QUASI-RANDOMNESS AND THE REGULARITY METHOD IN HYPERGRAPHS

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Emory University
Atlanta, GA

Outline

1. Ramsey Theorems for the Integers

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2. Graphs, Hypergraphs and its connection to Szemerédi's Theorem

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3. Regularity Method for Graphs

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4. Regularity Method for Hypergraphs and the Removal Lemma

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4. Regularity Method for Hypergraphs and the Removal Lemma
5. Extensions of the Removal Lemma

Partition Theorems for the Integers

Über die Kongruenz $x^m + y^m \equiv z^m \pmod{p}$.

Von I. Schur in Berlin.

Im folgenden will ich zeigen, daß der Dicksonsche Satz sich fast unmittelbar aus einem sehr einfachen Hilfssatz ergibt, der mehr der Kombinatorik als der Zahlentheorie angehört:

Hilfssatz. *Verteilt man die Zahlen 1, 2, ..., N irgendwie auf m Zeilen, so müssen, sobald $N > m!e$ wird, in mindestens einer Zeile zwei Zahlen vorkommen, deren Differenz in derselben Zeile enthalten ist.*[3])

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If $n > m!\mathrm{e}$, then any partition/coloring of $[n] = \{1, \ldots, n\}$ with m classes/colors yields one class containing a solution of $x + y = z$.

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BEWEIS EINER BAUDET'SCHEN VERMUTUNG

VON

BARTEL L. VAN DER WAERDEN

(in Hamburg).

BAUDET hat vermutet dass für jedes l gilt:

Behauptung 1 (l). Ist die unendliche Zahlenfolge 1, 2, 3, . . . in zwei fremde Klassen eingeteilt, so liegt in einer dieser Klassen eine arithmetische Progression von l Zahlen.

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Conjecture of Baudet (and Schur)

If the natural numbers are split into two classes, then one class contains arithmetic progressions with any given number of terms.

Partition Theorems for the Integers

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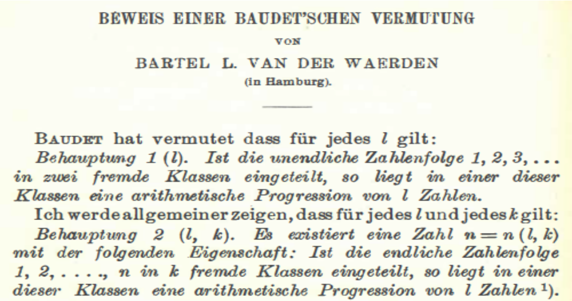
van der Waerden's Theorem 1927

For all integers $m \geq 2$ and $k \geq 3$ there exists an integer n such that any coloring of $[n]$ with m colors yields a monochromatic arithmetic progression with k terms.



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Which linear equations have such a partition-property?

Examples:

Schur:	x	$+$	y	$=$	z	✓	
van der Waerden:	x	$+$	y	$=$	$2z$	✓	(with $x \neq y$)

A General Perspective

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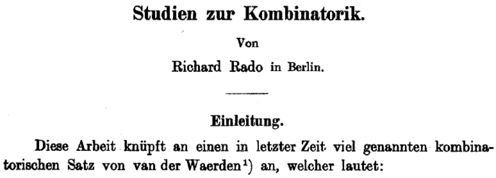
Rado's Theorem 1933

Characterization of (systems of) linear equations with the partition-property.



Studien zur Kombinatorik.

Von

Richard Rado in Berlin.

Einleitung.

Diese Arbeit knüpft an einen in letzter Zeit viel genannten kombinatorischen Satz von van der Waerden[1]) an, welcher lautet:

Density Versions

Partition Theorems:

$$\begin{array}{rccccccc} \text{Schur:} & x & + & y & = & z & \checkmark \\ \text{van der Waerden:} & x & + & y & = & 2z & \checkmark \end{array}$$

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- What is the size of the largest subset not containing a solution?

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- odd numbers contain no solution for Schur's equation
  $\Rightarrow$ there are sets of *density* 1/2 with no solution
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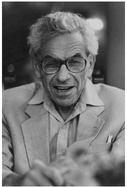
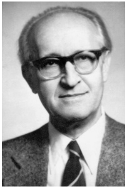


ON SOME SEQUENCES OF INTEGERS

PAUL ERDÖS *and* PAUL TURÁN*.

Consider a sequence of integers $a_1 < a_2 < \ldots \leqslant N$ containing no three terms for which $a_i - a_l = a_l - a_s$, *i.e.* a sequence containing no three consecutive members of an arithmetic progression. Such sequences we call A sequences belonging to N, or simply A sequences. We consider those with the maximum number of elements, and denote by $r = r(N)$



Question (Erdős & Turán, 1936)

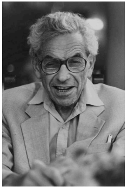
Set

$$r_k(n) = \max\{|A| : A \subseteq [n] \text{ containing no } k\text{-AP}\}.$$

Is it true that

$$\lim_{n\to\infty} \frac{r_k(n)}{n} = 0 \quad ?$$

Erdős-Turán Conjecture



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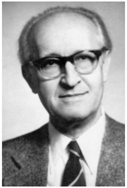

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Positive answer $\Longrightarrow$ van der Waerden's Theorem

Bounds for $r_k(n)$

Lower Bound

- Behrend (1946): $r_3(n) \geq \frac{n}{\exp(c\sqrt{\log n})}$
  in particular
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Szemerédi's Theorem 1975

For every integer $k \geq 3$ we have $r_k(n) = o(n)$.

Proofs of Szemerédi's Theorem

Different proofs of Szemerédi's Theorem have appeared:

- Combinatorics/Graph Theory (Szemerédi)
  → used van der Waerden's theorem
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We will discuss the hypergraph approach here.

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Remark

Other proofs appeared over the last decade:

- Elek and Szegedy using non-standard analysis

II. Graphs and hypergraphs

Definition

A k-uniform hypergraph (k-graph) $H^{(k)}$ on V is a pair (V, E), where

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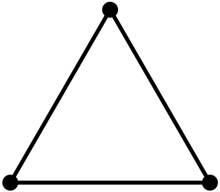
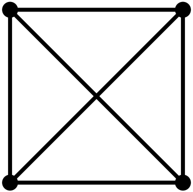


triangle = $K_3^{(2)}$



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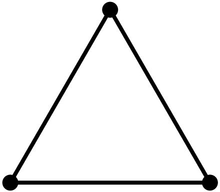

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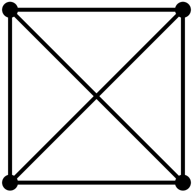

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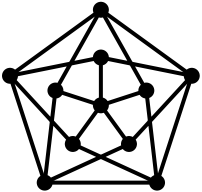

Grötzsch graph

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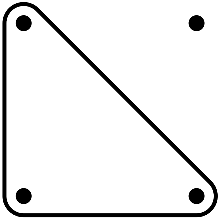

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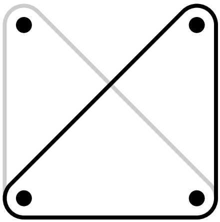

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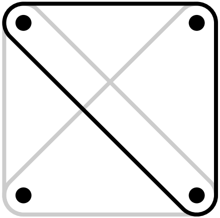

