* $\{T\}^{\perp}$ *is closed under pure submodules.*

It is open whether all 1-tilting modules are of finite type. However, they are always of countable type:

Theorem 4.6. [45] *Let* R *be a ring and* T *be a* 1*-tilting module. Then* T *is of countable type.*

The proof of 4.6 uses set-theoretic methods developed by Eklof, Fuchs and Shelah for the structure theory of so called Baer modules [57]. However, 4.6 holds in ZFC. In this sense, 4.6 says that the structure of 1-tilting modules, and classes, is purely an algebraic problem, depending only on the structure of countably and finitely presented modules rather than additional set-theoretic assumptions.

The counterpart of a tilting (right R-) module of finite type is a cotilting left R-module of cofinite type:

4.7. Cotilting modules of cofinite type. Let R be a ring. Let $\mathcal{C} \subseteq R$-Mod. Then $\mathfrak{C}$ is of *cofinite type* provided that there exist $n < \omega$ and a subset $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$ such that $\mathcal{C} = \mathcal{S}^{\intercal}$, where $\mathcal{S}^{\intercal} = \{M \in R\text{-Mod} \mid \operatorname{Tor}_i^R(S, M) = 0$ for all $S \in \mathcal{S}$ and all $0 < i \leq n\}$.

Let C be a cotilting left R-module. Then C is of *cofinite type* provided that the (cotilting) class ${}^{\perp}\{C\}$ is of cofinite type.

Applying 3.5, we can dualize 4.2:

Lemma 4.8. *Let R be a ring and $\mathcal{C}$ be a class of left R-modules of cofinite type. Then $\mathcal{C}$ is cotilting and definable.*

4.4 yields a characterization of 1-cotilting classes of cofinite type:

Lemma 4.9. [30] *Let R be a ring and $\mathcal{C}$ be a class of left R-modules. Then $\mathcal{C}$ is 1-cotilting of cofinite type iff there is a module $M \in \mathcal{P}_1$ such that $\mathcal{C} = \{M\}^{\intercal}$.*

Since classes of cofinite type are closed under direct limits, any cotilting module of cofinite type is pure-injective by 3.2.

The bijective correspondence between tilting classes of finite type and cotilting classes of cofinite type is mediated by resolving subclasses of mod-R:

Definition 4.10. Let R be a ring and $\mathcal{S} \subseteq$ mod-R. Then $\mathcal{S}$ is *resolving* provided that $\mathcal{P}_0^{<\omega} \subseteq \mathcal{S}$, $\mathcal{S}$ is closed under direct summands and extensions, and $\mathcal{S}$ is closed under kernels of epimorphisms.

A subclass $\mathcal{S} \subseteq \mathcal{P}_1^{<\omega}$ is resolving iff $\mathcal{S}$ is closed under extensions and direct summands, and $R \in \mathcal{S}$.

Theorem 4.11. [8] *Let R be a ring and $n < \omega$. There is a bijective correspondence among*

- *n-tilting classes of finite type,*
- *resolving subclasses of $\mathcal{P}_n^{<\omega}$,*
- *n-cotilting classes of cofinite type in R-Mod.*

Moreover, if T is an n-tilting module of finite type then T^* is an n-cotilting left R-module of cofinite type; in the correspondence of 4.11, the n-tilting class $\{T\}^{\perp}$ corresponds to the n-cotilting class ${}^{\perp}\{T^*\} = \{T\}^{\intercal}$, cf. [8].

There is a partial converse of 4.8:

Theorem 4.12. [30] *Let R be a left noetherian ring. Assume that $\mathcal{F}_1 = \mathcal{P}_1$ (this holds when R is (i) right perfect or (ii) right hereditary or (iii) 1-Gorenstein, for example). Then every 1-cotilting left R-module is of cofinite type.*

4.12 applies to the left artinian case:

4.13. 1-cotilting classes over left artinian rings. [30] Let R be a left artinian ring. Then 1-cotilting classes of left R-modules are of cofinite type, hence coincide with the classes of the form $\{M \in R\text{-Mod} \mid \operatorname{Tor}_1^R(S, M) = 0$ for all $S \in \mathcal{S}\}$ for some $\mathcal{S} \subseteq \mathcal{P}_1^{<\omega}$. Moreover, by 4.11, these classes correspond bijectively to the classes $\mathcal{S}'$ which are closed under extensions and direct summands, and satisfy $\mathcal{P}_0^{<\omega} \subseteq \mathcal{S}' \subseteq \mathcal{P}_1^{<\omega}$.

