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**Complexity of distances:
Reductions of distances between metric
and Banach spaces**

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Preprint No. 36-2020

PRAHA 2020

COMPLEXITY OF DISTANCES: REDUCTIONS OF DISTANCES BETWEEN METRIC AND BANACH SPACES

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ABSTRACT. We show that all the standard distances from metric geometry and functional analysis, such as Gromov-Hausdorff distance, Banach-Mazur distance, Kadets distance, Lipschitz distance, Net distance, and Hausdorff-Lipschitz distance have all the same complexity and are reducible to each other in a precisely defined way.

This is done in terms of descriptive set theory and is a part of a larger research program initiated by the authors in [8]. The paper is however targeted also to specialists in metric geometry and geometry of Banach spaces.

INTRODUCTION

Metric geometry and nonlinear geometry of Banach spaces are rapidly evolving fields connected to many different areas of mathematics including Riemannian geometry, Banach space theory, graph theory, computer science, etc. One of their feature is that they, as ‘metric disciplines’, quantitatively measure non-equivalence of the objects they work with by distances. Standard examples of such distances are the Gromov-Hausdorff distance between metric spaces and the Banach-Mazur distance between Banach spaces, but the list of useful distances is quite large and the study of those and their mutual relations inspired mathematicians to prove several deep theorems, most notably in the fields of the non-linear geometry of Banach spaces and geometry of Riemannian manifolds, see e.g. [10, 13, 15, 18, 22, 25, 29]. We have been also inspired by the influential book of Gromov ([17]), where many of the distances we work with were defined.

The issue we address in this paper is to compare the complexities, on the small scale, of the distances that have recently received attention in metric geometry and Banach space theory. For instance, we look for a constructive assignment of Banach spaces to metric spaces so that the Banach-Mazur distance of the assigned Banach spaces is small if and only if the Gromov-Hausdorff distance of the original metric spaces is small.

In order to provide such a comparison, given distances d_1 and d_2 on two classes of metric or Banach spaces \mathcal{P}_1 and \mathcal{P}_2 , respectively, we say that d_1 is *Borel-uniformly continuous reducible* to d_2 if there exists a construction $f : \mathcal{P}_1 \rightarrow \mathcal{P}_2$ which is a uniformly continuous embedding with respect to distances d_1 and d_2 , see Definition 34 for a more precise treatment.

2010 *Mathematics Subject Classification.* 03E15, 46B20, 54E50, (primary), 46B80 (secondary).

Key words and phrases. Gromov-Hausdorff distance, Banach-Mazur distance, Kadets distance, analytic pseudometrics, analytic equivalence relations.

The following is our main result, which therefore in a (certain precise) sense says that various problems ranging from linear classification of Banach spaces to large scale geometry of metric spaces are comparable in their difficulty.

Theorem 1. (1) *The following pseudometrics are mutually Borel-uniformly continuous bi-reducible: the **Gromov-Hausdorff distance** when restricted to Polish metric spaces, to metric spaces bounded from above, from below, from both above and below, to Banach spaces; the **Banach-Mazur distance** on Banach spaces, the **Lipschitz distance** on Polish metric spaces and Banach spaces; the **Kadets distance** on Banach spaces; the **Hausdorff-Lipschitz distance** on Polish metric spaces; the **net distance** on Banach spaces.*

(2) *The pseudometrics above are Borel-uniformly continuous reducible to the **uniform distance** on Banach spaces.*

To illustrate the meaning, for example the proof that the Banach-Mazur distance on Banach spaces (denoted by ρ_{BM}) is Borel-uniformly continuous reducible to the Lipschitz distance on Polish metric spaces (denoted by ρ_L) gives a Borel assignment (that is, a very constructive one, avoiding e.g. the axiom of choice) which assigns to a given Banach space X (separable, infinite-dimensional) a Polish metric space $M(X)$ in such a way that this assignment is a uniformly continuous embedding with respect to pseudometrics ρ_{BM} and ρ_L . In particular, $\rho_{BM}(X, Y) = 0$ if and only if $\rho_L(M(X), M(Y)) = 0$. Thus, the problem of whether two Banach spaces are close with respect to the Banach-Mazur distance may be transferred to the problem of whether certain metric spaces are close with respect to the Lipschitz distance.

We recall that there is an active and well established stream within descriptive set theory, called *invariant descriptive set theory* (IDST), whose aim is to provide such reductions for equivalence relations. So for example, it is known that the complexities of the equivalence relations of isomorphism of separable C^* -algebras, homeomorphism of metrizable compact spaces, and linear isometry of Banach spaces are the same (see [28],[31], [23]). On the other hand, they are strictly less complex than linear isomorphism of Banach spaces [12], isomorphism of separable operator spaces [1], and strictly more complex than the isomorphism of countable graphs. We refer to [14] as a general reference.

We follow this line of thought and our results are also written in the terms of IDST. However, our aim is to make it comprehensible and interesting also for researchers working in metric geometry and geometry of Banach spaces without particular knowledge of descriptive set theory. We must note that this paper naturally complements our paper [8], which is targeted to descriptive set theorists and to which we refer for additional motivation and some general results. However, since the readership of these two articles likely will not be exactly the same, we try to make this paper self-contained.

The paper is organized as follows. In Section 1, we present our examples of distances and prove basic facts about them that will be needed later. We also introduce some basic notions from descriptive set theory and repeat our

definitions from [8]. In Sections 2, 3, and 4 we concentrate the constructions of our reductions. The final Section 5 summarizes our findings and suggests some problems.

1. PRELIMINARIES AND BASIC RESULTS

The goal of this section is to recall several basic notions from descriptive set theory, such as coding of Polish metric spaces or Banach spaces, and to introduce the distances we work with in this paper. We also prove here several basic results about these distances which will be needed in further sections. The notation and terminology is standard, for the undefined notions see [11] for Banach spaces and [19] for descriptive set theory.

1.1. Coding of Polish metric spaces and Banach spaces. We begin with formalizing the class of all infinite Polish metric spaces as a standard Borel space. In most situations it will not be important how we formalize this class, but whenever it does become important we shall use the following definition.

Definition 2. By \mathcal{M} we denote the space of all metrics on \mathbb{N} . This gives \mathcal{M} a Polish topology inherited from $\mathbb{R}^{\mathbb{N} \times \mathbb{N}}$.

If p and q are positive real numbers, by \mathcal{M}_p , \mathcal{M}^q and \mathcal{M}_p^q respectively, we denote the space of metrics with values in $\{0\} \cup [p, \infty)$, $[0, q]$, and $\{0\} \cup [p, q]$ (assuming that $p < q$), respectively.

Remark 3. Every $f \in \mathcal{M}$ is then a code for Polish metric space M_f which is the completion of (\mathbb{N}, f) . Hence, in this sense we may refer to the set \mathcal{M} as to the standard Borel space of all infinite Polish metric spaces. This approach was used for the first time by Vershik [30] and further e.g. in [6], see also [14, page 324]. Another possible approach is to view all Polish metric spaces as the Effros-Borel space $F(\mathbb{U})$ of all closed subspaces of the Urysohn space \mathbb{U} . When one considers the space of all pseudometrics on \mathbb{N} then these two approaches are equivalent, see e.g. [14, Theorem 14.1.3]. Similarly, one can get a Borel isomorphism Θ between \mathcal{M} and $F(\mathbb{U}) \setminus F_{fin}(\mathbb{U})$, where $F_{fin}(\mathbb{U})$ denotes the Borel set of finite subsets of \mathbb{U} , such that $\Theta(f)$ is isometric to M_f for every $f \in \mathcal{M}$. Since the Borel set of finite metric spaces is not interesting from our point of view we will ignore it in the sequel.

Remark 4. Let (M, d) be a separable metric space. If there is no danger of confusion, we write $M \in \mathcal{M}$ by which we mean that the metric d restricted to a countable dense subset of M induces a metric $d' \in \mathcal{M}$. Analogously, if there is no danger of confusion, we write $M \in \mathcal{M}_p$, $M \in \mathcal{M}^q$ or $M \in \mathcal{M}_p^q$.

Next, we formalize the class of all infinite-dimensional separable Banach spaces as a standard Borel space. As in the case of infinite Polish metric spaces, the concrete coding of this space is usually not important. However, when we compute that certain maps from or into this space are Borel we adopt a coding analogous to that one for \mathcal{M} (and which is more similar to the general coding of metric structures from [2]).

Definition 5. Let us denote by V the vector space over \mathbb{Q} of all finitely supported sequences of rational numbers, that is, the unique infinite-dimensional

vector space over \mathbb{Q} with a countable Hamel basis $(e_n)_{n \in \mathbb{N}}$. By \mathcal{B}_0 we denote the space of all norms on the vector space V . This gives \mathcal{B}_0 a Polish topology inherited from \mathbb{R}^V . We shall consider only those norms for which its canonical extension to the real vector space c_{00} is still a norm; that is, norms for which the elements $(e_n)_n$ are not only \mathbb{Q} -linearly independent, but also \mathbb{R} -linearly independent. Let us denote the subset of such norms by \mathcal{B} .

Let us point out that \mathcal{B} is a G_δ subset \mathcal{B}_0 , thus a Polish space of its own. Indeed, it suffices to check that for $\|\cdot\| \in \mathcal{B}_0$ we have $\|\cdot\| \in \mathcal{B}$ if and only if for every fixed $n \in \mathbb{N}$ the elements e_1, \dots, e_n are \mathbb{R} -linearly independent in $(c_{00}, \|\cdot\|)$, which is an open subset of \mathcal{B}_0 . We show that the complement C , the set of those norms in \mathcal{B}_0 for which the elements e_1, \dots, e_n are \mathbb{R} -linearly dependent, is closed in \mathcal{B}_0 . Let $(\|\cdot\|_m)_{m \in \mathbb{N}} \subseteq C$ converge to $\|\cdot\|$. We show that $\|\cdot\| \in C$. For each $m \in \mathbb{N}$ there are $\alpha_1^m, \dots, \alpha_n^m \in \mathbb{R}$, not all of them zero, such that $\|\sum_{i=1}^n \alpha_i^m e_i\|_m = 0$. Without loss of generality, we may assume that $M_m = \max\{|\alpha_1^m|, \dots, |\alpha_n^m|\} = 1$. By passing to a subsequence if necessary, we may therefore assume that each α_i^m converges to some α_i , for $i \leq n$, where at least one of the limits is non-zero. It follows that $x = \sum_{i=1}^n \alpha_i e_i \neq 0$ and $\|x\| = 0$, showing that $\|\cdot\| \in C$.

Remark 6. Each norm $\nu \in \mathcal{B}$ is then a code for an infinite-dimensional Banach space X_ν which is the completion of (V, ν) . The completion is naturally a complete normed space over \mathbb{R} . This is the same as taking the canonical extension of ν to c_{00} and then taking the completion.

We refer the reader to another paper of the authors, [7], where the space \mathcal{B} is thoroughly investigated from the topological point of view, which is however not so relevant for our considerations here, where we are merely satisfied with the fact that \mathcal{B} is a standard Borel space.

Hence, we may refer to the set \mathcal{B} as to the standard Borel space of all infinite-dimensional separable Banach spaces. Another possible approach, introduced by Bossard [4], is to view all infinite-dimensional separable Banach spaces as the space $SB(X)$ of all closed linear infinite-dimensional subspaces of a universal separable Banach space X ; then it is a Borel subset of the Effros-Borel space $F(X)$, the interested reader is referred to the monograph [9] for further information. Similarly as in the case of Polish metric spaces, those two approaches are equivalent which is witnessed by Theorem 8.

It would be possible to get a coding of all separable Banach spaces, i.e. even finite-dimensional, if we considered the space of all pseudonorms on V . As in the case of Polish metric spaces, the Borel set of all finite-dimensional Banach spaces is not interesting from our point of view, so we will ignore it in the sequel.

Remark 7. If there is no danger of confusion, we write $X \in \mathcal{B}$ as a shortcut for “ X is an infinite-dimensional separable Banach space”.

Theorem 8. *For every universal separable Banach space X , there is a Borel isomorphism Θ between \mathcal{B} and $SB(X)$ such that $\Theta(\nu)$ is isometric to X_ν for every $\nu \in \mathcal{B}$.*

Proof. First, let us observe that whenever X and Y are universal separable Banach spaces, there is a Borel isomorphism Φ between $SB(X)$ and $SB(Y)$

such that $\Phi(Z)$ is isometric to Z for every $Z \in SB(X)$. Indeed, fix an isometry $i : X \rightarrow Y$. Then $SB(X) \ni Z \mapsto i(Z) \in SB(Y)$ defines a Borel injective map, let us call it Φ_1 , such that Z is isometric to $\Phi_1(Z)$ for every $Z \in SB(X)$. Next, we find an analogous Borel injective map $\Phi_2 : SB(Y) \rightarrow SB(X)$. Finally, using the usual proof of the Cantor-Bernstein Theorem (see e.g. [19, Theorem 15.7]), we find a Borel isomorphism Φ between $SB(X)$ and $SB(Y)$ whose graph lies in the union of the graph of Φ_1 and the inverse of the graph of Φ_2 .

Hence, we may without loss of generality assume that $X = C([0, 1]) \oplus_2 C([0, 1])$. Using the classical Kuratowski-Ryll-Nardzewski principle (see e.g. [9, Theorem 1.2]), we easily get a sequence of Borel maps $d_n : SB(X) \rightarrow X$ such that for every $Z \in SB(X)$ the sequence $(d_n(Z))_{n=1}^\infty$ is normalized, linearly independent and linearly dense in Z . Since all uncountable Polish metric spaces are Borel isomorphic, we may pick a Borel isomorphism j between $SB(X)$ and the interval $[1, 2]$. Now, we define a Borel injective map $\Theta_1 : SB(X) \rightarrow \mathcal{B}$ by putting for every $Z \in SB(X)$

$$\Theta_1(Z)(\alpha) = \left\| j(Z)\alpha_1 d_1(Z) + \sum_{i=2}^{\infty} \alpha_i d_i(Z) \right\|, \quad \alpha \in V.$$

Then Θ_1 is an injective Borel map from $SB(X)$ into \mathcal{B} such that $X_{\Theta_1(Z)}$ is isometric to Z for every $Z \in SB(X)$.

Next, by [20, Lemma 2.4], there is a Borel map $\widetilde{\Theta}_2 : \mathcal{B} \rightarrow SB(C([0, 1]))$ such that $\widetilde{\Theta}_2(\nu)$ is isometric to X_ν for every $\nu \in \mathcal{B}$. Pick a Borel isomorphism j between \mathcal{B} and the interval $[0, 1]$ and for every $\nu \in \mathcal{B}$ define $\Theta_2(\nu)$ as the Banach space of all $(j(\nu)f, \sqrt{1-j^2(\nu)}f) \in X$ where $f \in \widetilde{\Theta}_2(\nu)$. Then Θ_2 is an injective Borel map from \mathcal{B} into $SB(X)$ such that $\Theta_2(\nu)$ is isometric to X_ν for every $\nu \in \mathcal{B}$.