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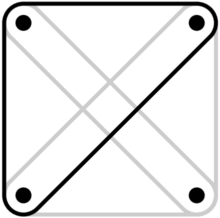

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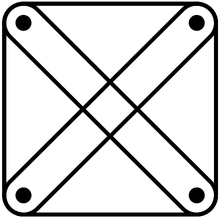

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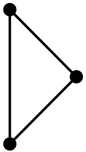

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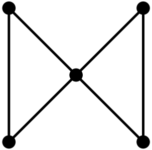

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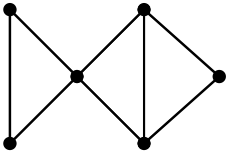

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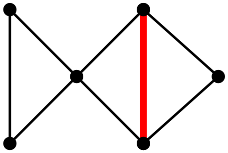

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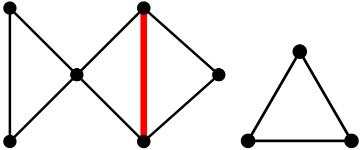

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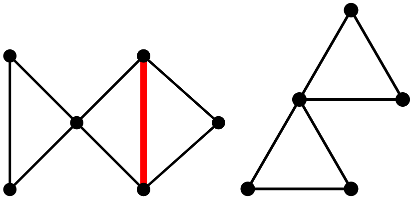

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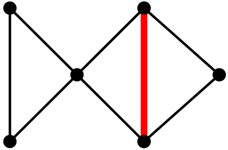

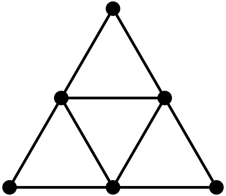

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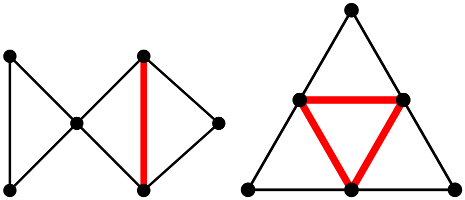

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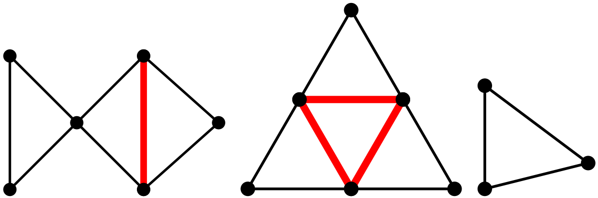

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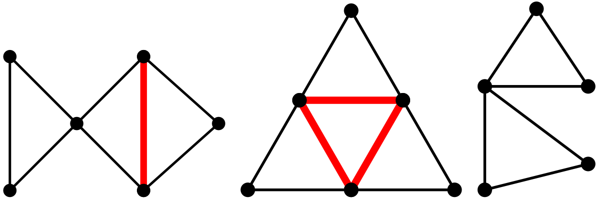

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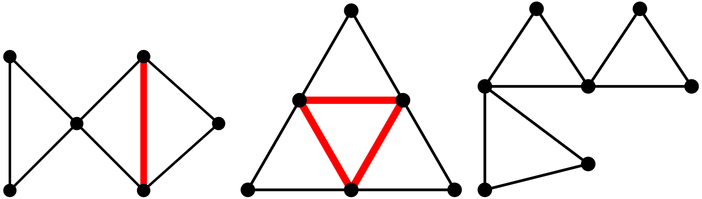

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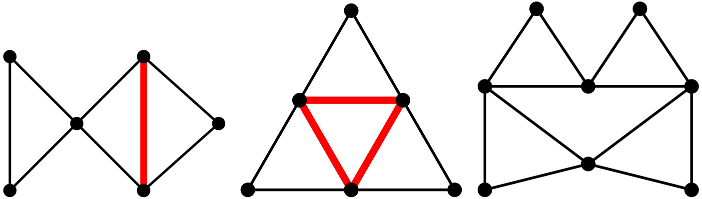

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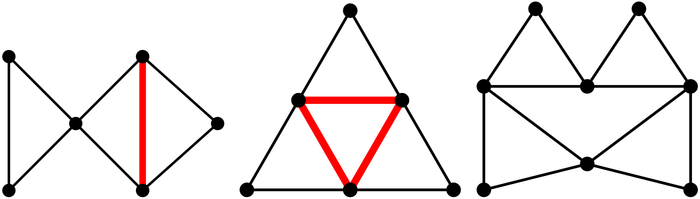

Question (Brown, Erdős & T. Sós, 1973)

How many edges can a simple triangle graph on n vertices have?

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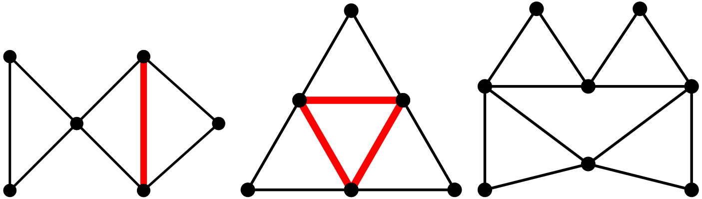

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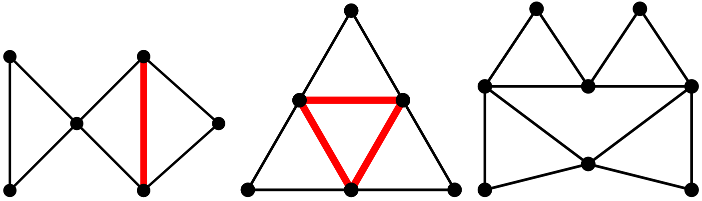

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Observation (Ruzsa & Szemerédi)

Theorem $\Longrightarrow$ Roth's Theorem ($r_3(n) = o(n)$)

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Consequently,

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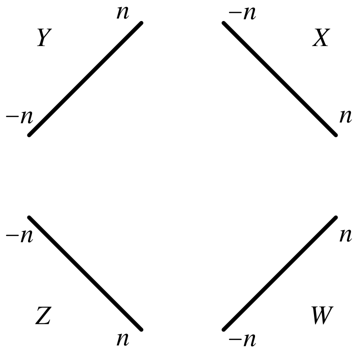

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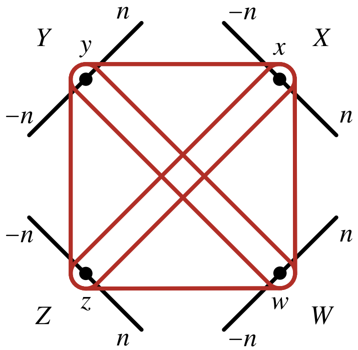

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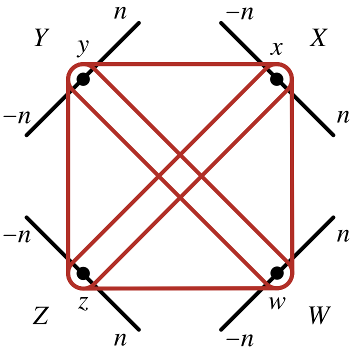