In general, the converse of 4.8 does not hold: there exist Prüfer domains with 1-cotilting modules that are not of cofinite type. We are going to discuss the recent results on the Prüfer and Dedekind domain cases in detail - these results extend [4]:

4.14. Tilting classes over Prüfer domains. [45], [84], [44] Let R be a Prüfer domain. Then all tilting modules have projective dimension ≤ 1, and they are of finite type. Moreover, for each 1-tilting class, $\mathcal{T}$, there is a set, $\mathcal{E}$, of non-zero finitely generated (projective) ideals of R such that $\mathcal{T}$ consists of all modules M satisfying $IM = M$ for all $I \in \mathcal{E}$ (or, equivalently, $\mathrm{Ext}^1_R(R/I, M) = 0$ for all $I \in \mathcal{E}$). This is proved in [45] and [44], using 4.6.

It follows that tilting classes correspond bijectively to finitely generated localizing systems of R in the sense of [63, §5.1]. Here, a filter, $\mathcal{I}$, of non-zero ideals of R is a *finitely generated localizing system* of R provided that $\mathcal{I}$ contains a basis consisting of finitely generated ideals, and $\mathcal{I}$ is multiplicatively closed.

Given such system $\mathcal{I}$, the corresponding tilting class consists of all modules M satisfying $IM = M$ for all $I \in \mathcal{I}$, cf. [84]. Note that by [63, 5.1], finitely generated localizing systems of R correspond bijectively to the overrings of R.

By 4.11, cotilting classes of cofinite type coincide with the classes of the form $\{M \mid \mathrm{Tor}^R_1(M, R/I) = 0 \text{ for all } I \in \mathcal{I}\}$ where $\mathcal{I}$ is a finitely generated localizing system of R.

However, by [43], there exist maximal valuation domains R such that the class of all Whitehead modules is 1-cotilting, but not of cofinite type.

A complete description is available for Dedekind domains:

4.15. Tilting and cotilting modules over Dedekind domains. [4], [45] Let R be a Dedekind domain. By 2.3, for each set of maximal ideals, P, there is a tilting module $T_P = R_P \oplus \bigoplus_{p \in P} E(R/p)$ with the corresponding tilting class $\{T_P\}^\perp = \{M \mid pM = M \text{ for all } p \in P\}$. Since localizing systems of ideals of R are determined by their prime ideals, by 4.14, any tilting module T is equivalent to T_P for a set of maximal ideals P, cf. [45]. (In the particular case when $R = \mathbb{Z}$, and R is a small Dedekind domain, this result was proved assuming V = L in [17] and [31], respectively).

By 4.11, cotilting classes of cofinite type are exactly the classes of the form $\mathcal{C}_P = \{M \mid \mathrm{Tor}^R_1(M, R/p) = 0 \text{ for all } p \in P\}$ for a set, P, of maximal ideals of R. Moreover, $\mathcal{C}_P = {}^\perp\{C_P\}$ where $C_P = \prod_{p \in P} J_p \oplus \bigoplus_{q \in \mathrm{Spec}(R) \setminus P} E(R/q)$ is a cotilting module. (Here, J_p denotes the completion of the localization of R at p).

By 4.12, all cotilting classes are of the form $\mathcal{C}_P$, and all cotilting modules are equivalent to the modules of the form C_P, for a set, P, of maximal ideals of R, cf. [4].

The analogy between modules over Dedekind domains and over tame hereditary algebras (cf. 2.3 and 2.4) extends to the tilting and cotilting setting, cf. [51] and [52].

5. Tilting approximations and the finitistic dimension conjectures

Let R be a ring and $\mathcal{C}$ be a class of modules. The *$\mathcal{C}$-dimension* of R is defined as the supremum of projective dimensions of all modules in $\mathcal{C}$.

If $\mathcal{C} = \text{Mod-}R$ then the $\mathcal{C}$-dimension is called the (right) *global dimension* of R; if $\mathcal{C} = \mathcal{P}$, it is called the *big finitistic dimension* of R. If $\mathcal{C}$ is the class of all finitely generated modules in $\mathcal{P}$ then the $\mathcal{C}$-dimension is called the *little finitistic dimension* of R. These dimensions are denoted by gl.dim(R), Fin.dim(R), and fin.dim(R), respectively.