Finally, using the usual proof of the Cantor-Bernstein Theorem (see e.g. [19, Theorem 15.7]), we find a Borel isomorphism Θ between \mathcal{B} and $SB(X)$ whose graph lies in the union of the graph of Θ_2 and the inverse of the graph of Θ_1 . \square

1.2. Distances between metric spaces and Banach spaces.

1.2.1. *Gromov-Hausdorff distance.* We start with the notion of Gromov-Hausdorff distance which has been investigated already in [8]. For the convenience of the reader, we repeat some facts about this distance here.

Definition 9 (Gromov-Hausdorff distance). Let (M, d_M) be a metric space and $A, B \subseteq M$ two non-empty subsets. The *Hausdorff distance* between A and B in M , $\rho_H^M(A, B)$, is defined as

$$\max \left\{ \sup_{a \in A} d_M(a, B), \sup_{b \in B} d_M(b, A) \right\},$$

where for an element $a \in M$ and a subset $B \subseteq M$, $d_M(a, B) = \inf_{b \in B} d_M(a, b)$.

Suppose now that M and N are two metric spaces. Their *Gromov-Hausdorff distance*, $\rho_{GH}(M, N)$, is defined as the infimum of the Hausdorff

distances of their isometric copies contained in a single metric space, that is

$$\rho_{GH}(M, N) = \inf_{\substack{\iota_M: M \hookrightarrow X \\ \iota_N: N \hookrightarrow X}} \rho_H^X(\iota_M(M), \iota_N(N)),$$

where ι_M and ι_N are isometric embeddings into a metric space X .

For two metrics $f, g \in \mathcal{M}$ we denote by $\rho_{GH}(f, g)$ the Gromov-Hausdorff distance between (\mathbb{N}, f) and (\mathbb{N}, g) , which is easily seen to be equal to the Gromov-Hausdorff distance between their completions M_f and M_g .

Let A and B be two sets. A *correspondence* between A and B is a binary relation $\mathcal{R} \subseteq A \times B$ such that for every $a \in A$ there is $b \in B$ such that $a\mathcal{R}b$, and for every $b \in B$ there is $a \in A$ such that $a\mathcal{R}b$.

Fact 10 (see e.g. Theorem 7.3.25. in [5]). *Let M and N be two metric spaces. For every $r > 0$ we have $\rho_{GH}(M, N) < r$ if and only if there exists a correspondence \mathcal{R} between M and N such that $\sup |d_M(m, m') - d_N(n, n')| < 2r$, where the supremum is taken over all $m, m' \in M$ and $n, n' \in N$ with $m\mathcal{R}n$ and $m'\mathcal{R}n'$.*

It is easier to work with bijections instead of correspondences. One may wonder in which situations we may do so. Let us define the corresponding concept and prove some results in this direction. Those will be used later.

Definition 11. By S_∞ we denote the set of all bijections from \mathbb{N} to \mathbb{N} . For two metrics on natural numbers $f, g \in \mathcal{M}$ and $\varepsilon > 0$, we consider the relation

$$f \simeq_\varepsilon g \iff \exists \pi \in S_\infty \forall \{n, m\} \in [\mathbb{N}]^2 : |f(\pi(n), \pi(m)) - g(n, m)| \leq \varepsilon.$$

We write $f \simeq g$ if $f \simeq_\varepsilon g$ for every $\varepsilon > 0$.

The following two observations are proved e.g. in [8, Lemma 15 and Lemma 16].

Lemma 12. *For any two metrics on natural numbers $f, g \in \mathcal{M}$ and any $\varepsilon > 0$ we have $\rho_{GH}(f, g) \leq \varepsilon$ whenever $f \simeq_{2\varepsilon} g$.*

Lemma 13. *Let $p > 0$ be a real number. For any two metrics on natural numbers $f, g \in \mathcal{M}_p$ we have $\rho_{GH}(f, g) = \inf\{r : f \simeq_{2r} g\}$ provided that $\rho_{GH}(f, g) < p/2$.*

Lemma 14. *Let $f, g \in \mathcal{M}$ define two perfect metric spaces, that is, spaces without isolated points. Then $\rho_{GH}(f, g) = \inf\{r : f \simeq_{2r} g\}$.*

Proof. By Lemma 12, $\rho_{GH}(f, g) \leq r$ whenever $f \simeq_{2r} g$. For the other inequality, suppose $\rho_{GH}(f, g) < r$ and fix s with $\rho_{GH}(f, g) < s < r$. By Fact 10, there is a correspondence $\mathcal{R} \subseteq \mathbb{N} \times \mathbb{N}$ witnessing that $\rho_{GH}(f, g) < s$. Now we recursively define a permutation $\pi \in S_\infty$. During the $(2n - 1)$ -th step of the recursion we ensure that n is in the domain of π and during the $2n$ -th step we ensure that n is in the range of π .

Pick an arbitrary $n \in \mathbb{N}$ such that $1\mathcal{R}n$ and set $\pi(1) = n$. If $n = 1$ then we have ensured that 1 is both in the domain and the range of π . If $n \neq 1$, then pick some $m \in \mathbb{N}$ such that $m\mathcal{R}1$ and set $\pi(m) = 1$. If the only integer m with the property that $m\mathcal{R}1$ is equal to 1, which has been already used, we pick an arbitrary $m' \in \mathbb{N}$ that has not been used yet and such that

$f(m, m') < r - s$. The existence of such m' follows since f is perfect. We set $\pi(m') = 1$. In the general $(2n - 1)$ -th step we proceed analogously. If n has not been added to the domain of π yet we pick some m that has not been added to the range of π yet and such that $n\mathcal{R}m$. Then we may set $\pi(n) = m$. If there is no such m , we pick an arbitrary m such that $n\mathcal{R}m$ and take an arbitrary m' with $g(m, m') < r - s$ that has not been added to the range of π yet and set $\pi(n) = m'$. The $2n$ -th step is done analogously.

When the recursion is finished we claim that for every n, m we have $|f(m, n) - g(\pi(m), \pi(n))| \leq 2r$ which is what we should prove. Suppose e.g. that $\pi(m)$, resp. $\pi(n)$ are such that there are m' , resp. n' with $g(m', \pi(m)) < r - s$ and $g(n', \pi(n)) < r - s$, and $m\mathcal{R}m'$ and $n\mathcal{R}n'$. The other cases are treated analogously. Then by the choice of \mathcal{R} we have

$$\begin{aligned} |f(m, n) - g(\pi(m), \pi(n))| &\leq |f(m, n) - g(m', n')| + |g(m', n') - g(\pi(m), n')| \\ &\quad + |g(\pi(m), n') - g(\pi(m), \pi(n))| \\ &< 2s + g(m', \pi(m)) + g(n', \pi(n)) < 2r. \end{aligned}$$

□

Remark 15. If $f, g \in \mathcal{M}$ define neither perfect metric spaces, nor do they belong to \mathcal{M}_p , for some $p > 0$, then \simeq_ε does not give good estimates for the Gromov-Hausdorff distance between f and g . Consider e.g. \mathbb{N} as a metric space with its standard metric and a metric space $C_k = \{m + 1/n : m \in \mathbb{N}, n \geq k\} \subseteq \mathbb{R}$, $k \geq 2$, with a metric inherited from \mathbb{R} . We have $\rho_{GH}(\mathbb{N}, C_k) \rightarrow 0$ as $k \rightarrow \infty$, but clearly there are no bijections between \mathbb{N} and C_k witnessing the convergence.

1.2.2. Kadets distance.

Definition 16 (Kadets distance). Suppose that X and Y are two Banach spaces. Their *Kadets distance*, $\rho_K(X, Y)$, is defined as the infimum of the Hausdorff distances of their unit balls over all isometric linear embeddings of X and Y into a common Banach space Z . That is

$$\rho_K(X, Y) = \inf_{\substack{\iota_X: X \hookrightarrow Z \\ \iota_Y: Y \hookrightarrow Z}} \rho_H^Z(\iota_X(B_X), \iota_Y(B_Y)),$$

where ι_X and ι_Y are linear isometric embeddings into a Banach space Z .

Similarly as the Gromov-Hausdorff distance, the Kadets distance may be expressed in terms of correspondences. First, call a subset $A \subseteq X$ of a real vector space \mathbb{Q} -homogeneous if it is closed under scalar multiplication by rationals. The following lemma generalizes [18, Theorem 2.3], which uses homogeneous maps. The proof is however very similar.

Lemma 17. *Let X and Y be Banach spaces and E and F be some dense \mathbb{Q} -homogeneous subsets of X and Y respectively. Then we have $\rho_K(X, Y) < \varepsilon$ if and only if there exist $\delta \in (0, \varepsilon)$ and a \mathbb{Q} -homogeneous correspondence $\mathcal{R} \subseteq E \times F$ with the property that for every $x \in E$ there is $y \in F$ with $x\mathcal{R}y$ and $\|y\|_Y \leq \|x\|_X$, for every $y \in F$ there is $x \in E$ with $x\mathcal{R}y$ and $\|x\|_X \leq \|y\|_Y$, and*

$$\left| \left\| \sum_{i \leq n} x_i \right\|_X - \left\| \sum_{i \leq n} y_i \right\|_Y \right| \leq (\varepsilon - \delta) \left(\sum_{i \leq n} \max\{\|x_i\|_X, \|y_i\|_Y\} \right)$$

for all $(x_i)_i \subseteq E$ and $(y_i)_i \subseteq F$, where for all i , $x_i \mathcal{R} y_i$.

Proof. If $\rho_K(X, Y) < \varepsilon$, then fix some $\delta \in (0, \varepsilon - \rho_K(X, Y))$ and some isometric embeddings of X and Y into a Banach space Z such that $\rho_H^Z(B_X, B_Y) < \varepsilon - \delta$. Then set $x \mathcal{R} y$, for $x \in E$ and $y \in F$, if and only if $\|x - y\|_Z \leq \max\{(\varepsilon - \delta)\|x\|_X, (\varepsilon - \delta)\|y\|_Y\}$.

Suppose conversely that we have such $\delta \in (0, \varepsilon)$ and $\mathcal{R} \subseteq E \times F$. Set E' to be the linear span of E , analogously F' to be the linear span of F . Then set $Z = E' \oplus F'$ and define a norm $\|\cdot\|_Z$ on Z as follows: for $(x, y) \in Z$ set

$$\|(x, y)\|_Z = \inf \left\{ \|x_0\|_X + \|y_0\|_Y + (\varepsilon - \delta) \left(\sum_{i \leq n} \max\{\|x_i\|_X, \|y_i\|_Y\} \right) : \right. \\ \left. x = x_0 + \sum_{i \leq n} x_i, y = y_0 - \sum_{i \leq n} y_i, x_0 \in E', y_0 \in F', x_i \mathcal{R} y_i \right\}.$$

It is clear that $\|\cdot\|_Z$ satisfies the triangle inequality. Moreover, it is easy to check, using the \mathbb{Q} -homogeneity of \mathcal{R} , that $\|\cdot\|_Z$ is \mathbb{Q} -homogeneous, i.e. for every $z \in Z$ and $q \in \mathbb{Q}$, $\|qz\|_Z = |q|\|z\|_Z$. By continuity, we also get the full homogeneity for all real scalars.

Let us check that for any $x \in E'$ we have $\|(x, 0)\|_Z = \|x\|_X$. Clearly, $\|(x, 0)\|_Z \leq \|x\|_X$. Suppose there is a strict inequality. Then we have

$$\|x_0\|_X + \|y_0\|_Y + (\varepsilon - \delta) \left(\sum_{i \leq n} \max\{\|x_i\|_X, \|y_i\|_Y\} \right) < \|x\|_X,$$

where $x_0 \in E'$, $x_1, \dots, x_n \in E$, $y_0 \in F'$, $y_1, \dots, y_n \in F$, $x = x_0 + \sum_{i \leq n} x_i$, $y = 0 = y_0 - \sum_{i \leq n} y_i$ and $x_i \mathcal{R} y_i$. However, by our assumption we have

$$\begin{aligned} \|x_0\|_X + \|y_0\|_Y + (\varepsilon - \delta) \left(\sum_{i \leq n} \max\{\|x_i\|_X, \|y_i\|_Y\} \right) &= \\ &= \|x_0\|_X + \left\| \sum_{i \leq n} y_i \right\|_Y + (\varepsilon - \delta) \left(\sum_{i \leq n} \max\{\|x_i\|_X, \|y_i\|_Y\} \right) \geq \\ &\geq \|x_0\|_X + \left\| \sum_{i \leq n} x_i \right\|_X \geq \|x\|_X, \end{aligned}$$

a contradiction. Analogously, we show that for every $y \in F'$ we have $\|y\|_Y = \|(0, y)\|_Z$. So E' and F' are isometrically embedded into Z . Now for any $x \in B_X \cap E$ by the assumption there is $y \in F$ such that $\|y\|_Y \leq \|x\|_X$ and $x \mathcal{R} y$. So

$$\|(x, -y)\|_Z \leq (\varepsilon - \delta)\|x\|_X$$

since x can be written as $x_0 + x$, where $x_0 = 0$, and $-y$ as $y_0 - y$, where $y_0 = 0$. Analogously, for every $y \in B_Y \cap F$ there is $x \in E$ such that $\|x\|_X \leq \|y\|_Y$ and $\|(x, -y)\|_Z \leq (\varepsilon - \delta)\|y\|_Y$. Finally we take the completion of Z and get a Banach space Z' to which X and Y linearly isometrically embed so that $\rho_H^{Z'}(B_X, B_Y) < \varepsilon$. \square

1.2.3. Lipschitz distance.

Definition 18 (Lipschitz distance). Let M and N be two metric spaces. Their *Lipschitz distance* is defined as

$$\rho_L(M, N) = \inf \{ \log \max\{\text{Lip}(T), \text{Lip}(T^{-1})\} : T : M \rightarrow N \text{ is bi-Lipschitz bijection} \},$$

where

$$\text{Lip}(T) = \sup_{m \neq n \in M} \frac{d_N(T(m), T(n))}{d_M(m, n)}$$

is the Lipschitz norm of T .