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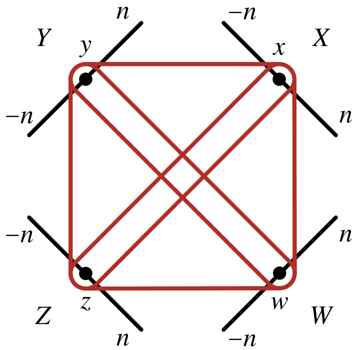

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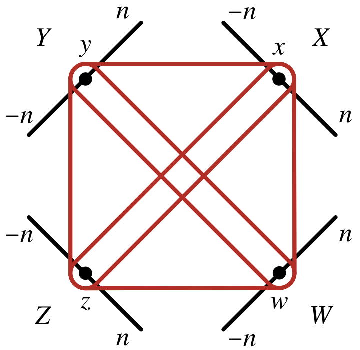

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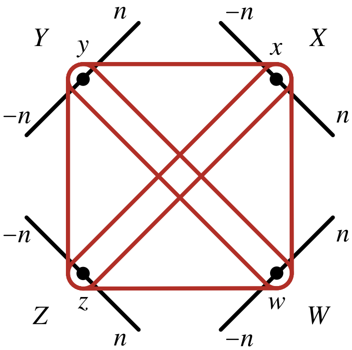

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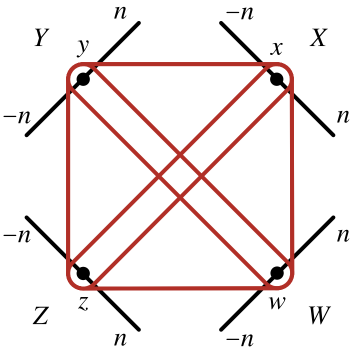

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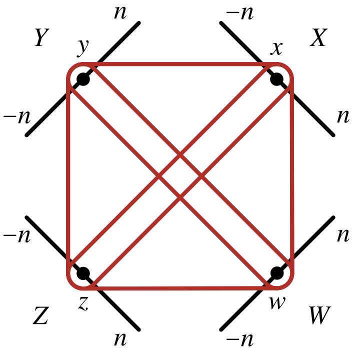

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$\Rightarrow H^{(3)}$ is a 3-uniform, simple clique hypergraph. □

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We have seen:

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Moreover:

Clique Union Lemma for hypergraphs

$\Longrightarrow$ Furstenberg-Katznelson theorem

A multidimensional version of Szemerédi's Theorem

Theorem (Furstenberg & Katznelson, 1978)

For every k and $d \in \mathbb{N}$ we have,

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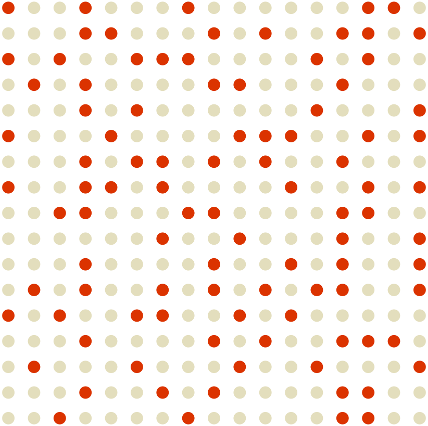

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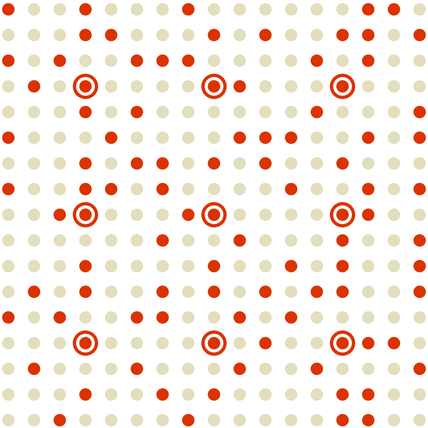

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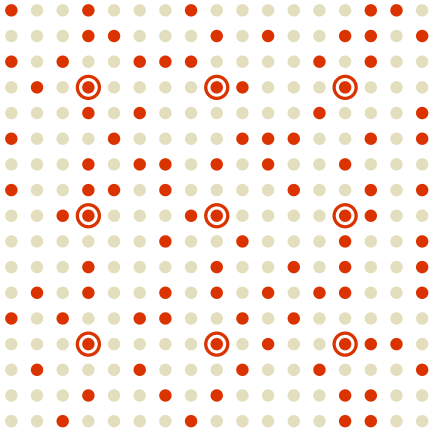

Solymosi showed:

Clique Union Lemma

$\Downarrow$

Furstenberg-Katznelson Theorem

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- Simple application yields Ruzsa-Szemerédi theorem (clique/triangle union lemma for graphs)

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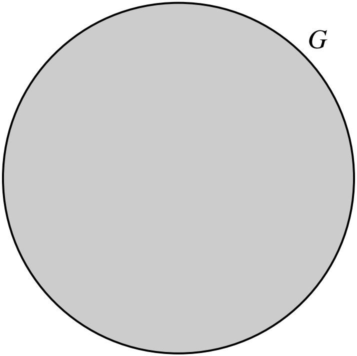
Every large graph G can be decomposed into "relatively few," mostly random-like (uniform edge distribution) bipartite subgraphs.



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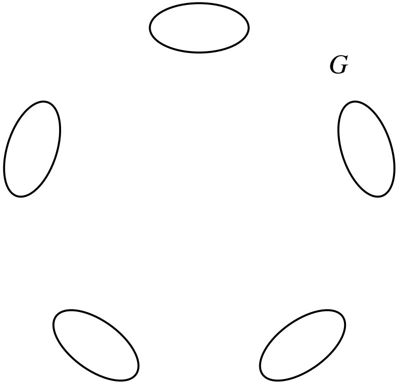

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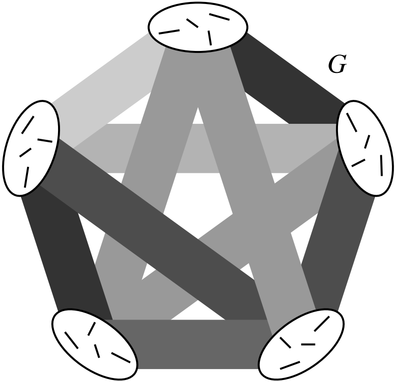

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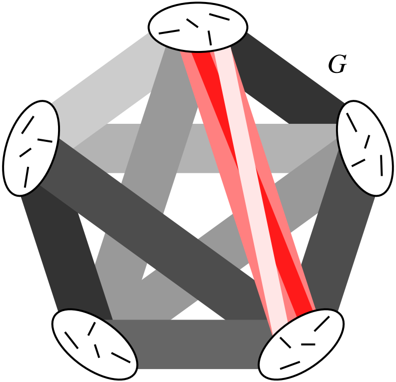
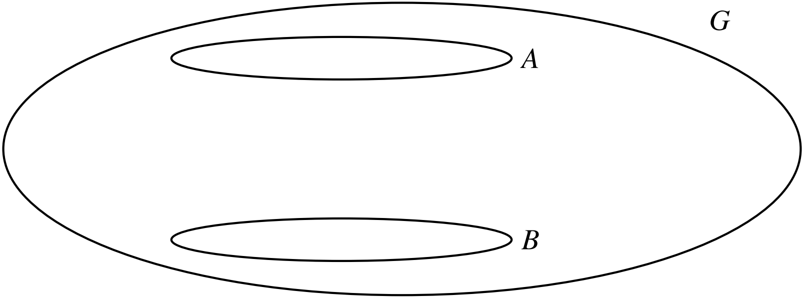