Clearly, fin.dim$(R) \leq$ Fin.dim$(R) \leq$ gl.dim(R) for any ring R. Moreover, if R has finite global dimension, then gl.dim(R) is attained on cyclic modules, so all the three dimensions coincide.

If R has infinite global dimension, then the finitistic dimensions take the role of the global dimension to provide a fine measure of complexity of the module category. For example, if $R = \mathbb{Z}_{p^n}$ for a prime integer p and $n > 1$, then R has infinite global dimension, but both finitistic dimensions are 0; they certainly reflect better the fact that R is of finite representation type.

In [39], Bass considered the following assertions

(I) fin.dim$(R) =$ Fin.dim(R)

(II) fin.dim(R) is finite

and proposed to investigate the validity of these assertions in dependence on the structure of the ring R. Later, (I) and (II) became known as the *first*, and the *second*, *finitistic dimension conjecture*, respectively.

In the case when R is commutative noetherian, Bass, Raynaud and Gruson proved that Fin.dim(R) coincides with the Krull dimension of R, so classical examples of Nagata can be used to provide counter-examples to the assertion (II). In case R is commutative local noetherian, Auslander and Buchweitz proved that fin.dim(R) coincides with the depth of R, so (I) holds iff R is a Cohen-Macaulay ring.

Assume that R is right artinian. Then the validity of (II) is still an open problem. However, Huisgen-Zimmermann proved that (I) need not hold even for monomial finite dimensional algebras, [69]. Smalø then constructed, for any $1 < n < \omega$, examples of finite dimensional algebras such that fin.dim$(R) = 1$ and Fin.dim$(R) = n$, [85].

There are many positive results available: (II) was proved for all monomial algebras in [68], for algebras of representation dimension ≤ 3 in [74] etc.

Moreover, (I) and (II) were proved for all algebras such that $\mathcal{P}^{<\omega}$ is contravariantly finite in [37] and [72]. In this section, we will use tilting approximations to give a simple proof of the latter result. Then we will prove (I) for all Iwanaga-Gorenstein rings.

In the rest of this section, R will be a right noetherian ring. We will denote by $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ the cotorsion pair cogenerated by the class $\mathcal{P}^{<\omega}$. By 1.12, $\mathfrak{C}$ is complete and hereditary; moreover, $\mathcal{P}^{<\omega} = \mathcal{A} \cap \text{mod-}R$.

The basic relation between tilting approximations and the finitistic dimension conjectures comes from [7]:

Theorem 5.1. [7] *Let R be a right noetherian ring. Then (II) holds iff $\mathfrak{C}$ is a tilting cotorsion pair. Moreover, if T is a tilting module such that $\{T\}^{\perp} = \mathcal{B}$, then fin.dim$(R)$ = proj.dim(T).*

5.2. The tilting module T in 5.1 is unique up to equivalence, and it is clearly of finite type. In principle, T can be constructed as in the proof of 2.8: that is, by an iteration of special $\mathcal{B}$-preenvelopes of R etc. yielding an Add(T)-coresolution of R, $0 \to R \to T_0 \to \cdots \to T_n \to 0$, and giving $T = \bigoplus_{i \leq n} T_i$. However, little is known of the (definable) class $\mathcal{B}$ in general, so this construction is of limited use. (The construction works fine for gl.dim$(R) < \infty$. Then $\mathcal{B} = \mathcal{I}_0$, so the Add$(T)$-coresolution above can be taken as the minimal injective coresolution of R.)

In the artinian case, we can compute fin.dim(R) using $\mathcal{A}$-approximations of all the (finitely many) simple modules. This is proved in [6], generalizing [37]:

Theorem 5.3. [6] *Let R be a right artinian ring and $\{S_0, \dots S_m\}$ be a representative set of all simple modules. For each $i \leq m$, take a special $\mathcal{A}$-preenvelope of S_i, $f_i : A_i \to S_i$. Then fin.dim$(R) = max_{i \leq m}$proj.dim(A_i).*

Moreover, all the modules A_i $(i \leq m)$ can be taken finitely generated iff $\mathcal{P}^{<\omega}$ is contravariantly finite. In this case (II) holds true, since $\mathcal{P}^{<\omega} = \mathcal{A} \cap$ mod-R.