Remark 19. The previous definition of the Lipschitz distance is from [5, Definition 7.2.1]. We note that Gromov in [17, Definition 3.1] defines the Lipschitz distance (between M and N) as

$$\inf \{ |\log \text{Lip}(T)| + |\log \text{Lip}(T^{-1})| : T : M \rightarrow N \text{ is bi-Lipschitz} \}.$$

Nevertheless, one can easily check that these two definitions give equivalent distances. Indeed, if we denote by ρ'_L the Lipschitz distance in the sense of Gromov, then we easily see that

$$\rho_L \leq \rho'_L \leq 2\rho_L.$$

More differently, Dutrieux and Kalton in [10] define the Lipschitz distance analogously to the definition of the Banach-Mazur distance, which we recall later, as ¹

$$\inf \{ \log \text{Lip}(T) \text{Lip}(T^{-1}) : T : M \rightarrow N \text{ is bi-Lipschitz} \}.$$

Denote this distance by ρ''_L . Clearly, ρ''_L is not equivalent with ρ_L since for example intervals $[0, 1]$ and $[0, 2]$ have distance zero only in ρ''_L . However, in [10] the authors work mainly with Banach spaces and if M and N are Banach spaces, it is easy to see that we have $\rho''_L(M, N) = \rho'_L(M, N)$. That follows from the fact that we may consider only those bi-Lipschitz maps such that both $\log \text{Lip}(T)$ and $\log \text{Lip}(T^{-1})$ are non-negative. Indeed, if say $\text{Lip}(T) < 1$, then we define $T' = T/\text{Lip}(T)$ and we get $\log \text{Lip}(T) + \log \text{Lip}(T^{-1}) = |\log \text{Lip}(T')| + |\log \text{Lip}((T')^{-1})|$.

However, for Banach spaces we have $\rho'_L(M, N) = 2\rho_L(M, N)$. This again follows after the appropriate rescaling of the maps $T : M \rightarrow N$.

One of the differences between the Gromov-Hausdorff distance and the Lipschitz distance on metric spaces is that for the former if M and N are metric spaces and M' , resp. N' their dense subsets, then $\rho_{GH}(M, N) = \rho_{GH}(M', N')$. That an analogous equality does not hold for the Lipschitz distance is witnessed by the following fact. We thank Benjamin Vejnar for providing us an example on which it is based.

Fact 20. *There exist metrics $d_M, d_N \in \mathcal{M}$ on \mathbb{N} such that their completions are isometric, however there is no bi-Lipschitz map between (\mathbb{N}, d_M) and (\mathbb{N}, d_N) .*

Proof. Let M be a Polish metric space, let G be the group of bi-Lipschitz autohomeomorphisms of M , and suppose there exists $m \in M$ such that $M \setminus G \cdot m$ is dense in M , where $G \cdot m$ is the orbit of m under the action

¹More precisely, they define it without the logarithm which we add in order to satisfy the triangle inequality.

of G on M . Let $(x_i)_i$ be some countable dense subset of M such that $\{x_i : i \in \mathbb{N}\} \cap G \cdot m = \emptyset$, and let $(y_j)_j$ be another countable dense subset of M such that $y_1 = m$. Then there is no bi-Lipschitz map between $(x_i)_i$ and $(y_j)_j$. Indeed, otherwise such a bi-Lipschitz map would extend to some bi-Lipschitz autohomeomorphism $g \in G$ and we would have $g \cdot y_1 = g \cdot m = x_k$, for some $k \in \mathbb{N}$, which is a contradiction.

To give a simple concrete example, consider $M = [0, 1]$ and $m = 0$. \square

It follows that we cannot in general for $d, p \in \mathcal{M}$ decide whether $\rho_L(M_d, M_p) < \varepsilon$ just by computing $\rho_L((\mathbb{N}, d), (\mathbb{N}, p))$. For a correspondence $\mathcal{R} \subseteq \mathbb{N}^2$ and $n \in \mathbb{N}$ we denote by $n\mathcal{R}$ the set $\{m \in \mathbb{N} : n\mathcal{R}m\}$ and by $\mathcal{R}n$ the set $\{m \in \mathbb{N} : m\mathcal{R}n\}$.

Lemma 21. *Let $d, p \in \mathcal{M}$. Then $\rho_L(M_d, M_p) < r$ if and only if there exists $r' < r$ and a sequence of correspondences $\mathcal{R}_i \subseteq \mathbb{N} \times \mathbb{N}$ decreasing in inclusion such that*

- (1) *for every $\varepsilon > 0$ there exists $i \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ we have $p\text{-diam}(n\mathcal{R}_i) < \varepsilon$ and $d\text{-diam}(\mathcal{R}_i n) < \varepsilon$;*
- (2) *for every $i \in \mathbb{N}$ and every $n, m, n', m' \in \mathbb{N}$ such that $d(n, m) \geq 2^{-i}$ and $n\mathcal{R}_i n'$ and $m\mathcal{R}_i m'$ we have $p(n', m') \leq \exp(r')d(n, m)$;*
- (3) *for every $i \in \mathbb{N}$ and every $n, m, n', m' \in \mathbb{N}$ such that $p(n, m) \geq 2^{-i}$ and $n'\mathcal{R}_i n$ and $m'\mathcal{R}_i m$ we have $d(n', m') \leq \exp(r')p(n, m)$.*

Proof. For the implication from the right to the left, for every $n \in \mathbb{N}$ we define $\phi(n) \in M_p$ and $\psi(n) \in M_d$ as the unique element of $\bigcap_i \overline{n\mathcal{R}_i}$ and $\bigcap_i \overline{\mathcal{R}_i n}$, respectively. We leave to the reader to verify the simple fact that $\phi : \mathbb{N} \rightarrow M_p$ is a Lipschitz map with Lipschitz constant less than $\exp(r)$, which therefore extends to a Lipschitz map $\bar{\phi} : M_d \rightarrow M_p$ with the same Lipschitz constant, and if $\bar{\psi}$ is defined analogously, then $\bar{\phi} = (\bar{\psi})^{-1}$.

For the other implication, suppose that we are given a bi-Lipschitz map $\phi : M_d \rightarrow M_p$ such that $L := \max\{\text{Lip}(\phi), \text{Lip}(\phi^{-1})\} < \exp(r)$ and pick $\varepsilon > 0$ with $L + \varepsilon < \exp(r)$. For every $i \in \mathbb{N}$, put $\varepsilon_i := \frac{\varepsilon}{2^{i+1}(1+L)}$ and define correspondence \mathcal{R}_i by

$$\mathcal{R}_i := \{(n, n') \in \mathbb{N} \times \mathbb{N} : \exists \tilde{n} \in \mathbb{N} \quad d(n, \tilde{n}) < \varepsilon_i \ \& \ p(\phi(\tilde{n}), n') < \varepsilon_i\}.$$

We claim that the correspondences $(\mathcal{R}_i)_i$ are as desired. It is easy to see that $\mathcal{R}_i \subseteq \mathbb{N} \times \mathbb{N}$ are correspondences decreasing in inclusion and that (1) is satisfied. We check condition (2) and find the number r' , the condition (3) is checked similarly. Fix some $i \in \mathbb{N}$ and $n, m, n', m' \in \mathbb{N}$ with $n\mathcal{R}_i n'$, $m\mathcal{R}_i m'$ and $d(n, m) \geq 2^{-i}$. Let \tilde{n} and \tilde{m} be natural numbers witnessing that $n\mathcal{R}_i n'$ and $m\mathcal{R}_i m'$, respectively. Then we have

$$\begin{aligned} p(n', m') &\leq 2\varepsilon_i + p(\phi(\tilde{n}), \phi(\tilde{m})) \leq 2\varepsilon_i + Ld(\tilde{n}, \tilde{m}) \\ &\leq 2\varepsilon_i + L(2\varepsilon_i + d(n, m)) = d(n, m) \left(L + \frac{2\varepsilon_i(1+L)}{d(n, m)} \right) \\ &\leq d(n, m)(L + 2^{i+1}\varepsilon_i(1+L)) = d(n, m)(L + \frac{\varepsilon}{i}), \end{aligned}$$

so if we put $r' = \log(L + \varepsilon)$ we get that (2) holds and $r' < r$. \square

1.2.4. *Banach-Mazur distance.*

Definition 22 (Banach-Mazur distance). We recall that if X and Y are Banach spaces, their (logarithmic) *Banach-Mazur distance* is defined as

$$\rho_{BM}(X, Y) = \inf \{ \log \|T\| \|T^{-1}\| : T : X \rightarrow Y \text{ is a linear isomorphism} \}.$$

In contrast to the Lipschitz distance, Banach-Mazur distance can be verified just by looking at isomorphisms that are defined on some fixed countable dense linear subspaces over \mathbb{Q} . That is made precise in the following lemma.²

Lemma 23. *Let X and Y be separable Banach spaces, let $(e_n)_{n \in \mathbb{N}}$ and $(f_n)_{n \in \mathbb{N}}$ be linearly independent and linearly dense sequences in X and Y , respectively, and put $V = \mathbb{Q} \text{span}\{e_n : n \in \mathbb{N}\}$, $W = \mathbb{Q} \text{span}\{f_n : n \in \mathbb{N}\}$.*

Then $\rho_{BM}(X, Y) < r$ if and only if there exists a surjective linear isomorphism $T : X \rightarrow Y$ with $\log \|T\| \|T^{-1}\| < r$ and $T(V) = W$.

Throughout the proof of the lemma (including the following claim), by an isomorphism we mean a surjective linear isomorphism.

Claim 24. *Let $T : X \rightarrow Y$ be an isomorphism and $v_1, \dots, v_n, v \in V$ be such that $Tv_j \in W$ for $1 \leq j \leq n$. Then, given $\eta > 0$, there is an isomorphism $S : X \rightarrow Y$ such that*

- $\|S - T\| \leq \eta$ and $\|S^{-1} - T^{-1}\| \leq \eta$,
- $Sv_j = Tv_j$ for $1 \leq j \leq n$,
- $Sv \in W$.

Proof. We consider two cases.

(1) Assume that v does not belong to the linear span of v_1, \dots, v_n . In this case, there is $x^* \in X^*$ such that $x^*(v) = 1$ and $x^*(v_j) = 0$ for $1 \leq j \leq n$. Let $\varepsilon > 0$ be such that $\varepsilon \leq \eta$, $\varepsilon < \|T^{-1}\|^{-1}$, $(\|T^{-1}\|^{-1} - \varepsilon)^{-1} \cdot \|T^{-1}\| \cdot \varepsilon \leq \eta$ and every linear operator $S : X \rightarrow Y$ with $\|S - T\| \leq \varepsilon$ is an isomorphism (which is possible, because the set of isomorphisms is open). Let $w \in W$ be such that $\|w - Tv\| \leq \varepsilon/\|x^*\|$, and let

$$Sx = Tx + x^*(x) \cdot (w - Tv), \quad x \in X.$$

Clearly, $Sv_j = Tv_j$ for $j \leq n$ and $Sv = w \in W$. At the same time, $\|S - T\| \leq \|x^*\| \|w - Tv\| \leq \varepsilon \leq \eta$. Note that $\|Sx\| \geq \|Tx\| - \varepsilon\|x\| \geq (\|T^{-1}\|^{-1} - \varepsilon)\|x\|$ for $x \in X$, and that S is an isomorphism with $\|S^{-1}\| \leq (\|T^{-1}\|^{-1} - \varepsilon)^{-1}$ in particular. Finally, we obtain $\|S^{-1} - T^{-1}\| = \|S^{-1}(T - S)T^{-1}\| \leq \|S^{-1}\| \|T - S\| \|T^{-1}\| \leq (\|T^{-1}\|^{-1} - \varepsilon)^{-1} \cdot \varepsilon \cdot \|T^{-1}\| \leq \eta$.

(2) Assume that, on the other hand, v belongs to the linear span of v_1, \dots, v_n . We just need to check that v belongs to the \mathbb{Q} -linear span of v_1, \dots, v_n as well, since then clearly $Tv \in W$ and the choice $S = T$ works. There are a large enough $m \in \mathbb{N}$ and rational numbers q^i, q_j^i such that

$$v = \sum_{i=1}^m q^i e_i, \quad v_j = \sum_{i=1}^m q_j^i e_i.$$

²After proving the lemma, we were told by Gilles Godefroy that a similar statement is already in [16]

For some real numbers $\alpha_1, \dots, \alpha_n$, we have $v = \sum_{j=1}^n \alpha_j v_j$. That is,

$$\sum_{i=1}^m q^i e_i = \sum_{j=1}^n \alpha_j \sum_{i=1}^m q_j^i e_i = \sum_{i=1}^m \left(\sum_{j=1}^n q_j^i \alpha_j \right) e_i.$$

As e_1, \dots, e_n are assumed to be linearly independent, we obtain

$$\sum_{j=1}^n q_j^i \alpha_j = q^i, \quad i = 1, \dots, m.$$

Hence, the system of linear equations $\sum_{j=1}^n q_j^i x_j = q^i, i = 1, \dots, m$, has a solution. It follows from the methods of solving systems of linear equations that it has a solution β_1, \dots, β_n consisting of rational numbers. By a similar computation as above, we can obtain $v = \sum_{j=1}^n \beta_j v_j$. \square

Proof of Lemma 23. Let $T_0 : X \rightarrow Y$ be an isomorphism with $\|T_0\| \|T_0^{-1}\| < e^r$. Let us pick a small enough $\varepsilon > 0$ such that $(\|T_0\| + \varepsilon)(\|T_0^{-1}\| + \varepsilon) < e^r$. We are going to find sequences T_1, T_2, \dots of isomorphisms, x_1, x_2, \dots of points in V and y_1, y_2, \dots of points in W such that

- $\|T_k - T_{k-1}\| \leq 2^{-k} \varepsilon$ and $\|T_k^{-1} - T_{k-1}^{-1}\| \leq 2^{-k} \varepsilon$,
- $T_k e_j = y_j$ and $T_k^{-1} f_j = x_j$ for $j \leq k$.

Let us assume that $k \in \mathbb{N}$ and that we have already found T_j, x_j and y_j for $j < k$. Applying Claim 24, we obtain an isomorphism $\tilde{T}_{k-1} : X \rightarrow Y$ such that

- $\|\tilde{T}_{k-1} - T_{k-1}\| \leq 2^{-k-1} \varepsilon$ and $\|\tilde{T}_{k-1}^{-1} - T_{k-1}^{-1}\| \leq 2^{-k-1} \varepsilon$,
- $\tilde{T}_{k-1} e_j = T_{k-1} e_j$ for $j < k$ and $\tilde{T}_{k-1} x_j = T_{k-1} x_j$ for $j < k$,
- $\tilde{T}_{k-1} e_k \in W$.

Let us put $y_k = \tilde{T}_{k-1} e_k$. Applying Claim 24 once more, we obtain an isomorphism $S_k : Y \rightarrow X$ such that

- $\|S_k - \tilde{T}_{k-1}^{-1}\| \leq 2^{-k-1} \varepsilon$ and $\|S_k^{-1} - \tilde{T}_{k-1}\| \leq 2^{-k-1} \varepsilon$,
- $S_k f_j = \tilde{T}_{k-1}^{-1} f_j$ for $j < k$ and $S_k y_j = \tilde{T}_{k-1}^{-1} y_j$ for $j \leq k$,
- $S_k f_k \in V$.

Let us put $x_k = S_k f_k$ and $T_k = S_k^{-1}$. Let us check that the choice works.