Graph Regularity Method – Basic Definitions

- graph $G = (V, E)$
- $A, B \subseteq V$, non-empty disjoint



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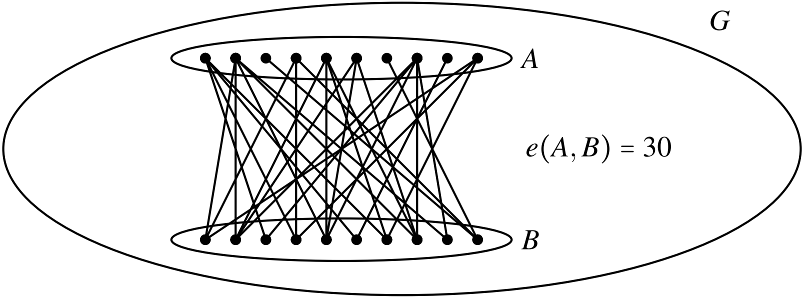

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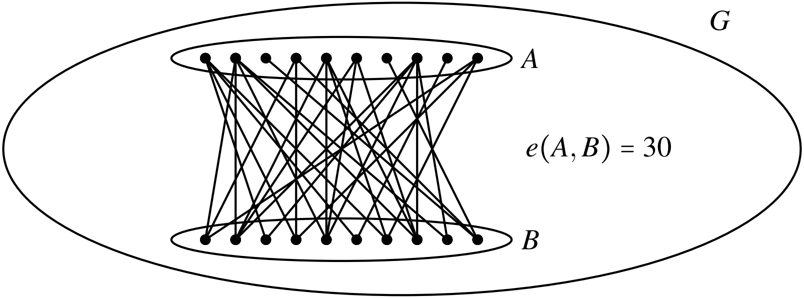

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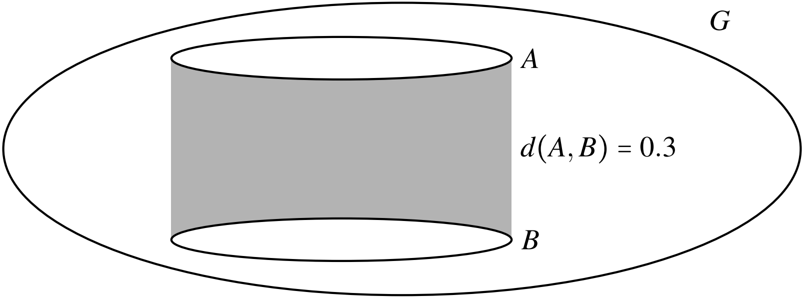

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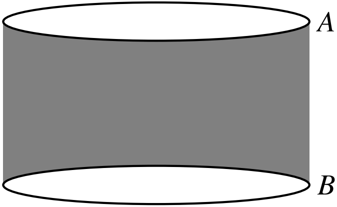

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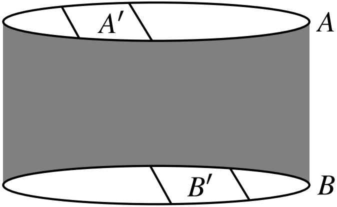

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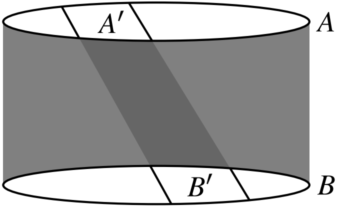

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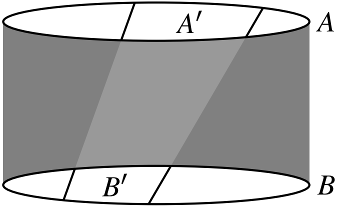

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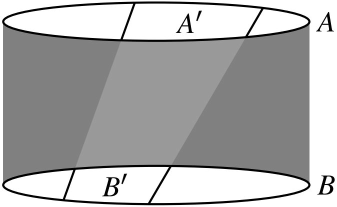

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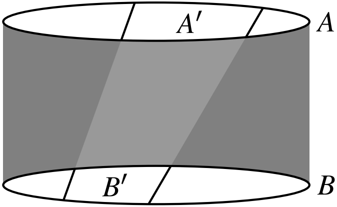

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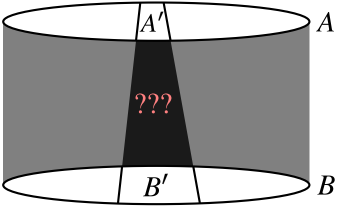

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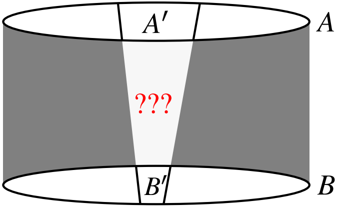

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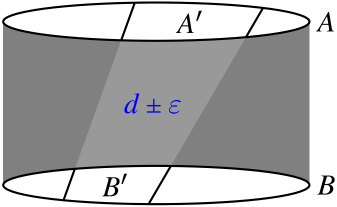
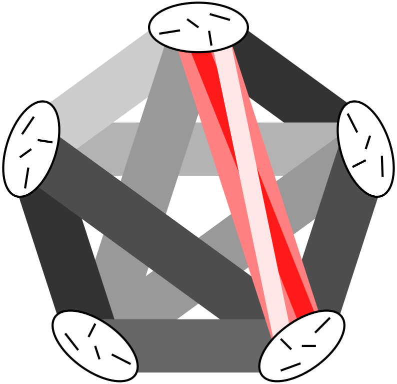
- sets threshold for $|A'|$ and $|B'|$
- measures uniformity of edge distribution



Regular Partition

Definition

A partition $V(G) = V_1 \cup \cdots \cup V_t$ is ε-regular if (V_i, V_j) are ε-regular for all but at most $\varepsilon\binom{t}{2}$ pairs i, j and $\big||V_i| - |V_j|\big| \leq 1$ for all pairs i, j.



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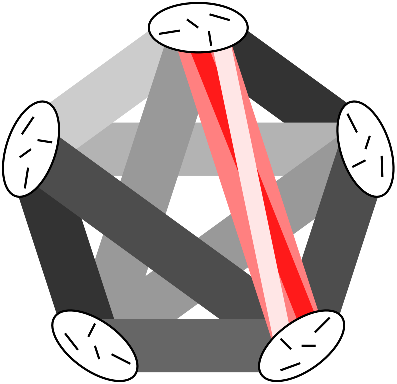
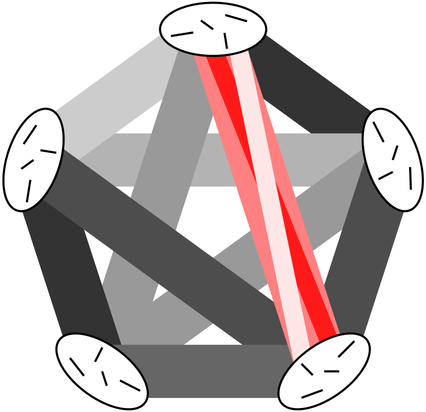


Regularity Lemma: Every "large" graph admits an ε-regular partition.