Now, we will relate pure-injectivity properties of the tilting module T from 5.1 to closure properties of the class $\mathcal{A}$.

A module M is *pure-split* if all pure submodules of M are direct summands; M is $\sum$*-pure-split* iff all modules in Add(M) are pure-split. For example, any $\sum$-pure-injective module is $\sum$-pure-split, [71].

A module M is *product complete* if Prod$(M) \subseteq$ Add(M). Any product complete module is $\sum$-pure-injective, [75].

The following is proved in [7] and [9]:

Lemma 5.4. *Let R be a right noetherian ring satisfying (II). Let T be the tilting module from 5.1. Then*

(1) *T is $\sum$-pure-split iff $\mathcal{A}$ is closed under direct limits.*
(2) *T is product complete iff $\mathcal{A}$ is closed under products iff $\mathcal{A}$ is definable.*
(3) *$\mathcal{A} = \mathcal{P}$ iff Add(T) is closed under cokernels of monomorphisms.*

5.5. The condition $\mathcal{A} = \mathcal{P}$ implies (I), since any module of finite projective dimension is then a direct summand in a $\mathcal{P}^{<\omega}$-filtered module, by 1.12. In fact, the proof of the first finitistic dimension conjectures in 5.6 and 5.8 below is based on proving that $\mathcal{A} = \mathcal{P}$. However, (I) may hold even if $\mathcal{A} \subsetneq \mathcal{P}$, see [9].

Theorem 5.6. [7] *Let R be an artin algebra such that (II) holds. Let T be the tilting module from 5.1. Then T can be taken finitely generated iff $\mathcal{P}^{<\omega}$ is contravariantly finite. In this case, (I) holds.*

5.3 and 5.6 now give

Corollary 5.7. [37], [72] *Let R be an artin algebra such that $\mathcal{P}^{<\omega}$ is contravariantly finite. Then (I) and (II) hold for R.*

All right serial artin algebras satisfy the assumption of 5.7, see [70]. However, there are finite dimensional algebras R with fin.dim(R) = Fin.dim$(R) = 1$ such that $\mathcal{P}^{<\omega}$ is not contravariantly finite, for example the IST-algebra [73]; for those algebras, T is an infinitely generated 1-tilting module.

Finally, we turn to Iwanaga-Gorenstein rings (see 2.5). Let $n < \omega$ and R be n-Gorenstein. Then $\mathcal{P} = \mathcal{I} = \mathcal{P}_n = \mathcal{I}_n$. In particular, there exist cotorsion pairs $\mathfrak{D} = (\mathcal{P}, \mathcal{GI})$ and $\mathfrak{E} = (\mathcal{GP}, \mathcal{I})$. The modules in $\mathcal{GI}$ are called *Gorenstein injective*, the ones in $\mathcal{GP}$ *Gorenstein projective.* The kernel of $\mathfrak{D}$ equals $\mathcal{I}_0$, the kernel of $\mathfrak{E}$ is $\mathcal{P}_0$, cf. [60]. Clearly, Fin.dim$(R) = n$, so (II) holds.

By [8], also (I) holds:

Theorem 5.8. [8] *Let R be an Iwanaga-Gorenstein ring. Then (I) holds true. Moreover, the tilting module T from 5.1 can be taken of the form $T = \bigoplus_{i \leq n} I_i$ where $0 \to R \to I_0 \to \cdots \to I_n \to 0$ is the minimal injective coresolution of R.*

If R in 5.8 is an artin algebra, then T is finitely generated. So by 5.6, Iwanaga-Gorenstein artin algebras give yet another example of algebras with $\mathcal{P}^{<\omega}$ contravariantly finite, [37].

NOTE: The Dissertation consists of the papers (1)-(7) listed below. The references (8)-(31) concern related work of the author in the past decade, the remaining references refer to related works of other authors.

References

[1] J. Trlifaj, *Every $*$-module is finitely generated*, J. Algebra **169** (1994), 392-398.

[2] J. Trlifaj, *Whitehead test modules*, Trans. Amer. Math. Soc. **348** (1996), 1521-1554.

[3] P. C. Eklof and J. Trlifaj, *How to make Ext vanish*, Bull. London Math. Soc. **33** (2001), 31–41.