We have $\|T_k - T_{k-1}\| = \|S_k^{-1} - T_{k-1}\| \leq \|S_k^{-1} - \tilde{T}_{k-1}\| + \|\tilde{T}_{k-1} - T_{k-1}\| \leq 2^{-k-1} \varepsilon + 2^{-k-1} \varepsilon = 2^{-k} \varepsilon$ and $\|T_k^{-1} - T_{k-1}^{-1}\| = \|S_k - T_{k-1}^{-1}\| \leq \|S_k - \tilde{T}_{k-1}^{-1}\| + \|\tilde{T}_{k-1}^{-1} - T_{k-1}^{-1}\| \leq 2^{-k-1} \varepsilon + 2^{-k-1} \varepsilon = 2^{-k} \varepsilon$. For $j < k$, we have $T_k e_j = S_k^{-1} \tilde{T}_{k-1}^{-1} \tilde{T}_{k-1} e_j = S_k^{-1} \tilde{T}_{k-1}^{-1} T_{k-1} e_j = S_k^{-1} \tilde{T}_{k-1}^{-1} y_j = S_k^{-1} S_k y_j = y_j$ and $T_k^{-1} f_j = S_k f_j = \tilde{T}_{k-1}^{-1} f_j = \tilde{T}_{k-1}^{-1} T_{k-1} T_{k-1}^{-1} f_j = \tilde{T}_{k-1}^{-1} T_{k-1} x_j = \tilde{T}_{k-1}^{-1} \tilde{T}_{k-1} x_j = x_j$. Finally, $T_k e_k = S_k^{-1} \tilde{T}_{k-1}^{-1} \tilde{T}_{k-1} e_k = S_k^{-1} \tilde{T}_{k-1}^{-1} y_k = S_k^{-1} S_k y_k = y_k$ and $T_k^{-1} f_k = S_k f_k = x_k$.

So, the sequences T_k, x_k and y_k are found. Clearly, the sequence T_0, T_1, \dots is Cauchy and has a limit T with $\|T - T_0\| \leq \sum_{k=1}^{\infty} 2^{-k} \varepsilon = \varepsilon$. Similarly, the sequence $T_0^{-1}, T_1^{-1}, \dots$ has a limit S with $\|S - T_0^{-1}\| \leq \varepsilon$. Moreover, $TS = \lim_{k \rightarrow \infty} T_k T_k^{-1} = \lim_{k \rightarrow \infty} I = I$, and so T is an isomorphism with $T^{-1} = S$. It follows that

$$\|T\| \|T^{-1}\| \leq (\|T_0\| + \varepsilon)(\|T_0^{-1}\| + \varepsilon) < e^r.$$

At the same time, $Te_j = y_j \in W$ and $T^{-1}f_j = x_j \in V$ for every j . Hence, we arrive at $T(V) = W$. \square

The last three distances we shall present are all related to the coarse (or large scale) geometry of metric (and Banach) spaces. We refer the reader to [5, Chapter 8] or the monograph [24] for an introduction into this subject.

1.2.5. *Hausdorff-Lipschitz and net distances.* Gromov defines in [17, Definition 3.19] a distance defined as some variation of both the Gromov-Hausdorff and Lipschitz distances.

Definition 25 (Hausdorff-Lipschitz distance). For metric spaces M and N , their *Hausdorff-Lipschitz distance* is defined as

$$\rho_{HL}(M, N) = \inf \{ \rho_{GH}(M, M') + \rho_L(M', N') + \rho_{GH}(N', N) : M', N' \text{ metric spaces} \}.$$

The Hausdorff-Lipschitz distance corresponds to the notion of *quasi-isometry* or *coarse Lipschitz equivalence*, because for metric spaces M and N we have $\rho_{HL}(M, N) < \infty$ if and only if the spaces M and N are quasi-isometric, or coarse Lipschitz equivalent (see e.g. [5, Section 8.3] for further information). For information about coarse geometry of Banach spaces we refer to the survey [21] or the monograph [26].

Following [3, Definition 10.18], by an (a, b) -net in a metric space M , where a, b are positive reals, we mean a subset $\mathcal{N} \subseteq M$ such that for every $m \neq n \in \mathcal{N}$ we have $d(m, n) \geq a$, and for every $x \in M$ there exists $n \in \mathcal{N}$ with $d(x, n) < b$. If the constants a and b are not important, we just call the subset \mathcal{N} a *net*. Observe that a maximal ε -separated subset $\mathcal{N} \subseteq M$ (which exists by Zorn's lemma) is an $(\varepsilon, \varepsilon)$ -net. Dutrieux and Kalton [10] consider the net distance which we define as follows (let us note that a slightly different definition of ρ_L is used in [10]).

Definition 26 (Net distance). The *net distance* between two Banach spaces X and Y is defined as

$$\rho_N(X, Y) = \inf \{ \rho_L(\mathcal{N}_X, \mathcal{N}_Y) : \mathcal{N}_X, \mathcal{N}_Y \text{ are nets in } X, Y \text{ respectively} \}.$$

The next observation is in a sense quantitative version of [5, Proposition 8.3.4], where it is proved that two metric spaces are quasi-isometric if and only if they have Lipschitz equivalent nets.

Proposition 27. *For Banach spaces X and Y we have $\rho_N(X, Y) = \rho_{HL}(X, Y)$.*

Proof. Fix Banach spaces X and Y and a positive real K . Suppose that $\rho_N(X, Y) < K$. So there exist (a, b) -net $\mathcal{N}_X \subseteq X$ and (a', b') -net $\mathcal{N}_Y \subseteq Y$ and a bi-Lipschitz map $T : \mathcal{N}_X \rightarrow \mathcal{N}_Y$ with $\log \max\{\text{Lip}(T), \text{Lip}(T^{-1})\} < K$. Take any $\varepsilon > 0$. By rescaling the nets \mathcal{N}_X and \mathcal{N}_Y by a sufficiently large constant C if necessary, that is, taking $\mathcal{N}_X/C = \{x/C : x \in \mathcal{N}_X\}$ and \mathcal{N}_Y/C , we may suppose that the nets \mathcal{N}_X and \mathcal{N}_Y are (a, ε) -net, resp. (a', ε) -net. Then we clearly have $\rho_{GH}(X, \mathcal{N}_X) \leq \varepsilon$ and $\rho_{GH}(Y, \mathcal{N}_Y) \leq \varepsilon$, so

$$\rho_{HL}(X, Y) \leq \rho_{GH}(X, \mathcal{N}_X) + \rho_L(\mathcal{N}_X, \mathcal{N}_Y) + \rho_{GH}(\mathcal{N}_Y, Y) < K + 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it shows that $\rho_{HL}(X, Y) \leq K$.

Conversely, suppose that $\rho_{HL}(X, Y) < K$. So there exist metric spaces X' and Y' such that $\rho_{GH}(X, X') + \rho_L(X', Y') + \rho_{GH}(Y', Y) < K$. By Fact 10

there are correspondences $\mathcal{R}_X \subseteq X \times X'$ and $\mathcal{R}_Y \subseteq Y' \times Y$ witnessing that $\rho_{GH}(X, X') < K$ and $\rho_{GH}(Y', Y) < K$. Let $C > 0$ be a sufficiently large constant, more precisely specified later, and find some C -maximal separated set \mathcal{N}_X in X , which is therefore a (C, C) -net. Since C is large, for every $n \neq m \in \mathcal{N}_X$ we have that $\{x \in X' : n\mathcal{R}_X x\} \cap \{x \in X' : m\mathcal{R}_X x\} = \emptyset$, so we pick some injective map $f_1 : \mathcal{N}_X \rightarrow X'$ such that for every $n \in \mathcal{N}_X$ we have $n\mathcal{R}_X f_1(n)$. Since $\rho_L(X', Y') < K$ there exists a bi-Lipschitz map $T : X' \rightarrow Y'$ with $\max\{\text{Lip}(T), \text{Lip}(T^{-1})\} < \exp(K)$. Again since C is large enough it follows that for every $n \neq m \in \mathcal{N}_X$ we have that $\{y \in Y : (T \circ f_1)(n)\mathcal{R}_Y y\} \cap \{y \in Y : (T \circ f_1)(m)\mathcal{R}_Y y\} = \emptyset$, so we pick some injective map $f_2 : (T \circ f_1)[\mathcal{N}_X] \rightarrow Y$ such that for every $z \in (T \circ f_1)[\mathcal{N}_X]$ we have $z\mathcal{R}_Y f_2(z)$. Set $\phi = f_2 \circ T \circ f_1 : \mathcal{N}_X \rightarrow Y$. It follows the range of ϕ is a net \mathcal{N}_Y in Y . Let us compute the Lipschitz constant of ϕ and ϕ^{-1} . For any $n \neq m \in \mathcal{N}_X$ we have

$$\begin{aligned} \|\phi(n) - \phi(m)\|_Y &\leq d_{Y'}((T \circ f_1)(n), (T \circ f_1)(m)) + 2K \\ &< \exp(K)d_{X'}(f_1(n), f_1(m)) + 2K \\ &\leq \exp(K)(\|n - m\|_X + 2K) + 2K \\ &\leq \left(\exp(K) + \frac{2K(\exp(K)+1)}{C} \right) \|n - m\|_X. \end{aligned}$$

However, $\frac{2K(\exp(K)+1)}{C} \rightarrow 0$ as $C \rightarrow \infty$. The computation of $\text{Lip}(\phi^{-1})$ is analogous, so we get that $\rho_N(X, Y) \leq K$, and we are done. \square

Remark 28. Note that in Proposition 27 the only geometric property of Banach spaces that we used in the proof is that any rescaling of a Banach space X is isometric to X . Spaces with this property are called *cones* [5, Definition 8.2.1]. So we have proved that if ρ_N was defined in an obvious way on metric spaces, it would coincide with ρ_{HL} on cones.

Our next result shows it is possible to express the Hausdorff-Lipschitz distance, up to uniform equivalence, in terms of correspondences. This observation will be used further.

Definition 29. Let $d, e \in \mathcal{M}$ and $\varepsilon > 0$. We say that d and e are $HL(\varepsilon)$ -close if there exists a correspondence $\mathcal{R} \subseteq \mathbb{N} \times \mathbb{N}$ such that for every $i, i', j, j' \in \mathbb{N}$ with $i\mathcal{R}j$ and $i'\mathcal{R}j'$ we have

$$e(j, j') \leq d(i, i') + \varepsilon \cdot \max\{1, d(i, i')\}, \quad (1)$$

$$d(i, i') \leq e(j, j') + \varepsilon \cdot \max\{1, e(j, j')\}. \quad (2)$$

Lemma 30. *There are continuous functions $\varphi_i : (0, \infty) \rightarrow (0, \infty)$, $i \in \{1, 2\}$, such that $\lim_{\varepsilon \rightarrow 0} \varphi_i(\varepsilon) = 0$ and, whenever $d, e \in \mathcal{M}$ and $\varepsilon > 0$ are given, we have*

$$\rho_{HL}(d, e) < \varepsilon \Rightarrow d \text{ and } e \text{ are } HL(\varphi_1(\varepsilon))\text{-close};$$

$$d \text{ and } e \text{ are } HL(\varepsilon)\text{-close} \Rightarrow \rho_{HL}(d, e) < \varphi_2(\varepsilon).$$

Proof. First, let us assume that $\rho_{HL}(d, e) < \varepsilon$, that is, there are $d', e' \in \mathcal{M}$ with $\rho_{GH}(d, d') + \rho_L(d', e') + \rho_{GH}(e', e) < \varepsilon$. By Fact 10, there are correspondences $\mathcal{R}_1 \subseteq \mathbb{N} \times \mathbb{N}$ and $\mathcal{R}_3 \subseteq \mathbb{N} \times \mathbb{N}$ witnessing that $\rho_{GH}(d, d') < \varepsilon$

and $\rho_{GH}(e', e) < \varepsilon$. Further, let $f : M_{d'} \rightarrow M_{e'}$ be a bi-Lipschitz bijection witnessing that $\rho_L(d', e') < \varepsilon$. Consider now the correspondence

$$\mathcal{R} := \{(i, j) \in \mathbb{N} \times \mathbb{N} : \text{there are } k, l \in \mathbb{N} \text{ such that } (i, l) \in \mathcal{R}_1, (k, j) \in \mathcal{R}_3, \\ d'(l, f^{-1}(k)) < \varepsilon \text{ and } e'(f(l), k) < \varepsilon\}.$$

This is indeed a correspondence since given $i \in \mathbb{N}$ we find l with $(i, l) \in \mathcal{R}_1$, pick $k \in \mathbb{N}$ with $e'(f(l), k) < \min\{\varepsilon, \frac{\varepsilon}{\text{Lip}(f^{-1})}\}$ and find j with $(k, j) \in \mathcal{R}_3$; thus, we have $(i, j) \in \mathcal{R}$ and similarly for every $j \in \mathbb{N}$ there is i with $(i, j) \in \mathcal{R}$.

Fix $i, i', j, j' \in \mathbb{N}$ with $i\mathcal{R}j$ and $i'\mathcal{R}j'$. Then there are $l, l' \in \mathbb{N}$ and $k, k' \in \mathbb{N}$ with $i\mathcal{R}_1l, i'\mathcal{R}_1l', k\mathcal{R}_3j, k'\mathcal{R}_3j', d'(l, f^{-1}(k)) < \varepsilon, e'(f(l), k) < \varepsilon, d'(l', f^{-1}(k')) < \varepsilon$ and $e'(f(l'), k') < \varepsilon$. We have

$$\begin{aligned} d(i, i') &\leq d'(l, l') + 2\varepsilon \leq d'(f^{-1}(k), f^{-1}(k')) + 4\varepsilon \leq \text{Lip}(f^{-1})e'(k, k') + 4\varepsilon \\ &\leq \exp(\varepsilon)(e(j, j') + 2\varepsilon) + 4\varepsilon \\ &= e(j, j') + (\exp(\varepsilon) - 1)e(j, j') + 2\varepsilon(\exp(\varepsilon) + 2). \end{aligned}$$

By symmetry, similar inequality holds when the roles of d and e are changed. Hence, if $\varphi_1(\varepsilon) = \exp(\varepsilon) - 1 + 2\varepsilon\exp(\varepsilon) + 4\varepsilon$, then d and e are $HL(\varphi_1(\varepsilon))$ -close.