Regularity Lemma

Szemerédi's Regularity Lemma

$\forall \varepsilon > 0\ \exists T_0, N_0$ s.t. every graph G on $n \geq N_0$ vertices admits an ε-regular partition $V(G) = V_1 \cup \cdots \cup V_t$ with $1/\varepsilon \leq t \leq T_0$.



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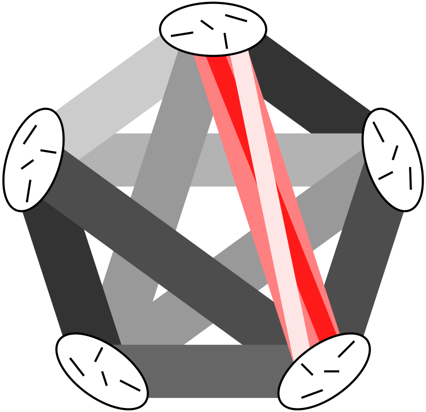
- Edges in irregular pairs:
  at most $\varepsilon n^2/2$.



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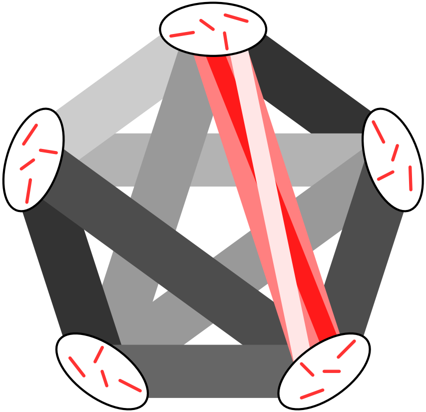

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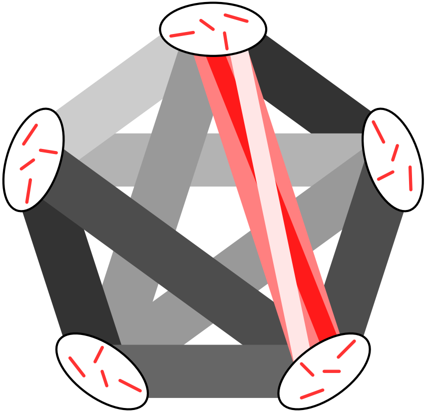
Number of "uncontrolled edges" is at most εn^2.

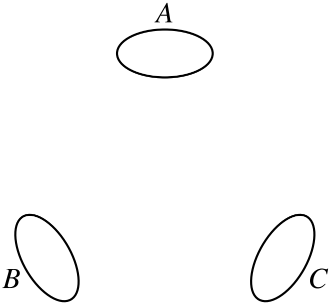


Triangle Counting Lemma

If A, B, $C \subseteq V$ are disjoint vertex sets such that each pair is ε-regular with density $\geq d$, then the number of triangles is at least

$$(1 - o(1))d^3|A||B||C|$$

where $o(1) \to 0$ as $\varepsilon \to 0$.



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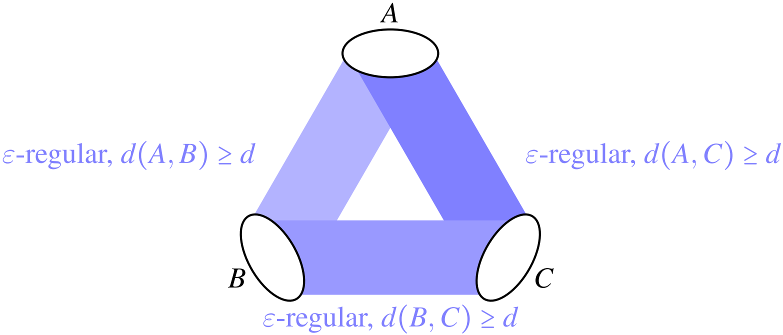

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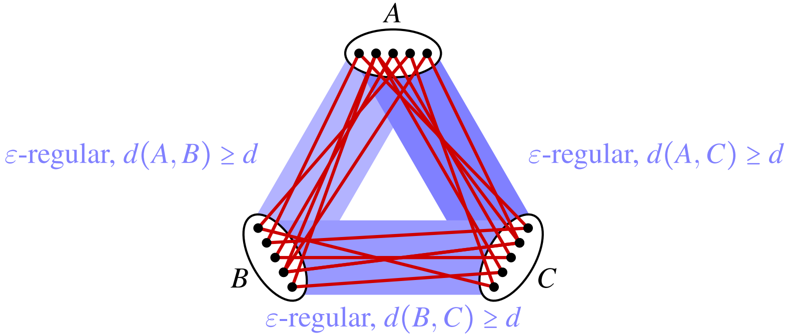

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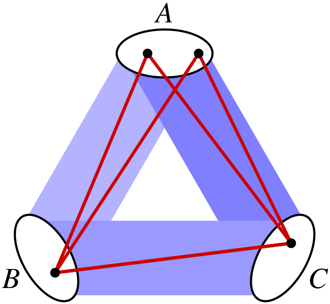

In particular, there is some edge contained in at least two triangles.

Proof of the Ruzsa-Szemerédi Theorem

Theorem (Ruzsa & Szemerédi)

$\forall \delta > 0\ \exists\ n_0$ such that every simple triangle graph G on $n \geq n_0$ vertices has less than δn^2 edges.

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Proof:

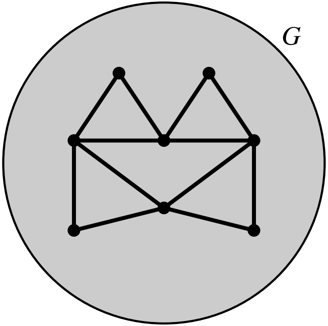
- Combined application of Regularity Lemma and Counting Lemma with the following choice of constants
- Given $\delta > 0$, we choose d and ε such that

$$\varepsilon \ll d \ll \delta$$

and let n_0 be sufficiently large, so that the Regularity Lemma and the Counting Lemma can be applied.

Proof of the Ruzsa-Szemerédi Theorem (cont'd)

- Let $G = (V, E)$ be a simple triangle graph with $|V| = n \geq n_0$ and $|E| \geq \delta n^2$.