[4] P. C. Eklof and J. Trlifaj, *Covers induced by Ext*, J. Algebra **231** (2000), 640–651.

[5] L. Angeleri Hügel, A. Tonolo and J. Trlifaj, *Tilting preenvelopes and cotilting precovers*, Algebras and Repres. Theory **4** (2001), 155-170.

[6] J. Trlifaj, *Approximations and the little finitistic dimension of artinian rings*, J. Algebra **246** (2001), 343-355.

[7] L. Angeleri Hügel and J. Trlifaj, *Tilting theory and the finitistic dimension conjectures*, Trans. Amer. Math. Soc. **354**(2002), 4345-4358.

[8] L. Angeleri Hügel, D. Herbera and J. Trlifaj, *Tilting modules and Gorenstein rings*, preprint.

[9] L. Angeleri Hügel and J. Trlifaj, *Direct limits of modules of finite projective dimension*, in "Rings, Modules, Algebras, and Abelian Groups", LNPAM 286, M. Dekker (2004).

[10] R.Colpi, A.Tonolo and J.Trlifaj, *Partial cotilting modules and the lattices induced by them*, Comm. Algebra **25** (1997), 3225–3237.

[11] R.Colpi and J.Trlifaj, *Classes of generalized $*$-modules*, Comm. Algebra **22**(1994), 3985-39.

[12] R.Colpi and J.Trlifaj, *Tilting modules and tilting torsion theories*, J. Algebra **178** (1995), 614–634.

[13] P.C.Eklof, K.R.Goodearl and J.Trlifaj, *Dually slender modules and steady rings*, Forum Math. **9** (1997), 61-74.

[14] P. C. Eklof, S. Shelah and J.Trlifaj, *On the cogeneration of cotorsion pairs*, to appear in J. Algebra (2004).

[15] P.C.Eklof and J.Trlifaj, *Gamma invariants for dense lattices*, Alg. Universalis **40** (1998), 427-445.

[16] L. Fuchs, L. Salce and J. Trlifaj, *On strongly flat modules over Matlis domains*, in "Rings, Modules, Algebras, and Abelian Groups", LNPAM 286, M. Dekker (2004).

[17] R. Göbel and J.Trlifaj, *Cotilting and a hierarchy of almost cotorsion groups*, J. Algebra **224** (2000), 110-122.

[18] R. Göbel and J.Trlifaj, *Large indecomposable roots of Ext*, J. Pure Appl. Algebra **157** (2001), 241-246.

[19] S.Shelah and J.Trlifaj, *Spectra of the Γ-invariant of uniform modules*, J. Pure and Appl. Algebra **162** (2001), 367-379.

[20] J. Trlifaj, *Almost $*$-modules need not be finitely generated*, Comm. Algebra **21** (1993), 2453-2462.

[21] J. Trlifaj, *Similarities and differences between abelian groups and modules over non-perfect rings*, in Abelian Group Theory (Proc.Conf.Oberwolfach'93), Contemp. Math. **171** (1994), 397-406.

[22] J. Trlifaj, *Strong incompactness for some non-perfect rings*, Proc. Amer. Math. Soc. **123** (1995), 21-25.

[23] J. Trlifaj, *Two problems of Ziegler and uniform modules over regular rings*, in Abelian Groups and Modules (Proc.ICAGM'95), M.Dekker, New York 1996, 373-383.

[24] J. Trlifaj, *Modules over non-perfect rings*, in Advances in Algebra and Model Theory, Gordon & Breach, Philadelphia 1997, 471-492.

[25] J. Trlifaj, *Dimension estimates for representable equivalences of module categories*, J. Algebra **193** (1997), 660-676.

[26] J. Trlifaj, *Uniform modules, Γ-invariants and Ziegler spectra of regular rings*, in Infinite Abelian Groups and Modules (Proc.ICAGM'98), Trends in Mathematics, Birkhäuser, Basel 1999, 327-340.

[27] J. Trlifaj, *Cotorsion theories induced by tilting and cotilting modules*, Abelian Groups, Rings and Modules, Contemporary Math. **273** (2001), 285-300.

[28] J. Trlifaj, *Local splitters for bounded cotorsion theories*, Forum Math. **14** (2002), 315-324.

[29] J. Trlifaj, *Ext and inverse limits*, Baer Centenial Volume, Illinois J. Math. **47** (2003), 529-538.