Conversely, let $\mathcal{R} \subseteq \mathbb{N} \times \mathbb{N}$ be a correspondence witnessing that d and e are $HL(\varepsilon)$ -close. Put $\delta = \varepsilon + \sqrt{\varepsilon}$. Let \mathcal{N}_d be a maximal δ -separated set in (\mathbb{N}, d) . For every $i \in \mathcal{N}_d$, we pick some $r(i) \in \mathbb{N}$ such that $i\mathcal{R}r(i)$. Then we put $\mathcal{N}_e := \{r(i) : i \in \mathcal{N}_d\}$. Clearly,

$$\rho_{GH}((\mathbb{N}, d), \mathcal{N}_d) \leq \delta.$$

We claim that for every $j \in \mathbb{N}$ there is $j' \in \mathcal{N}_e$ with $e(j, j') < \delta + \varepsilon \cdot \max\{1, \delta\}$, which gives

$$\rho_{GH}((\mathbb{N}, e), \mathcal{N}_e) \leq \delta + \varepsilon \cdot \max\{1, \delta\}.$$

Indeed, if $j \in \mathbb{N}$ is given, there is i with $i\mathcal{R}j$. Pick $i' \in \mathcal{N}_d$ with $d(i, i') < \delta$. Using (1), we obtain $e(j, r(i')) < \delta + \varepsilon \cdot \max\{1, \delta\}$.

Now, let us compute the Lipschitz constant for r and r^{-1} . Consider $i, i' \in \mathcal{N}_d, i \neq i'$. If $d(i, i') \geq 1$, by (1), we get $e(r(i), r(i')) \leq (1 + \varepsilon)d(i, i')$. If $d(i, i') \leq 1$, by (1) and using that \mathcal{N}_d is δ -separated, we get $e(r(i), r(i')) \leq d(i, i') + \varepsilon \leq (1 + \frac{\varepsilon}{\delta})d(i, i')$. Hence, $\text{Lip}(r) \leq \max\{1 + \varepsilon, 1 + \frac{\varepsilon}{\delta}\}$. Note that for every $k, k' \in \mathcal{N}_d, k \neq k'$, with $e(r(k), r(k')) \leq 1$, by (2), we have $e(r(k), r(k')) \geq d(k, k') - \varepsilon \geq \delta - \varepsilon$; hence, similar computation gives $\text{Lip}(r^{-1}) \leq \max\{1 + \varepsilon, 1 + \frac{\varepsilon}{\delta - \varepsilon}\} = 1 + \max\{\varepsilon, \sqrt{\varepsilon}\}$. Thus, we have $\rho_L(\mathcal{N}_d, \mathcal{N}_e) \leq \log(1 + \max\{\varepsilon, \sqrt{\varepsilon}\})$. Finally, if

$$\varphi_2(\varepsilon) = 2\varepsilon + 2\sqrt{\varepsilon} + \log(1 + \max\{\varepsilon, \sqrt{\varepsilon}\}) + \varepsilon \cdot \max\{1, \varepsilon + \sqrt{\varepsilon}\},$$

we get

$$\rho_{HL}(d, e) \leq \delta + \log(1 + \max\{\varepsilon, \sqrt{\varepsilon}\}) + \delta + \varepsilon \cdot \max\{1, \delta\} = \varphi_2(\varepsilon).$$

□

1.2.6. *Uniform distance.* The following definition comes from [10]³.

Definition 31 (Uniform distance). Let X and Y be Banach spaces. If $u : X \rightarrow Y$ is uniformly continuous, we put

$$\text{Lip}_\infty u := \inf_{\eta > 0} \sup \left\{ \frac{\|u(x) - u(y)\|}{\|x - y\|} : \|x - y\| \geq \eta \right\}.$$

The *uniform distance* between X and Y is defined as

$$\rho_U(X, Y) = \inf \left\{ \log((\text{Lip}_\infty u)(\text{Lip}_\infty u^{-1})) : u : X \rightarrow Y \text{ is uniform homeomorphism} \right\}.$$

Let us note the easy fact that we have

$$\text{Lip}_\infty u = \inf \left\{ A > 0 : \exists B > 0 \forall x, y \in X : \|u(x) - u(y)\| \leq A\|x - y\| + B \right\}.$$

The following is an analogue of Lemma 21.

Lemma 32. *Let $\mu, \nu \in \mathcal{B}$. Then $\rho_U(X_\mu, X_\nu) < r$ if and only if there exist $B > 0$, $r' \in (0, r)$ and a sequence of correspondences $\mathcal{R}_i \subseteq V \times V$ decreasing in inclusion such that*

- (1) *for every $i \in \mathbb{N}$ and every $v, w, v', w' \in V$ such that $v\mathcal{R}_i w$ and $v'\mathcal{R}_i w'$ we have $\nu(w - w') \leq \exp(r')\mu(v - v') + B$;*
- (2) *for $i \in \mathbb{N}$ and every $v, w, v', w' \in V$ such that $v\mathcal{R}_i w$ and $v'\mathcal{R}_i w'$ we have $\mu(v - v') \leq \nu(w - w') + B$;*
- (3) *for every $\varepsilon > 0$ there exist $\delta > 0$ and $i \in \mathbb{N}$ such that for every $v, v' \in V$ with $\mu(v - v') < \delta$ we have $\nu(w - w') < \varepsilon$ whenever $v\mathcal{R}_i w$ and $v'\mathcal{R}_i w'$;*
- (4) *for every $\varepsilon > 0$ there exist $\delta > 0$ and $i \in \mathbb{N}$ such that for every $w, w' \in V$ with $\nu(w - w') < \delta$ we have $\mu(v - v') < \varepsilon$ whenever $v\mathcal{R}_i w$ and $v'\mathcal{R}_i w'$.*

Proof. For the implication from the right to the left, for every $n \in \mathbb{N}$ we define $\phi(n) \in X_\nu$ and $\psi(n) \in X_\mu$ as the unique element of $\bigcap_i n\mathcal{R}_i$ and $\bigcap_i \overline{\mathcal{R}_i}n$, respectively. We leave to the reader to verify the simple fact that $\phi : (V, \mu) \rightarrow X_\nu$ is a uniformly continuous map with $\text{Lip}_\infty \phi \leq \exp(r')$, which therefore extends to a uniformly continuous map $\bar{\phi} : X_\mu \rightarrow X_\nu$ and if $\bar{\psi}$ is defined analogously, then $\bar{\phi} = (\bar{\psi})^{-1}$ and $\text{Lip}_\infty \bar{\phi} \text{Lip}_\infty \bar{\psi} < \exp(r)$.

For the other implication, suppose that we are given a uniform homeomorphism $u : X_\mu \rightarrow X_\nu$ such that $\text{Lip}_\infty u^{-1} = 1$ and $\text{Lip}_\infty u < \exp(r')$ for some $r' < r$. For every $i \in \mathbb{N}$ define correspondence \mathcal{R}_i by

$$\mathcal{R}_i := \left\{ (v, w) \in V \times V : \exists \tilde{v} \in V \quad \mu(v - \tilde{v}) < \frac{1}{i} \ \& \ \nu(u(\tilde{v}) - w) < \frac{1}{i} \right\}.$$

It is straightforward to check that $\mathcal{R}_i \subseteq \mathbb{N} \times \mathbb{N}$ are correspondences decreasing in inclusion satisfying all the conditions from the lemma. We omit further details, because this is similar to the proof of Lemma 21. \square

1.3. Analytic pseudometrics and reductions between them. Here we recall the notions introduced in [8], where we refer the interested reader for more information.

³More precisely, they define it without the logarithm which we add in order to satisfy the triangle inequality.

Definition 33. Let X be a standard Borel space. A pseudometric $\rho : X \times X \rightarrow [0, \infty]$ is called *an analytic pseudometric*, resp. *a Borel pseudometric*, if for every $r > 0$ the set $\{(x, y) \in X^2 : \rho(x, y) < r\}$ is analytic, resp. Borel.

We emphasize that pseudometrics in our definition may attain ∞ as a value.

Let us observe that distances considered in the previous subsection may be thought of as examples of analytic pseudometrics.

1. Gromov-Hausdorff distance Equip the Polish space \mathcal{M} with the Gromov-Hausdorff distance ρ_{GH} defined in Definition 9. We also consider the pseudometric ρ_{GH} on the space \mathcal{B} of codes for separable Banach spaces, denoted there by $\rho_{GH}^{\mathcal{B}}$. Note that for Banach spaces X and Y , $\rho_{GH}^{\mathcal{B}}(X, Y)$ is defined as the Gromov-Hausdorff distance of the unit balls B_X and B_Y (see e.g. the introduction in [18]). Both ρ_{GH} and $\rho_{GH}^{\mathcal{B}}$ are analytic, see [8, Proposition 17].

2. Kadets distance Equip the Polish space \mathcal{B} with the Kadets distance ρ_K defined in Definition 16. Using Lemma 17, it is not difficult to check that ρ_K is analytic on \mathcal{B} . We leave the details to the reader.

3. Lipschitz distance Equip the Polish spaces \mathcal{M} and \mathcal{B} with the Lipschitz distance ρ_L introduced in Definition 18, where for $d, p \in \mathcal{M}$ and $\mu, \nu \in \mathcal{B}$ by $\rho_L(d, p)$ and $\rho_L(\mu, \nu)$ we understand $\rho_L(M_d, M_p)$ and $\rho_L(X_\mu, X_\nu)$, respectively. We leave it to the reader to verify, using Lemma 21, that ρ_L is analytic on \mathcal{M} as well as on \mathcal{B} . Whenever we consider the pseudometric ρ_L on \mathcal{B} and we want to emphasize it, we write $\rho_L^{\mathcal{B}}$ instead of just ρ_L .

4. Banach-Mazur distance Equip the Polish space \mathcal{B} by the Banach-Mazur distance ρ_{BM} defined in Definition 22. We leave it to the reader to verify, using Lemma 23, that ρ_{BM} is an analytic pseudometric on \mathcal{B} .

5. Hausdorff-Lipschitz and net distances Equip the Polish spaces \mathcal{M} and \mathcal{B} with the Hausdorff-Lipschitz distance ρ_{HL} from Definition 25. It is easy to check that for $d, p \in \mathcal{M}$ we then have

$$\rho_{HL}(d, p) = \inf\{\rho_{GH}(d, e_1) + \rho_L(e_1, e_2) + \rho_{GH}(e_2, p) : e_1, e_2 \in \mathcal{M}\}.$$

Analogously, for elements from \mathcal{B} . It therefore follows from the fact that ρ_{GH} and ρ_L are analytic that ρ_{HL} is analytic as well.

Moreover, equip the Polish space \mathcal{B} with the net distance ρ_N from Definition 26. It is clearly analytic as it coincides there with ρ_{HL} .

6. Uniform distance Equip the Polish space \mathcal{B} with the uniform distance ρ_U from Definition 31. We leave it to the reader to verify, using Lemma 32, that ρ_U is an analytic pseudometric on \mathcal{B} .

Now we recall the notion of reducibility between analytic pseudometrics as is was introduced in [8].

Definition 34. Let X , resp. Y be standard Borel spaces and let ρ_X , resp. ρ_Y be analytic pseudometrics on X , resp. on Y . We say that ρ_X is *Borel-uniformly continuous reducible* to ρ_Y , $\rho_X \leq_{B,u} \rho_Y$ in symbols, if there exists a Borel function $f : X \rightarrow Y$ such that, for every $\varepsilon > 0$ there are $\delta_X > 0$ and $\delta_Y > 0$ satisfying

$$\forall x, y \in X : \quad \rho_X(x, y) < \delta_X \Rightarrow \rho_Y(f(x), f(y)) < \varepsilon$$

and

$$\forall x, y \in X : \quad \rho_Y(f(x), f(y)) < \delta_Y \Rightarrow \rho_X(x, y) < \varepsilon.$$

In this case we say that f is a *Borel-uniformly continuous reduction*. If $\rho_X \leq_{B,u} \rho_Y$ and $\rho_Y \leq_{B,u} \rho_X$, we say that ρ_X is *Borel-uniformly continuous bi-reducible* with ρ_Y and write $\rho_X \sim_{B,u} \rho_Y$.

Moreover, if f is injective we say it is an *injective Borel-uniformly continuous reduction*.

If f is an isometry from the pseudometric space (X, ρ_X) into (Y, ρ_Y) , we say it is a *Borel-isometric reduction*.

If there is $C > 0$ such that for every $x, y \in X$ we have

$$\rho_Y(f(x), f(y)) \leq C\rho_X(x, y) \quad \text{and} \quad \rho_X(x, y) \leq C\rho_Y(f(x), f(y)),$$

we say that f is a *Borel-Lipschitz reduction*.

If there are $\varepsilon > 0$ and $C > 0$ such that for every $x, y \in X$ we have

$$\begin{aligned} \rho_X(x, y) < \varepsilon &\implies \rho_Y(f(x), f(y)) \leq C\rho_X(x, y) \\ \text{and } \rho_Y(f(x), f(y)) < \varepsilon &\implies \rho_X(x, y) \leq C\rho_Y(f(x), f(y)), \end{aligned}$$

we say that f is a *Borel-Lipschitz on small distances reduction*.

The definition of a Borel-uniformly continuous reduction seems to be the most useful one. Sometimes we are able to demonstrate the reducibility between some pseudometrics by maps with stronger properties and this is the reason why we mentioned the remaining notions above.

2. REDUCTIONS BETWEEN PSEUDOMETRICS ON SPACES OF METRIC SPACES

In this section we prove the reducibility results between pseudometrics on the spaces of metric spaces.

By [8, Theorem 11], we have the following.

Theorem 35. *Let $0 < p < q$. The following pseudometrics are mutually Borel-uniformly continuous bi-reducible: ρ_{GH} , $\rho_{GH} \upharpoonright \mathcal{M}_p$, $\rho_{GH} \upharpoonright \mathcal{M}^q$, $\rho_{GH} \upharpoonright \mathcal{M}_p^q$.*

Next, we show that Lipschitz and Gromov-Hausdorff distances are Borel-uniformly continuous bi-reducible.

Theorem 36. *Fix real numbers $p < q$. Then the identity map on \mathcal{M}_p^q is a Borel-uniformly continuous reduction from ρ_{GH} to ρ_L and from ρ_L to ρ_{GH} .*

Moreover, the identity is not only Borel-uniformly continuous, but also Borel-Lipschitz on small distances.

Proof. Take some $d, e \in \mathcal{M}_p^q$ and suppose that $\rho_{GH}(d, e) < \varepsilon < p/2$. By Lemma 13 there is a permutation $\pi \in S_\infty$ witnessing that $d \simeq_{2\varepsilon} e$. The permutation π also defines a bi-Lipschitz map between (\mathbb{N}, d) and (\mathbb{N}, e) . Let us compute the Lipschitz constant of π . We have

$$\text{Lip}(\pi) = \sup_{m \neq n \in \mathbb{N}} \frac{e(\pi(m), \pi(n))}{d(m, n)} \leq \sup_{m \neq n \in \mathbb{N}} \frac{d(m, n) + 2\varepsilon}{d(m, n)} \leq 1 + \frac{2\varepsilon}{p}.$$

The same argument shows that $\text{Lip}(\pi^{-1}) \leq 1 + \frac{2\varepsilon}{p}$ and so we have $\rho_L(d, e) \leq \log(1 + \frac{2\varepsilon}{p}) \leq \frac{2\varepsilon}{p}$.