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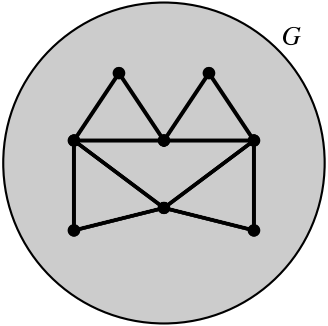

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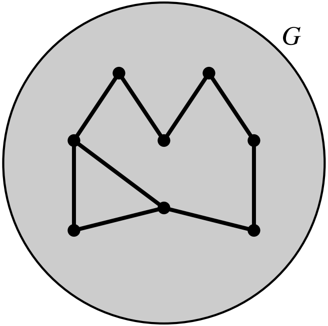

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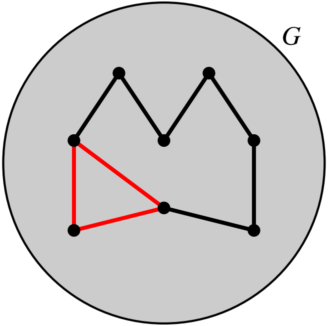

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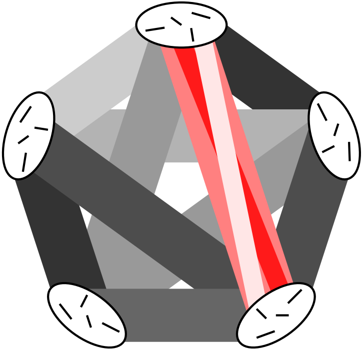

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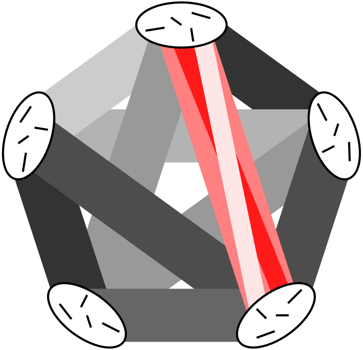

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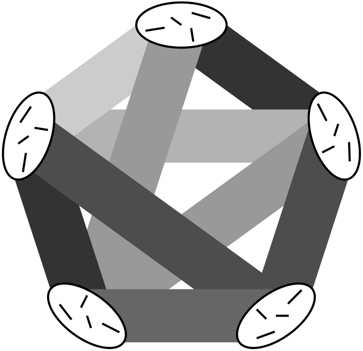

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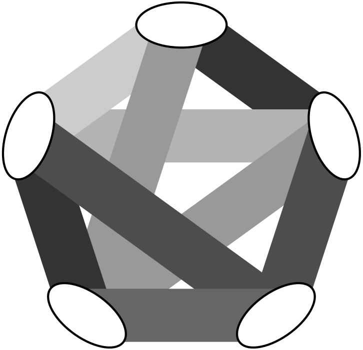

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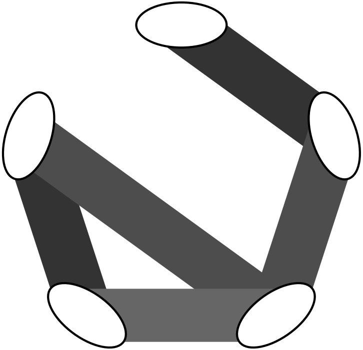

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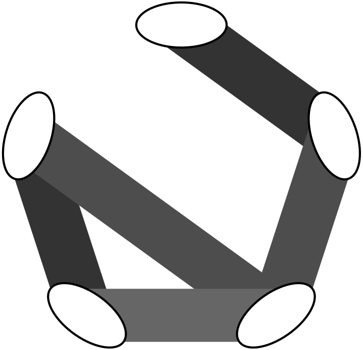

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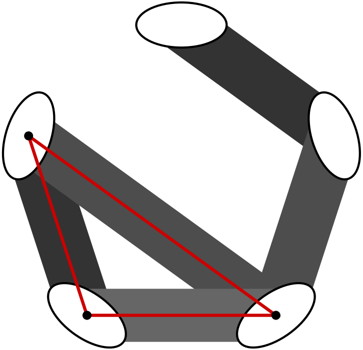

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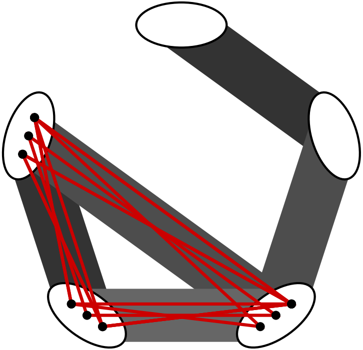

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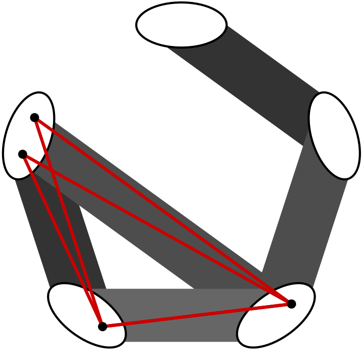

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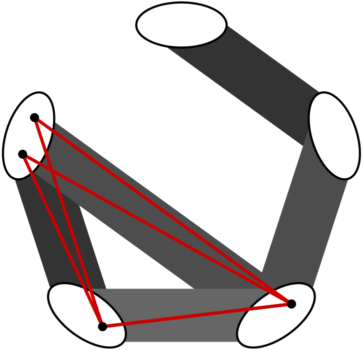
$\Rightarrow$ $|E| < \delta n^2$ □



Removal Lemma

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Theorem (Triangle Removal Lemma)

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Theorem (Removal Lemma)

For every $\varepsilon > 0$ and every graph F with ℓ vertices, there is some $c > 0$ and n_0 such that any graph G on $n \geq n_0$ vertices satisfies:

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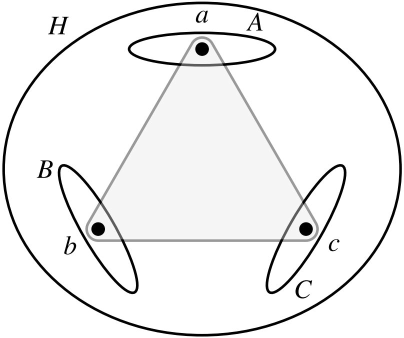
Removal Lemma (informal version)

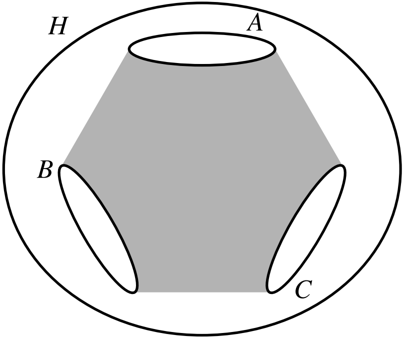
If G contains only “a few” copies of F, then one can remove “a few” edges from G to obtain an F-free graph.

IV. Regularity Method for Hypergraphs



Number of edges:

$$e(A, B, C) = |\{\{a, b, c\} \in E(H) : a \in A, b \in B, c \in C\}|$$



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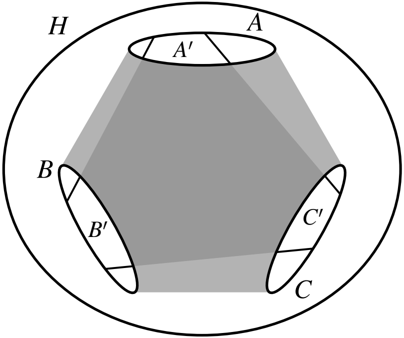
Density:

$$d(A,B,C) = \frac{e(A,B,C)}{|A||B||C|}$$

Weak Regularity for 3-Uniform Hypergraphs



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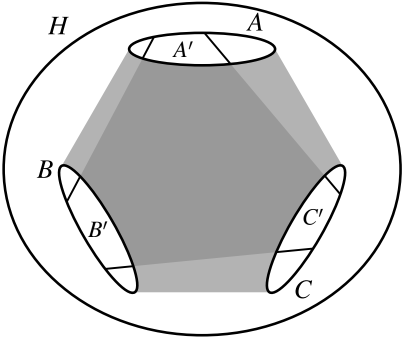
ε-regularity:

For all $A' \subset A$, $B' \subset B$, $C' \subset C$ with $|A'| \geq \varepsilon|A|$, $|B'| \geq \varepsilon|B|$, and $|C'| \geq \varepsilon|C|$

$$|d(A,B,C) - d(A',B',C')| < \varepsilon$$

Weak Regularity for 3-Uniform Hypergraphs



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Regularity Lemma: easy to prove (simple extension of graph case)

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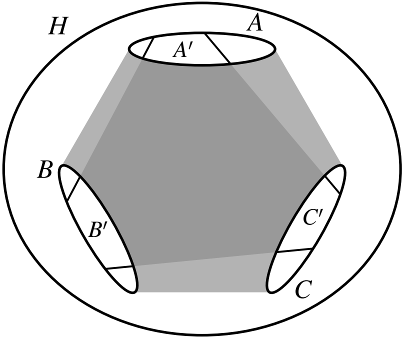

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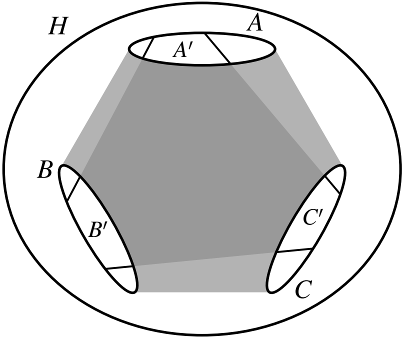
Regularity Lemma: easy to prove (simple extension of graph case)

Counting Lemma: fails to be true (too weak a notion of regularity)

For example, for every $\varepsilon > 0$ there exist $K_4^{(3)}$-free 4-partite 3-uniform hypergraphs with density $1/2 - o(1)$ and all its 3-partite subhypergraphs being ε-regular.

Weak Regularity for 3-Uniform Hypergraphs



Number of edges:

$$e(A,B,C) = |\{\{a,b,c\} \in E(H) : a \in A, b \in B, c \in C\}|$$

Density:

$$d(A,B,C) = \frac{e(A,B,C)}{|A||B||C|}$$

ε-regularity:

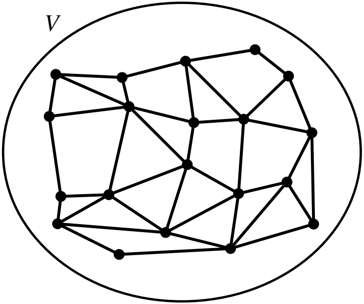
For all $A' \subset A$, $B' \subset B$, $C' \subset C$ with $|A'| \geq \varepsilon|A|$, $|B'| \geq \varepsilon|B|$, and $|C'| \geq \varepsilon|C|$

$$|d(A,B,C) - d(A',B',C')| < \varepsilon$$

Regularity Lemma: easy to prove (simple extension of graph case)

Counting Lemma: fails to be true (too weak a notion of regularity)

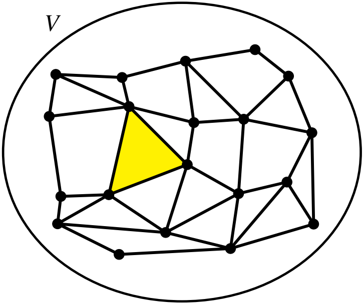
Counterexamples for the Counting Lemma suggest that hyperedge distribution must be uniform on pairs (and not only on vertices).



Setup:

- given graph $G = (V, E_G)$

Regularity of 3-Uniform Hypergraphs Respecting Pairs



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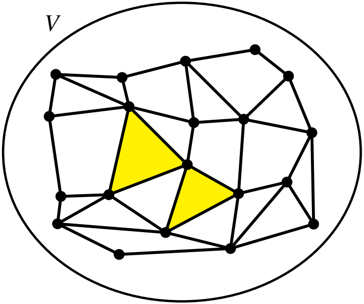

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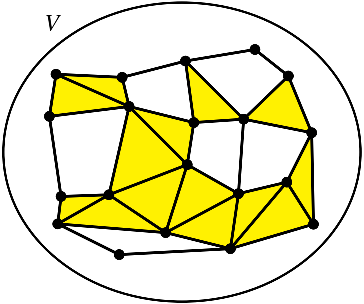

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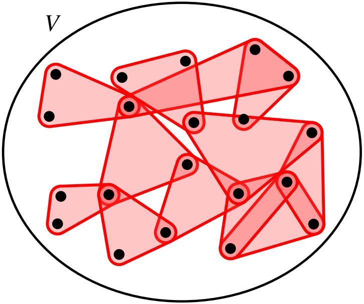

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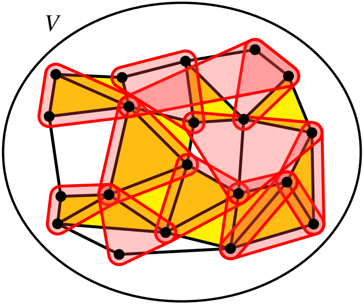

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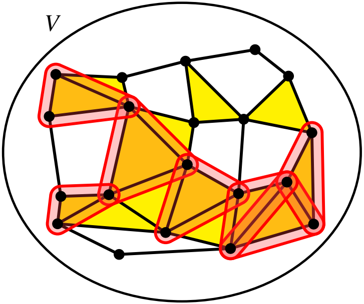

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$$d(H \mid G) = \frac{|E_H \cap \mathcal{K}_3(G)|}{|\mathcal{K}_3(G)|}$$

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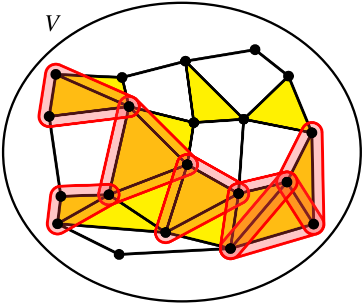

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Definition (H is ε-regular with respect to G)

For all subgraphs $G' \subseteq G$ with $|\mathcal{K}_3(G')| \geq \varepsilon |\mathcal{K}_3(G)|$ we have

$$\left| d(H \mid G) - d(H \mid G') \right| < \varepsilon .$$

Regularity Lemma for k-Uniform Hypergraphs

Regularity Lemma for k-uniform hypergraphs H provides a family of partitions of vertices, pairs, 3-sets, ..., $(k-1)$-sets such that:

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H is regular

Hyperedges of H are uniformly distributed with respect to the partition classes of the $(k-1)$-sets.

Removal Lemma for hypergraphs

- Regularity Method for hypergraphs yields:

Theorem (Clique Union Lemma)

Every k-uniform, simple clique hypergraph on n vertices has $o(n^k)$ edges.