[30] J. Trlifaj, *Infinite dimensional tilting modules and cotorsion pairs*, in Handbook of Tilting Theory, Camb. Univ. Press, Cambridge (to appear).

[31] J. Trlifaj and S.L. Wallutis, *Tilting modules over small Dedekind domains*, J. Pure Appl. Algebra **172** (2002), 109-117. Corrigendum: **183** (2003), 329-331.

[32] S.T. Aldrich, E. Enochs, O. Jenda and L. Oyonarte, *Envelopes and covers by modules of finite injective and projective dimensions*, J. Algebra **242**(2001), 447-459.

[33] L. Angeleri Hügel and F. Coelho, *Infinitely generated tilting modules of finite projective dimension*, Forum Math **13** (2001), 239-250.

[34] L. Angeleri Hügel and F. Coelho, *Infinitely generated complements to partial tilting modules*, Math. Proc. Camb. Phil. Soc. **132**(2002), 89-96.

[35] L.Angeleri Hügel, D.Happel and H.Krause, *Introduction*, in Handbook of Tilting Theory, Camb. Univ. Press, Cambridge (to appear).

[36] M. Auslander and R. Buchweitz, *Homological Theory of Cohen-Macaulay Approximations*, Mem.Soc.Math. de France **38**, (1989), 5–37.

[37] M. Auslander and I. Reiten, *Applications of contravariantly finite subcategories*, Adv.Math. **86**, (1991), 111–152.

[38] M. Auslander and S. O. Smalø, *Preprojective modules over artin algebras*, J. Algebra **66** (1980), 61-122.

[39] H. Bass, *Finitistic dimension and a homological generalization of semiprimary rings*, Trans. Amer. Math. Soc. **95** (1960), 466-488.

[40] S. Bazzoni, *Cotilting modules are pure-injective*, Proc. Amer. Math. Soc. **131**(2003), 3665-3672.

[41] S. Bazzoni, *n-cotilting and n-tilting modules*, to appear in J. Algebra.

[42] S. Bazzoni, *n-cotilting modules and pure-injectivity*, to appear in Bull. London Math. Soc.

[43] S. Bazzoni, *An example of a* 1*-cotilting module which is not of cofinite type*, manuscript.

[44] S. Bazzoni, *Tilting modules over commutative domains*, manuscript.

[45] S. Bazzoni, P. C. Eklof and J. Trlifaj, *Tilting cotorsion pairs*, preprint.

[46] S. Bazzoni and L. Salce, *Strongly flat covers*, J. London Math. Soc. **66**(2002), 276-294.

[47] S. Bazzoni and L. Salce, *On strongly flat modules over integral domains*, to appear in Rocky Mountain J. Math.

[48] S. Bazzoni and L. Salce, *Almost perfect domains*, Colloq. Math. **95**(2003), 285-301.

[49] A. Beligiannis and I. Reiten, *Homological aspects of torsion theories*, preprint.

[50] L. Bican, R. El Bashir, and E. Enochs, *All modules have flat covers*, Bull. London Math. Soc. **33**(2001), 385–390.

[51] A.B. Buan and H. Krause, *Cotilting modules over tame hereditary algebras*, preprint.

[52] A.B. Buan and H. Krause, *Tilting and cotilting for quivers of type $\tilde{A}_n$*, preprint.

[53] R.Colpi, G.D'Este and A.Tonolo, *Quasi-tilting modules and counter equivalences*, J.Algebra **191** (1997), 461-494.

[54] R. Colpi and K.R. Fuller, *Cotilting dualities*, in Handbook of Tilting Theory, Camb. Univ. Press, Cambridge (to appear).

[55] W. Crawley-Boevey, *Infinite dimensional modules in the representation theory of finite dimensional algebras*, CMS Conf. Proc. **23** (1998), 29–54.

[56] B.Eckmann and A.Schopf, *Über injective Moduln*, Arch. Math. (Basel) **4** (1953), 75–78.

[57] P. C. Eklof and A. H. Mekler, **Almost Free Modules**, Revised edition, North-Holland (2002).

[58] P. C. Eklof and S. Shelah, *On the existence of precovers*, Illinois J. Math. **47** (2003), 173-188.

[59] E. Enochs, *Torsion free covering modules*, Proc. Amer. Math. Soc. **14** (1963), 884-889.