Conversely, suppose that $\rho_L(d, e) < \varepsilon < 1$. Then there is a bi-Lipschitz map $\pi : (\mathbb{N}, d) \rightarrow (\mathbb{N}, e)$ such that $\max\{\text{Lip}(\pi), \text{Lip}(\pi^{-1})\} \leq 1 + \delta$, where $1 + \delta < \exp(\varepsilon)$. So for any $m, n \in \mathbb{N}$ we have

$$\begin{aligned} |d(m, n) - e(\pi(m), \pi(n))| &\leq \max\{|(1 + \delta)e(\pi(m), \pi(n)) - e(\pi(m), \pi(n))|, \\ &|(1 + \delta)d(m, n) - d(m, n)|\} = \max\{\delta e(\pi(m), \pi(n)), \delta d(m, n)\} \leq \delta q. \end{aligned}$$

Thus, we have $d \simeq_{\delta q} e$ and, by Lemma 12, $\rho_{GH}(d, e) \leq \frac{\delta q}{2} < \frac{q(\exp(\varepsilon)-1)}{2} \leq \frac{q(\exp(1)-1)}{2} \varepsilon$. \square

Theorem 37. *There is an injective Borel-uniformly continuous reduction from ρ_L on \mathcal{M} to ρ_{GH} on \mathcal{M} .*

Moreover, the reduction is not only Borel-uniformly continuous, but also Borel-Lipschitz on small distances.

Proof. For every $d \in \mathcal{M}$, we define a metric \tilde{d} on $(\mathbb{N} \times \mathbb{Z}) \cup \{\clubsuit\}$ by

$$\begin{aligned} \tilde{d}((i, k), (j, l)) &= |10k - 10l| + \min\{1, 2^{\min\{k, l\}}d(i, j)\}, \\ \tilde{d}((i, k), \clubsuit) &= |10k + 4| + 1. \end{aligned}$$

We leave it to the reader to verify the elementary fact that \tilde{d} is a metric.

Since $d \mapsto \tilde{d}$ is an injective continuous mapping from \mathcal{M} into $\mathbb{R}^{((\mathbb{N} \times \mathbb{Z}) \cup \{\clubsuit\})^2}$, it is easy to show that there is an injective continuous mapping $f : \mathcal{M} \rightarrow \mathcal{M}$ such that $M_{f(d)}$ is isometric to the completion of $((\mathbb{N} \times \mathbb{Z}) \cup \{\clubsuit\}, \tilde{d})$. Hence, to prove the theorem, it is sufficient to show that

$$\rho_{GH}(\tilde{d}, \tilde{e}) \leq (\exp \rho_L(d, e)) - 1$$

and

$$\rho_{GH}(\tilde{d}, \tilde{e}) < 1/4 \quad \Rightarrow \quad \rho_L(d, e) \leq \log(1 + 24\rho_{GH}(\tilde{d}, \tilde{e}))$$

for every $d, e \in \mathcal{M}$.

Assume that $(\exp \rho_L(d, e)) - 1 < \varepsilon$ and pick some $L : M_d \rightarrow M_e$ with $\text{Lip } L < 1 + \varepsilon$ and $\text{Lip } L^{-1} < 1 + \varepsilon$. We define a correspondence

$$\mathcal{R} = \{(\clubsuit, \clubsuit)\} \cup \{((i, k), (j, k)) : d(i, L^{-1}(j)) < 2^{-k-1}\varepsilon, e(L(i), j) < 2^{-k-1}\varepsilon\}.$$

Our aim is to show that $|\tilde{d}(a, a') - \tilde{e}(b, b')| < 2\varepsilon$ whenever $a\mathcal{R}b$ and $a'\mathcal{R}b'$. In the case that $a = a' = \clubsuit$ (and so $b = b' = \clubsuit$), this is trivial. Assume that $a \neq \clubsuit = a'$ (and so $b \neq \clubsuit = b'$), and denote $a = (i, k), b = (j, k)$. We want to show that $||10k + 4| + 1 - |10k + 4| - 1| < 2\varepsilon$, which is obvious. We obtain the same conclusion in the case $a = \clubsuit \neq a'$. So, assume that $a \neq \clubsuit \neq a'$ (and so $b \neq \clubsuit \neq b'$), and denote $a = (i, k), b = (j, k), a' = (i', k'), b' = (j', k')$. We want to show that $||10k - 10k'| + \min\{1, 2^{\min\{k, k'}\}}d(i, i')\} - |10k - 10k'| - \min\{1, 2^{\min\{k, k'}\}}e(j, j')\}| < 2\varepsilon$, which can be slightly simplified to

$$|\min\{1, 2^{\min\{k, k'}\}}d(i, i')\} - \min\{1, 2^{\min\{k, k'}\}}e(j, j')\}| < 2\varepsilon.$$

Due to the symmetry, it is sufficient to show that the number under the absolute value is less than 2ε . If $1 \leq 2^{\min\{k, k'}\}}e(j, j')$, then we just write

$\min\{1, 2^{\min\{k, k'\}}d(i, i')\} - 1 \leq 0 < 2\varepsilon$. In the opposite case that $1 > 2^{\min\{k, k'\}}e(j, j')$, we write

$$\begin{aligned} d(i, i') &\leq d(L^{-1}(j), L^{-1}(j')) + d(i, L^{-1}(j)) + d(i', L^{-1}(j')) \\ &< (1 + \varepsilon)e(j, j') + 2^{-k-1}\varepsilon + 2^{-k'-1}\varepsilon \\ &\leq (1 + \varepsilon)e(j, j') + 2^{-\min\{k, k'\}-1}\varepsilon + 2^{-\min\{k, k'\}-1}\varepsilon \end{aligned}$$

and

$$\begin{aligned} &\min\{1, 2^{\min\{k, k'\}}d(i, i')\} - \min\{1, 2^{\min\{k, k'\}}e(j, j')\} \\ &\leq 2^{\min\{k, k'\}}d(i, i') - 2^{\min\{k, k'\}}e(j, j') \\ &< 2^{\min\{k, k'\}}(1 + \varepsilon)e(j, j') + 2^{-1}\varepsilon + 2^{-1}\varepsilon - 2^{\min\{k, k'\}}e(j, j') \\ &= 2^{\min\{k, k'\}}\varepsilon \cdot e(j, j') + \varepsilon < \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

By Fact 10, this completes the verification of $\rho_{GH}(\tilde{d}, \tilde{e}) \leq \varepsilon$.

Now, assume that $\rho_{GH}(\tilde{d}, \tilde{e}) < \varepsilon < 1/4$ for some $d, e \in \mathcal{M}$. This is witnessed by a correspondence $\mathcal{R} \subseteq [(\mathbb{N} \times \mathbb{Z}) \cup \{\clubsuit\}]^2$ provided by Fact 10. We verify first that $\clubsuit\mathcal{R}\clubsuit$, showing that there is no m with $m\mathcal{R}\clubsuit$ but \clubsuit . If $m\mathcal{R}\clubsuit$, then there are points which have distance to m in $(5 - 2\varepsilon, 5 + 2\varepsilon)$ and in $(7 - 2\varepsilon, 7 + 2\varepsilon)$. If two points have distance in $(3, 7)$, then these points belong to $(\mathbb{N} \times \{0\}) \cup \{\clubsuit\}$. As $(5 - 2\varepsilon, 5 + 2\varepsilon) \subseteq (3, 7)$, we obtain $m \in (\mathbb{N} \times \{0\}) \cup \{\clubsuit\}$. The case $m \in \mathbb{N} \times \{0\}$ is also excluded, since distances of these points to other points belong to the set $[0, 1] \cup \{5\} \cup [10, \infty)$, which is disjoint from $(7 - 2\varepsilon, 7 + 2\varepsilon)$.

It follows that

$$(i, k)\mathcal{R}(j, l) \Rightarrow k = l. \quad (3)$$

Indeed, we have $2 > 2\varepsilon > |\tilde{d}((i, k), \clubsuit) - \tilde{e}((j, l), \clubsuit)| = ||10k + 4| - |10l + 4||$, and this is possible only if $k = l$.

By (3), the relations

$$\mathcal{R}_k = \{(i, j) \in \mathbb{N}^2 : (i, k)\mathcal{R}(j, k)\}, \quad k \in \mathbb{Z},$$

are correspondences. Let us show that

$$i\mathcal{R}_k j, i'\mathcal{R}_k j' \ \& \ d(i, i') \leq 2^{-k-1} \Rightarrow e(j, j') \leq d(i, i') + 2^{1-k}\varepsilon. \quad (4)$$

Using $\tilde{e}((j, k), (j', k)) < \tilde{d}((i, k), (i', k)) + 2\varepsilon$, we obtain

$$\min\{1, 2^k e(j, j')\} < \min\{1, 2^k d(i, i')\} + 2\varepsilon \leq 2^k d(i, i') + 2\varepsilon.$$

In particular, using $d(i, i') \leq 2^{-k-1}$, the minimum on the left hand side is less than $1/2 + 2\varepsilon$, hence less than 1 and equal to $2^k e(j, j')$. Thus, (4) follows.

Further, we show that

$$i\mathcal{R}_k j \ \& \ i\mathcal{R}_{k'} j' \Rightarrow e(j, j') \leq 2^{1-\min\{k, k'\}}\varepsilon. \quad (5)$$

Using $|\tilde{e}((j, k), (j', k')) - \tilde{d}((i, k), (i, k'))| < 2\varepsilon$, we obtain $||10k - 10k'| + \min\{1, 2^{\min\{k, k'\}}e(j, j')\} - |10k - 10k'|| < 2\varepsilon$. That is,

$$\min\{1, 2^{\min\{k, k'\}}e(j, j')\} < 2\varepsilon,$$

which gives (5).

Let us consider the decreasing sequence of correspondences given by

$$\mathcal{R}_s^* = \bigcup_{k=s}^{\infty} \mathcal{R}_k = \{(i, j) \in \mathbb{N}^2 : (\exists k \geq s)((i, k)\mathcal{R}(j, k))\}, \quad s = 1, 2, \dots,$$

and let us observe that

$$e\text{-diam}(i\mathcal{R}_s^*) \leq 2^{1-s}\varepsilon, \quad d\text{-diam}(\mathcal{R}_s^*j) \leq 2^{1-s}\varepsilon. \quad (6)$$

The first inequality follows from (5), the second one holds due to the symmetry.

We claim moreover that

$$i\mathcal{R}_s^*j, i'\mathcal{R}_s^*j' \ \& \ d(i, i') \geq 2^{-s-2} \quad \Rightarrow \quad e(j, j') \leq (1 + 24\varepsilon)d(i, i'). \quad (7)$$

Let $k \in \mathbb{Z}$ be such that $2^{-k-2} \leq d(i, i') < 2^{-k-1}$. We have $k \leq s$, and so $i\mathcal{R}_k^*j, i'\mathcal{R}_k^*j'$ in particular. Let us pick $n, n' \in \mathbb{N}$ such that $i\mathcal{R}_kn, i'\mathcal{R}_kn'$. We obtain from (4) that $e(n, n') \leq d(i, i') + 2^{1-k}\varepsilon$. Also, we obtain from (6) that $e(j, n) \leq 2^{1-k}\varepsilon$ and $e(j', n') \leq 2^{1-k}\varepsilon$. By the triangle inequality,

$$e(j, j') \leq d(i, i') + 2^{1-k}\varepsilon + 2^{1-k}\varepsilon + 2^{1-k}\varepsilon = d(i, i') + 24 \cdot 2^{-k-2}\varepsilon \leq (1 + 24\varepsilon)d(i, i'),$$

which gives (7). Let us note that, due to the symmetry, we have also

$$i\mathcal{R}_s^*j, i'\mathcal{R}_s^*j' \ \& \ e(j, j') \geq 2^{-s-2} \quad \Rightarrow \quad d(i, i') \leq (1 + 24\varepsilon)e(j, j'). \quad (8)$$

Finally, applying Lemma 21 and using (6), (7) and (8), we obtain $\rho_L(d, e) \leq \log(1 + 24\varepsilon)$. \square

Theorem 38. *The pseudometrics ρ_{GH} on \mathcal{M} and ρ_L on \mathcal{M} are Borel-uniformly continuous bi-reducible.*

Proof. By Theorem 35 and Theorem 36 we get

$$\rho_{GH} \leq_{B,u} (\rho_{GH} \upharpoonright \mathcal{M}_2^4) \leq_{B,u} (\rho_L \upharpoonright \mathcal{M}_2^4) \leq_{B,u} \rho_L.$$

For the other direction, we use Theorem 37. \square

Finally, using an analogous proof, we obtain that “the coarse Lipschitz distance” on metric spaces is reducible to the Gromov-Hausdorff distance. We will see later it is actually bi-reducible with it (Theorem 46).

Theorem 39. *There is an injective Borel-uniformly continuous reduction from ρ_{HL} on \mathcal{M} to ρ_{GH} on \mathcal{M} .*

Proof. Denote by \mathbb{N}^- the set $\{k \in \mathbb{Z} : k \leq 0\}$. For every $d \in \mathcal{M}$, we define a metric \tilde{d} on $(\mathbb{N} \times \mathbb{N}^-) \cup \{\clubsuit\}$ by

$$\begin{aligned} \tilde{d}((i, k), (j, l)) &= |10k - 10l| + \min\{1, 2^{\min\{k, l\}}d(i, j)\}, \\ \tilde{d}((i, k), \clubsuit) &= |10k + 4| + 1. \end{aligned}$$

Note that this is the same construction which we used already in the proof of Theorem 37 with the exception that the underlying set is $(\mathbb{N} \times \mathbb{N}^-) \cup \{\clubsuit\}$ and in the proof of Theorem 37 it is $(\mathbb{N} \times \mathbb{Z}) \cup \{\clubsuit\}$. Hence, to prove the theorem, it is sufficient to show that for every $\varepsilon > 0$ there are $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\rho_{GH}(\tilde{d}, \tilde{e}) < \delta_1 \Rightarrow \rho_{HL}(d, e) \leq \varepsilon$$

and

$$\rho_{HL}(d, e) < \delta_2 \Rightarrow \rho_{GH}(\tilde{d}, \tilde{e}) \leq \varepsilon$$

for every $d, e \in \mathcal{M}$.

By Lemma 30, there exists $\delta' > 0$ such that

$$\begin{aligned} d \text{ and } e \text{ are } HL(\delta')\text{-close} &\Rightarrow \rho_{HL}(d, e) < \varepsilon, \\ \rho_{HL}(d, e) < \delta' &\Rightarrow d \text{ and } e \text{ are } HL(\varepsilon)\text{-close.} \end{aligned}$$

We claim that it suffices to put $\delta_1 = \min\{\frac{1}{5}, \frac{\delta'}{24}\}$ and $\delta_2 = \delta'$.