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Theorem (Removal Lemma for Hypergraphs)

For every $\varepsilon > 0$ and every k-uniform hypergraph F with ℓ vertices, there is some $c > 0$ and n_0 such that any hypergraph H on $n \geq n_0$ vertices satisfies:

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Question

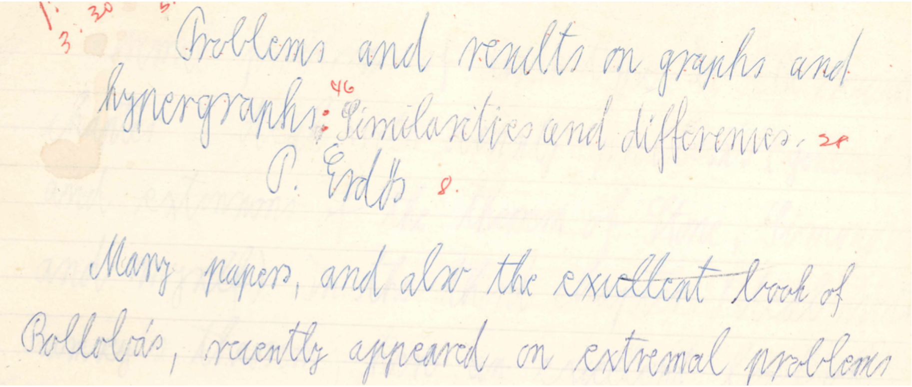
Can F be replaced by a (possibly infinite) family $\mathcal{F}$, i.e., *either H is close to containing no $F \in \mathcal{F}$ or H contains many copies of some $F \in \mathcal{F}$?*

V. Generalizations of the Removal Lemma

Problems and Results on Graphs and Hypergraphs: Similarities and Differences

Paul Erdös

Many papers and also the excellent book of Bollobás, recently appeared on extremal problems on graphs. Two survey papers of Simonovits are in the press and Brown, Simonovits and I have several papers, some appeared, some in the press and some in preparation on this subject.



Problems and results on graphs and hypergraphs: similarities and differences.

P. Erdős

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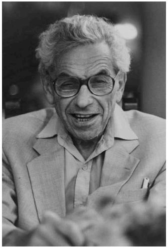
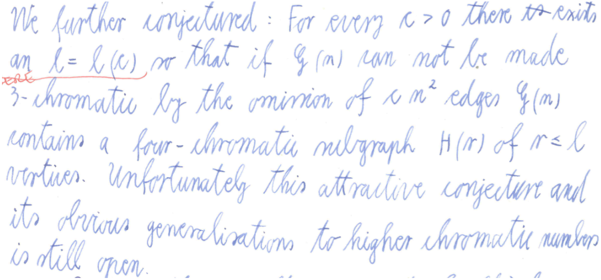
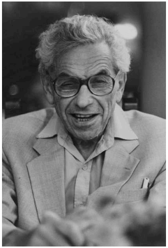
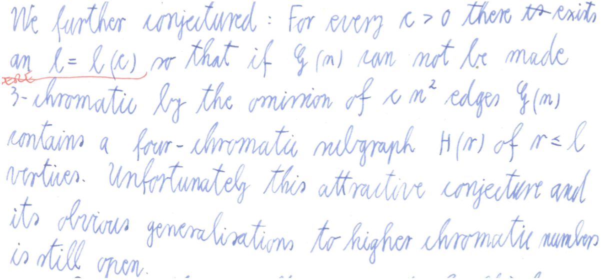



We further conjectured: For every $c > 0$ there exists an $\ell = \ell(c)$ so that if $\mathcal{G}(n)$ can not be made 3-chromatic by the omission of cn^2 edges $\mathcal{G}(n)$ contains a four-chromatic subgraph $H(r)$ of $r \leq \ell$ vertices. Unfortunately this attractive conjecture and its obvious generalisations to higher chromatic numbers is still open.

Conjecture (Erdős, 1983)

$\forall \varepsilon > 0\ \exists\, c, n_0, \ell$, such that every graph G with $n \geq n_0$ vertices satisfies:





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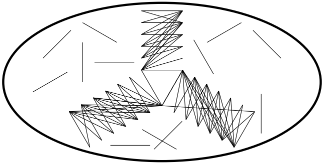

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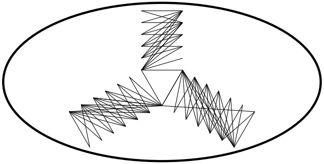
Removal Lemma for Infinite Families



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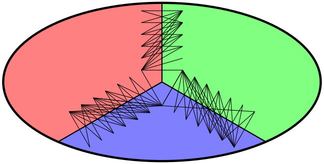

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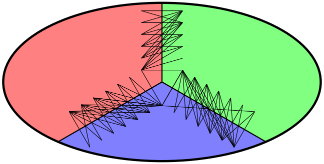

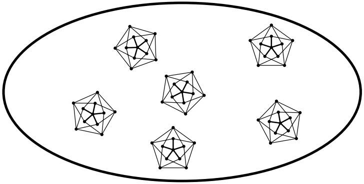

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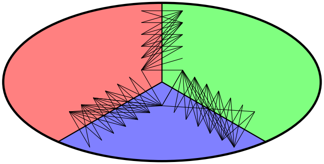

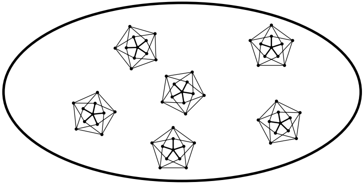

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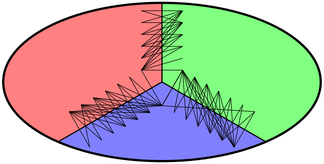

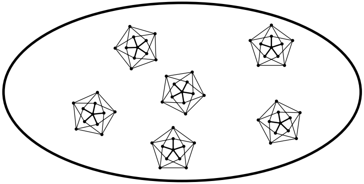

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$\forall \varepsilon > 0 \; \exists \, c, n_0, \ell$, such that every graph G with $n \geq n_0$ vertices satisfies:

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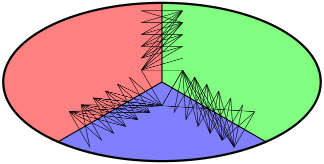

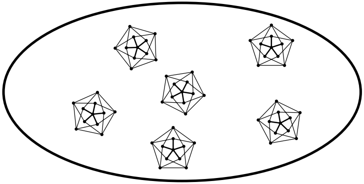

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- Conjecture ≙ Removal Lemma for infinite family $\mathcal{F}_4 = \{F \colon \chi(F) \geq 4\}$
- Bollobás, Erdős, Simonovits, and Szemerédi solved it for $\mathcal{F}_3$ in 1978
- $\mathcal{F}_r$ for any $r \geq 4$ was confirmed jointly with Duke in 1985

Removal Lemma for Infinite Families

Theorem (Alon & Shapira, 2005)

Let $\mathcal{F}$ be a possibly infinite family of graphs.
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G is ε-far from being $\mathcal{F}$-free $\Rightarrow G$ contains "many" copies of a "small" $F \in \mathcal{F}$.

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- Application in Theoretical Computer Science in the area of Property Testing (introduced by Rubinfeld and Sudan in 1996 and Goldreich, Goldwasser, and Ron in 1998)

Open Problem

An Open Problem

Ruzsa-Szemerédi Theorem

$\forall \delta > 0 \;\; \exists n_0$ such that any simple triangle graph G on $n \geq n_0$ vertices satisfies $e(G) \leq \delta n^2$.

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Is it true that

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Problem

Improve these bounds!

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$$\mathrm{RSz}(\delta) \overset{???}{\leq} 2^{2^{2^{1/\delta}}}$$

where the height of the tower is independent of δ?

Thank you!