[60] E. Enochs and O. Jenda, **Relative Homological Algebra**, de Gruyter (2000).

[61] A. Facchini, *A tilting module over commutative integral domains*, Comm. Algebra **15** (1987), 2235-2250.

[62] C. Faith, **Algebra: Rings, Modules and Categories I**, Springer, New York 1973.

[63] M. Fontana, J.A. Huckaba, and I.J. Papick, **Prüfer domains**, M.Dekker (1997).

[64] L. Fuchs and L.Salce, **Modules over Non-Noetherian Domains**, AMS (2001).

[65] L. Fuchs and L.Salce, *S-divisible modules over domains*, Forum Math. **4**(1992), 383-394.

[66] R. Göbel and S. Shelah, *Cotorsion theories and splitters*, Trans. Amer. Math. Soc. **352**(2000), 5357–5379.

[67] R. Göbel, S. Shelah and S.L. Wallutis, *On the lattice of cotorsion theories*, J. Algebra **238** (2001), 292-313.

[68] L. L. Green, E. E. Kirkman, and J. J. Kuzmanovich, *Finitistic dimension of finite dimensional monomial algebras*, J. Algebra **136** (1991), 37–51.

[69] B. Huisgen Zimmermann, *Homological domino effects and the first finitistic dimension conjecture*, Invent. Math., **108** (1992), 369-383.

[70] B. Huisgen Zimmermann, *Syzigies and homological dimensions of left serial rings*, Methods in Module Theory, LNPAM **140**, M.Dekker, New York 1993, 161-174.

[71] B. Huisgen Zimmermann, *Purity, algebraic compactness, direct sum decompositions, and representation type*, Trends in Mathematics, Birkhäuser, Basel 2000, 331-368.

[72] B. Huisgen Zimmermann and S. Smalø, *A homological bridge between finite and infinite dimensional representations*, Algebras and Repres. Theory **4** (2001), 155-170.

[73] K. Igusa, S. O. Smalø and G. Todorov, *Finite projectivity and contravariant finiteness*, Proc. Amer. Math. Soc. **109** (1990), 937-941.

[74] K. Igusa and G. Todorov, *On the finitistic global dimension conjecture for artin algebras*, preprint.

[75] C. Jensen and H. Lenzing, **Model Theoretic Algebra**, ALA 2, Gordon & Breach, Amsterdam 1989.

[76] H. Krause and O. Solberg, *Applications of cotorsion pairs*, to appear in J. London Math. Soc.

[77] M. Matsumura, **Commutative Ring Theory**, 5th ed., CSAM **8**, Cambridge Univ. Press, Cambridge 1994.

[78] M. Prest, **Model Theory and Modules**, LMSLN **130**, Cambridge Univ. Press, Cambridge 1988.

[79] I. Reiten, *Tilting and homologically finite subcategories with applications to quasihereditary algebras*, in Handbook of Tilting Theory, Camb. Univ. Press, Cambridge (to appear).

[80] I. Reiten and C.M. Ringel, *Infinite dimensional representations of canonical algebras*, preprint.

[81] C.M. Ringel, *Infinite dimensional representations of finite dimensional hereditary algebras*, Symposia Math. **23** (1979), Academic Press, 321–412.

[82] C.M. Ringel, *Infinite length modules. Some examples as introduction*, Trends in Mathematics, Birkhäuser, Basel 2000, 1-74

[83] L. Salce, *Cotorsion theories for abelian groups*, Symposia Math. **23** (1979), Academic Press, 11–32.

[84] L. Salce, *Tilting modules over domains*, preprint.

[85] S. Smalø, *Homological differences between finite and infinite dimensional representations of algebras*, Trends in Mathematics, Birkhäuser, Basel 2000, 425-439.

[86] L. Unger, *The collection of tilting modules and its combinatorics*, in Handbook of Tilting Theory, Camb. Univ. Press, Cambridge (to appear).

[87] T. Wakamatsu, *Stable equivalence of self-injective algebras and a generalization of tilting modules*, J. Algebra **134** (1990), 298-325.

[88] R. Warfield, *Purity and algebraic compactness for modules*, Pacific J. Math. **28** (1969), 699–719.

[89] J. Xu, **Flat Covers of Modules**, Lecture Notes in Math. 1634, Springer-Verlag (1996).