Assume that $\rho_{GH}(\tilde{d}, \tilde{e}) < \delta_1$. This is witnessed by a correspondence $\mathcal{R} \subseteq [(\mathbb{N} \times \mathbb{N}^-) \cup \{\clubsuit\}]^2$. Then, using verbatim the same arguments as in the proof of Theorem 37, the relation $\mathcal{R}_0 = \{(i, j) \in \mathbb{N}^2 : (i, 0)\mathcal{R}(j, 0)\}$ is a correspondence and whenever $i\mathcal{R}_0j$ and $i'\mathcal{R}_0j'$, we have

$$\begin{aligned} d(i, i') \leq \frac{1}{2} &\Rightarrow e(j, j') \leq d(i, i') + 2\delta_1, \\ d(i, i') \geq \frac{1}{4} &\Rightarrow e(j, j') \leq (1 + 24\delta_1)d(i, i'); \end{aligned}$$

and similarly for the symmetric situation when the roles of d and e are changed. In particular, \mathcal{R}_0 witnesses the fact that d and e are $HL(24\delta_1)$ -close and since $24\delta_1 \leq \delta'$, we have $\rho_{HL}(d, e) < \varepsilon$.

Assume that $\rho_{HL}(d, e) < \delta_2$. Then d and e are $HL(\varepsilon)$ -close, which is witnessed by a correspondence $\mathcal{R}' \subseteq \mathbb{N}^2$. Similarly as in the proof of Theorem 37, we define a correspondence

$$\mathcal{R} = \{(\clubsuit, \clubsuit)\} \cup \{((i, k), (j, k)) : i\mathcal{R}'j, k \leq 0\}.$$

Our aim is to show that $|\tilde{d}(a, a') - \tilde{e}(b, b')| < 2\varepsilon$ whenever $a\mathcal{R}b$ and $a'\mathcal{R}b'$. Using verbatim the same arguments as in the proof of Theorem 37, it is sufficient to show that for $a = (i, k)$, $b = (j, k)$, $a' = (i', k')$, $b' = (j', k')$ with $1 > 2^{\min\{k, k'\}}e(j, j')$, $i\mathcal{R}'j$, $i'\mathcal{R}'j'$ we have

$$2^{\min\{k, k'\}}d(i, i') - 2^{\min\{k, k'\}}e(j, j') < 2\varepsilon. \quad (9)$$

Fix a, a', b, b' as above. If $e(j, j') \leq 1$, using that \mathcal{R}' witnesses d and e are $HL(\varepsilon)$ -close, we get $2^{\min\{k, k'\}}(d(i, i') - e(j, j')) < 2\varepsilon$. On the other hand, if $e(j, j') \geq 1$, we get $2^{\min\{k, k'\}}(d(i, i') - e(j, j')) \leq 2^{\min\{k, k'\}}((1 + \varepsilon)e(j, j') - e(j, j')) = 2^{\min\{k, k'\}}\varepsilon e(j, j') < \varepsilon$. Hence, (9) holds and so the correspondence \mathcal{R} witnesses that $\rho_{GH}(\tilde{d}, \tilde{e}) \leq \varepsilon$. \square

3. REDUCTIONS FROM PSEUDOMETRICS ON \mathcal{B} TO PSEUDOMETRICS ON \mathcal{M}

We start with a reduction from the Banach-Mazur distance to the Lipschitz distance. An essential ingredient is Lemma 23.

Theorem 40. *There is a Borel-uniformly continuous reduction from ρ_{BM} to ρ_L on \mathcal{M}_p^q , where $0 < p < q$.*

Moreover, the reduction is not only Borel-uniformly continuous, but also Borel-Lipschitz on small distances.

Proof. Without loss of generality, we assume that $p = 2$ and $q = 15$. The structure of the proof is the following. First, we describe a construction which to each $\nu \in \mathcal{B}$ assigns a metric space M_ν . Next, we show that for $\nu, \lambda \in \mathcal{B}$ we have $\rho_L(M_\nu, M_\lambda) \leq \rho_{BM}(\nu, \lambda)$ and

$$\rho_L(M_\nu, M_\lambda) < \log\left(\frac{4}{3}\right) \implies \rho_{BM}(\nu, \lambda) \leq 2\rho_L(M_\nu, M_\lambda).$$

Finally, we show it is possible to make such an assignment in a Borel way.

Fix some countable sequence $(c_i)_{i \in \mathbb{N}}$ of positive real numbers such that for every positive real number $r > 0$ there exists $i \geq 7$ such that $c_i \cdot r \in (2, 9/4)$. Also, let $\pi : \mathbb{Q} \rightarrow \mathbb{N} \setminus \{1\}$ be some bijection and let \preceq be some linear order on V . To each $\nu \in \mathcal{B}$ we assign a countable metric space M_ν with the following underlying set:

$$V \cup \{p_{a,b}^{m,k} : a \preceq b \in V, m \geq 7, k \leq m\} \cup \\ \cup \{f_{a,q}^m : a \in V, q \in \mathbb{Q}, m \leq \pi(q)\} \cup \{x_{a,b}^i : a \preceq b \in V, i \leq 3\}.$$

The metric d_ν on M_ν is defined as follows.

- For every $a \neq b \in V$ we set $d_\nu(a, b) = 15$.
- For every $a \preceq b \in V$, $m \geq 7$ we define the number $K_{a,b}^m$ to be $\max\{2, \min\{3, c_m \cdot \nu(a-b)\}\}$. Then we set $d_\nu(a, p_{a,b}^{m,1}) = d_\nu(p_{a,b}^{m,1}, p_{a,b}^{m,2}) = \dots = d_\nu(p_{a,b}^{m,m}, b) = K_{a,b}^m$.
- For every $a \in V$ and $q \in \mathbb{Q}$ we set $d_\nu(a, f_{a,q}^1) = 7$, $d_\nu(f_{a,q}^1, f_{a,q}^2) = \dots = d_\nu(f_{a,q}^{\pi(q)}, qa) = 10$.
- For every $a \preceq b \in V$ we set $d_\nu(a, x_{a,b}^1) = d_\nu(b, x_{a,b}^2) = d_\nu(x_{a,b}^1, x_{a,b}^3) = d_\nu(x_{a,b}^2, x_{a,b}^3) = d_\nu(x_{a,b}^3, a+b) = 5$.
- On the rest of M_ν^2 , we take the greatest extension of d_ν defined above with 15 as the upper bound, which is nothing but the graph metric (bounded by 15).

We shall call the pairs of elements from M_ν , for which the distance was defined directly before taking the extension, *edges*. In order to simplify some notation, whenever we write $p_{b,a}^{m,k}$, where $a \preceq b$, we mean the element $p_{a,b}^{m,k}$. Also by $p_{a,b}^{m,0}$ we mean the element a , and by $p_{a,b}^{m,m+1}$ we mean the element b . We shall call the pairs $p_{a,b}^{m,k}, p_{a,b}^{m,k+1}$ *neighbors*.

Consider two norms $\nu, \lambda \in \mathcal{B}$. Denote the elements of M_λ by $V \cup \{q_{a,b}^{m,k} : a \preceq b \in V, m \geq 7, k \leq m\} \cup \{g_{a,q}^m : a \in V, q \in \mathbb{Q}, m \leq \pi(q)\} \cup \{y_{a,b}^i : a \preceq b \in V, i \leq 3\}$ and the numbers $\max\{2, \min\{3, c_m \cdot \lambda(a-b)\}\}$ by $L_{a,b}^m$.

We claim that $\rho_L(M_\nu, M_\lambda) \leq \rho_{BM}(\nu, \lambda)$. If $\rho_{BM}(\nu, \lambda) < \varepsilon$, by Lemma 23, there exists a surjective \mathbb{Q} -linear isomorphism $T : (V, \nu) \rightarrow (V, \lambda)$ with $\|T\| \|T^{-1}\| < \exp(\varepsilon)$. Fix $\varepsilon' > 0$. We may assume that $\min\{\|T\|, \|T^{-1}\|\} \geq 1 - \varepsilon'$. We use T to define a bi-Lipschitz bijection $T' : M_\nu \rightarrow M_\lambda$. For every $a \in V$ we set $T'(a) = T(a)$ and for all elements of the form $p_{a,b}^{m,k}$, $f_{a,q}^m$, and $x_{a,b}^i$, with appropriate indices, whenever $T(a) \preceq T(b)$, we set $T'(p_{a,b}^{m,k}) = q_{T(a), T(b)}^{m,k}$, $T'(f_{a,q}^m) = g_{T(a), q}^m$, and $T'(x_{a,b}^i) = y_{T(a), T(b)}^i$; similarly, if $T(b) \prec T(a)$, we set $T'(p_{a,b}^{m,k}) = q_{T(b), T(a)}^{m+1-k, k}$, $T'(f_{a,q}^m) = g_{T(a), q}^m$, $T'(x_{a,b}^1) = y_{T(b), T(a)}^2$, $T'(x_{a,b}^2) = y_{T(b), T(a)}^1$ and $T'(x_{a,b}^3) = y_{T(b), T(a)}^3$. Let us compute the Lipschitz

constants of T' . If $a = b$, then obviously $\frac{L_{T(a),T(b)}^m}{K_{a,b}^m} = 1$. Otherwise, we have

$$\begin{aligned} \frac{L_{T(a),T(b)}^m}{K_{a,b}^m} &= \frac{\max\{2, \min\{3, c_m \cdot \lambda(T(a) - T(b))\}\}}{\max\{2, \min\{3, c_m \cdot \nu(a - b)\}\}} \\ &\leq \max\left\{1, \frac{\lambda(T(a) - T(b))}{\nu(a - b)}\right\} \leq \max\{1, \|T\|\} \leq \|T\| + \varepsilon', \end{aligned}$$

where in the first inequality we used the easy fact that for $x, y > 0$ we have $\frac{\max\{2, \min\{3, x\}\}}{\max\{2, \min\{3, y\}\}} \leq \max\{1, \frac{x}{y}\}$. It follows that $\text{Lip}(T') \leq \|T\| + \varepsilon'$. Indeed, it follows from the definition of T' that it maps edges onto edges. Moreover, for every edge $(x, y) \in M_\nu^2$ we have $d_\lambda(T'(x), T'(y)) \leq (\|T\| + \varepsilon')d_\nu(x, y)$, so the same inequality extends to the graph metrics – the extensions of d_ν and d_λ on the whole M_ν and M_λ respectively. We obtain in particular that $\text{Lip}(T) \leq (\|T\| + \varepsilon')\|T^{-1}\|/(1 - \varepsilon')$. Since an analogous inequality holds for $\text{Lip}((T')^{-1})$ and $\varepsilon' > 0$ was arbitrary, we have $\rho_L(M_\nu, M_\lambda) \leq \log(\|T\|\|T^{-1}\|) < \varepsilon$. Thus, we conclude that $\rho_L(M_\nu, M_\lambda) \leq \rho_{BM}(\nu, \lambda)$.

Conversely, assume that $\exp(\rho_L(M_\nu, M_\lambda)) < 4/3$, that is, there exists a bijection $T : M_\nu \rightarrow M_\lambda$ with $\text{Lip}(T) < 4/3$ and $\text{Lip}(T^{-1}) < 4/3$. We will show that $\rho_{BM}(\nu, \lambda) \leq 2\rho_L(M_\nu, M_\lambda)$.

First we claim that T maps $V \subseteq M_\nu$ bijectively onto $V \subseteq M_\lambda$. Indeed, the points $a \in V \subseteq M_\nu$ are characterized as those points x of M_ν for which there exist infinitely many points $y \in M_\nu$ with $\nu(x - y) \leq 3$. On the other hand, the points from $M_\nu \setminus V$ are characterized as those points x of M_ν for which there are at most two points distinct from x of distance less than 4 from x . Since $\text{Lip}(T) < 4/3$, we get $T(V) \subseteq V$ and similarly we have $T^{-1}(V) \subseteq V$, which proves the claim. We denote by S the induced bijection between (V, ν) and (V, λ) .

We claim that S is \mathbb{Q} -linear. Let us check that it is homogeneous for all rationals, that is, $S(qa) = qS(a)$ for all $a \in V$ and $q \in \mathbb{Q}$, which in particular gives that $S(0) = 0$. For each $a \in V \subseteq M_\nu$ and $q \in \mathbb{Q}$ there is a path of points $a, f_{a,q}^1, \dots, f_{a,q}^{\pi(q)}, qa$. The map T must send this path to some path $T(a), g_{T(a),q'}^1, \dots, g_{T(a),q'}^{\pi(q')}, q'T(a)$. However, q' is determined by the length of the path which must be the same as the length of the former path. Therefore $q' = q$ and $S(qa) = T(qa) = qT(a) = qS(a)$. Next, we show that for $a \neq b \in V$ we have $S(a + b) = S(a) + S(b)$. There is a “triangle of paths” formed by the points $a, b, x_{a,b}^1, x_{a,b}^2, x_{a,b}^3, a + b$. T must preserve this triangle, so it maps it to a triangle formed by the points $T(a), T(b), y_{T(a),T(b)}^1, y_{T(a),T(b)}^2, y_{T(a),T(b)}^3, T(a) + T(b)$. That shows that $S(a + b) = T(a + b) = T(a) + T(b) = S(a) + S(b)$.

It remains to compute the Lipschitz constant of S , resp. S^{-1} , as a map from (V, ν) to (V, λ) . In order to do it, we claim that for every $a \preceq b$, $a \neq b$, $m \geq 7$ and $k \leq m$ we have $T(p_{a,b}^{m,k}) = q_{S(a),S(b)}^{m,k}$ if $S(a) \preceq S(b)$, and $T(p_{a,b}^{m,k}) = q_{S(b),S(a)}^{m,m+1-k}$ if $S(b) \preceq S(a)$. We only treat the former case, the other is treated analogously. First observe that $T(p_{a,b}^{m,1}) = q_{S(a),b'}^{m',k'}$,

for some m' and b' , and $k' = 1$ or $k' = m'$. Indeed, $p_{a,b}^{m,1}$ is a neighbor of a , so $d_\nu(a, p_{a,b}^{m,1}) \leq 3$. Therefore $d_\lambda(S(a), T(p_{a,b}^{m,1})) < 4$, so $S(a)$ and $T(p_{a,b}^{m,1})$ are also neighbors. Analogously, we show that for every $0 \leq k \leq m$ we have that $T(p_{a,b}^{m,k})$ and $T(p_{a,b}^{m,k+1})$ are neighbors, which implies that T indeed maps the ‘path’ $a, p_{a,b}^{m,1}, p_{a,b}^{m,2}, \dots, p_{a,b}^{m,m}, b$ onto the path $S(a), q_{S(a),S(b)}^{m,1}, q_{S(a),S(b)}^{m,2}, \dots, q_{S(a),S(b)}^{m,m}, S(b)$.

We are now ready to compute the Lipschitz constants. We do it for S . Pick some $a \preceq b$, $a \neq b$. We want to compute $\frac{\lambda(S(a)-S(b))}{\nu(a-b)}$. We consider only the case when $S(a) \preceq S(b)$, the other case is analogous. By the choice of $(c_i)_{i \in \mathbb{N}}$, there exists $m \geq 7$ such that $c_m \cdot \nu(a-b) \in (2, 9/4)$. It follows that $d_\nu(a, p_{a,b}^{m,1}) \in (2, 9/4)$, so we have

$$d_\lambda(S(a), q_{S(a),S(b)}^{m,1}) = d_\lambda(T(a), T(p_{a,b}^{m,1})) \leq \text{Lip}(T)d_\nu(a, p_{a,b}^{m,1}) < 3,$$

which implies that

$$\lambda(S(a) - S(b)) \leq \frac{d_\lambda(S(a), q_{S(a),S(b)}^{m,1})}{c_m} \leq \frac{\text{Lip}(T)d_\nu(a, p_{a,b}^{m,1})}{c_m} = \text{Lip}(T)\nu(a-b).$$

That shows that $\|S\| \leq \text{Lip}(T)$. Analogously, we get $\|S^{-1}\| \leq \text{Lip}(T^{-1})$; hence, we have $\rho_{BM}(\nu, \lambda) \leq 2 \log \max\{\text{Lip}(T), \text{Lip}(T^{-1})\}$. Considering all bi-Lipschitz maps T with $\text{Lip}(T) < 4/3$ and $\text{Lip}(T^{-1}) < 4/3$, we obtain $\rho_{BM}(\nu, \lambda) \leq 2\rho_L(M_\nu, M_\lambda)$ whenever $\rho_L(M_\nu, M_\lambda) < \log(4/3)$.

Finally, to verify that the map $\mathcal{B} \ni \nu \rightarrow (M_\nu, d_\nu)$ is Borel, let us denote by N the underlying set of M_ν (which is the same for every $\nu \in \mathcal{B}$). Now it suffices to fix some bijection $\phi : \mathbb{N} \rightarrow N$ and check that the distances in M_ν depend on distances of ν in a continuous (when considering ν as a member of \mathbb{R}^V) way. \square

A consequence of the last theorem and Theorem 36 is that the Banach-Mazur distance is Borel-uniformly continuous reducible to the Gromov-Hausdorff distance. We will see later it is actually bi-reducible with it.

Corollary 41. *We have $\rho_{BM} \leq_{B,u} \rho_{GH}$.*

Next we show that ‘the coarse Lipschitz distance’ on Banach spaces is reducible to the Gromov-Hausdorff distance. Again, we will see later it is actually bi-reducible with it (Theorem 46).

The reduction is obtainable already from Theorem 39. However, the proof which follows is in this concrete case more natural and gives a slightly better result, that is, the reduction is even Borel-Lipschitz on small distances.

Theorem 42. *There is an injective Borel-uniformly continuous reduction from ρ_{HL} , equivalently ρ_N , on \mathcal{B} to ρ_{GH} on \mathcal{M} .*

Moreover, the reduction is not only Borel-uniformly continuous, but also Borel-Lipschitz on small distances.

Proof. To every separable Banach space $(X, \|\cdot\|_X)$ we associate a metric space (X, d_X) whose underlying set is unchanged, and for every $x, y \in X$ we set $d_X(x, y) = \min\{\|x - y\|_X, 1\}$. We claim that the map $(X, \|\cdot\|_X) \rightarrow (X, d_X)$ is the desired reduction.

Fix some separable Banach spaces X and Y . Suppose first that $\rho_{HL}(X, Y) = \rho_N(X, Y) < K$, for some $K > 0$, where the first equality follows from Proposition 27. So there exist nets $\mathcal{N}_X \subseteq X$ and $\mathcal{N}_Y \subseteq Y$ and a bi-Lipschitz map $T : \mathcal{N}_X \rightarrow \mathcal{N}_Y$ with $\log \max\{\text{Lip}(T), \text{Lip}(T^{-1})\} < K$. Pick any $\varepsilon > 0$. By rescaling the nets \mathcal{N}_X and \mathcal{N}_Y if necessary we may assume (as in the proof of Proposition 27) the nets are an (a, ε) -net, resp. an (a', ε) -net, for some $a, a' > 0$. Since (\mathcal{N}_X, d_X) and (\mathcal{N}_Y, d_Y) belong to $\mathcal{M}_{\min(a, a')}^1$ we get from Theorem 36 that $\rho_{GH}((\mathcal{N}_X, d_X), (\mathcal{N}_Y, d_Y)) \leq (\exp(K) - 1)/2$. Since $\rho_{GH}((X, d_X), (\mathcal{N}_X, d_X)) \leq \varepsilon$, $\rho_{GH}((Y, d_Y), (\mathcal{N}_Y, d_Y)) \leq \varepsilon$, and since ε was arbitrary, we get that $\rho_{GH}((X, d_X), (Y, d_Y)) \leq \frac{\exp(K)-1}{2} \leq \frac{\exp(1)-1}{2}K$ whenever $K < 1$.

Conversely, suppose that $\rho_{GH}((X, d_X), (Y, d_Y)) < K$, where $K < 1/4$. By Lemma 14 there exists a bijection $\phi : X \rightarrow Y$ witnessing the Gromov-Hausdorff distance, i.e. for every $x, y \in X$ we have $|d_X(x, y) - d_Y(\phi(x), \phi(y))| < 2K$ (note that although Lemma 14 was stated only for countable dense subsets of perfect metric spaces, by transfinite recursion it can be proved also for the completions). We aim to show that ϕ is large scale bi-Lipschitz for $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$. Pick any $x, y \in X$ with $\|x - y\|_X \geq 1$. Find points $x_0 = x, x_1, x_2, \dots, x_{n-1}, x_n = y$ such that $\sum_{i=0}^{n-1} \|x_i - x_{i+1}\| = \|x - y\|$, $n \leq 3\|x - y\|_X$ and for every $i < n$ we have $\|x_i - x_{i+1}\| \leq 1/2$. Notice that for every $i < n$ we have $\|x_i - x_{i+1}\|_X = d_X(x_i, x_{i+1}) \leq 1/2$. So $d_Y(\phi(x_i), \phi(x_{i+1})) \leq d_X(x_i, x_{i+1}) + 2K < 1$, therefore $\|\phi(x_i) - \phi(x_{i+1})\|_Y = d_Y(\phi(x_i), \phi(x_{i+1})) \leq \|x_i - x_{i+1}\|_X + 2K$.

Now we compute

$$\begin{aligned} \|\phi(x) - \phi(y)\|_Y &\leq \sum_{i=0}^{n-1} \|\phi(x_i) - \phi(x_{i+1})\|_Y \leq \|x - y\|_X + 2K(3\|x - y\|_X) \\ &= (1 + 6K)\|x - y\|_X. \end{aligned}$$

Along with the analogous computations for ϕ^{-1} we get that $\max\{\text{Lip}_1(\phi), \text{Lip}_1(\phi^{-1})\} \leq 1 + 6K$, where

$$\text{Lip}_1(\phi) = \sup_{\substack{x, y \in X \\ \|x - y\|_X \geq 1}} \frac{\|\phi(x) - \phi(y)\|_Y}{\|x - y\|_X}.$$

Now it suffices to choose some maximal 2-separated set \mathcal{N}_X in $(X, \|\cdot\|_X)$, which is a net in X . Its image $\phi[\mathcal{N}_X]$, denoted by \mathcal{N}_Y , is a net in Y . Indeed, we claim that for each $x \neq y \in \mathcal{N}_X$ we have $\|\phi(x) - \phi(y)\|_Y > 1$. Otherwise, there is $z \in Y$ such that $\|\phi(x) - z\|_Y \leq 1/2$ and $\|z - \phi(y)\|_Y \leq 1/2$. This implies that $\|x - y\|_X \leq \|x - \phi^{-1}(z)\|_X + \|\phi^{-1}(z) - y\|_X < \|\phi(x) - z\|_Y + \|z - \phi(y)\|_Y + 4K \leq 2$, a contradiction. Finally, we claim that for every $y \in Y$ we can find $y' \in \mathcal{N}_Y$ with $\|y - y'\|_Y < 4$. Pick any $y \in Y$. Then there exist $x_1, x_2, x_3 \in X$ and $x' \in \mathcal{N}_X$ such that $\max\{\|\phi^{-1}(y) - x_1\|_X, \|x_1 - x_2\|_X, \|x_2 - x_3\|_X, \|x_3 - x'\|_X\} \leq 1/2$. Set $y' = \phi(x')$. We get that $\|y - y'\|_Y \leq \|y - \phi(x_1)\|_Y + \|\phi(x_1) - \phi(x_2)\|_Y + \|\phi(x_2) - \phi(x_3)\|_Y + \|\phi(x_3) - y'\|_Y \leq 2 + 8K < 4$. So we have verified that \mathcal{N}_Y is a net. It is bi-Lipschitz with \mathcal{N}_X as witnessed by ϕ . So we get the estimate $\rho_{HL}((X, \|\cdot\|_X), (Y, \|\cdot\|_Y)) = \rho_N((X, \|\cdot\|_X), (Y, \|\cdot\|_Y)) \leq \log(1 + 6K) \leq 6K$.

Finally, we observe that the map $(X, \|\cdot\|_X) \rightarrow (X, d_X)$ can be viewed as a Borel function from \mathcal{B} to \mathcal{M} . Recall that elements of \mathcal{B} are norms on a countable infinite-dimensional \mathbb{Q} -vector space denoted by V . By fixing a bijection $f : V \rightarrow \mathbb{N}$ we associate to each $\|\cdot\| \in \mathcal{B}$ a metric $d \in \mathcal{M}$ such that for every $n, m \in \mathbb{N}$ we have $d(n, m) = \min\{1, \|f^{-1}(n) - f^{-1}(m)\|\}$. This is clearly Borel. \square

Remark 43. Observe that the only geometric property of Banach spaces that we used in the proof, besides that Banach spaces are cones so that ρ_{HL} and ρ_N agree on them (see Remark 28), was that Banach spaces are geodetic metric spaces. Clearly, it is sufficient that they are length spaces, i.e. between every two points x, y there is a path of length $d(x, y) + \varepsilon$, where $\varepsilon > 0$ is arbitrary. Therefore it follows from the proof of Theorem 42 that there is a reduction from ρ_{HL} (or ρ_N) on cones that are length spaces to ρ_{GH} on metric spaces.

Finally, we present the proof of the reduction that involves the Kadets distance.

Theorem 44. *There is an injective Borel-uniformly continuous reduction from ρ_K on \mathcal{B} to ρ_{GH} on \mathcal{M} .*

Moreover, the reduction is not only Borel-uniformly continuous, but also Borel-Lipschitz on small distances.

We need the following lemma first.

Lemma 45. *Let X and Y be two separable Banach spaces and fix countable dense subsets $(x_i)_i$ and $(y_i)_i$ of the spheres S_X and S_Y respectively. Then $\rho_K(X, Y) < \varepsilon$, for some $\varepsilon > 0$, implies that there exists a bijection $\pi \in S_\infty$ such that for every finite $F \subseteq \mathbb{N}$ and every $(\delta_i)_{i \in F} \subseteq \{-1, 1\}$ we have*

$$\left| \left\| \sum_{i \in F} \delta_i x_i \right\|_X - \left\| \sum_{i \in F} \delta_i y_{\pi(i)} \right\|_Y \right| < 2|F|\varepsilon.$$

Proof. We may suppose that X and Y are subspaces of a Banach space Z and that we have $\rho_H^Z(B_X, B_Y) < \varepsilon$. First we claim that for every $x \in S_X$ there exists $y \in S_Y$ such that $\|x - y\| < 2\varepsilon$. Analogously, for every $y \in S_Y$ there exists such $x \in S_X$. Indeed, by definition for every $x \in S_X$ there exists $y' \in B_Y$ with $\|x - y'\| < \varepsilon$. So we can take $y = y'/\|y'\|$ and we have $\|y - y'\| < \varepsilon$, so we are done by the triangle inequality. Now since S_X and S_Y are perfect metric spaces, by a back-and-forth argument (see e.g. the proof of Lemma 14), we get a bijection $\pi \in S_\infty$ such that for every $i \in \mathbb{N}$ we have $\|x_i - y_{\pi(i)}\| < 2\varepsilon$. We claim that π is as desired.

Take any finite subset $F \subseteq \mathbb{N}$ and $(\delta_i)_{i \in F} \subseteq \{-1, 1\}$. Then we have

$$\begin{aligned} \left| \left\| \sum_{i \in F} \delta_i x_i \right\| - \left\| \sum_{i \in F} \delta_i y_{\pi(i)} \right\| \right| &\leq \left\| \sum_{i \in F} \delta_i (x_i - y_{\pi(i)}) \right\| \leq \sum_{i \in F} \|x_i - y_{\pi(i)}\| < \\ &< 2|F|\varepsilon, \end{aligned}$$

and we are done. \square

Proof of Theorem 44. The structure of the proof is the following. First, we describe a construction which to each separable Banach space X assigns

a metric space M_X . Next, we show that for every two separable Banach spaces X and Y we have $\rho_{GH}(M_X, M_Y) \leq 2\rho_K(X, Y)$ and

$$\rho_{GH}(X, Y) < 1 \implies \rho_K(X, Y) \leq 17\rho_{GH}(M_X, M_Y).$$

Finally, we show it is possible to make such an assignment in a Borel way.

Let X be a separable Banach space. Fix a countable dense subset $D_X = \{x_i : i \in \mathbb{N}\} \subseteq S_X$ of the unit sphere of X that is symmetric, that is, for every $x \in D_X$ we also have $-x \in D_X$. Set $M_X = D_X \cup \{p_{F,k} : F \in [\mathbb{N}]^{<\omega} \setminus \{\emptyset\}, k \in F\}$. We define a metric d_X on M_X as follows:

$$\begin{aligned} d_X(x_i, x_j) &= \|x_i - x_j\|_X, \\ d_X(x_i, p_{F,k}) &= 10 + \|x_i - x_k\|_X, \\ d_X(p_{F,i}, p_{F,j}) &= 15 + \frac{\|\sum_{k \in F} x_k\|_X}{|F|}, \quad i \neq j \in F, \\ d_X(p_{F,i}, p_{G,j}) &= 20, \quad F \neq G. \end{aligned}$$

Fix separable Banach spaces X and Y . The space $M_Y = D_Y \cup \{q_{F,k} : F \in [\mathbb{N}]^{<\omega} \setminus \{\emptyset\}, k \in F\}$, where $D_Y = \{y_i : i \in \mathbb{N}\} \subseteq S_Y$ is symmetric countable dense, is constructed analogously as M_X .

We claim that for every $\varepsilon > 0$ with $\rho_K(X, Y) < \varepsilon$ we have $\rho_{GH}(M_X, M_Y) \leq 2\varepsilon$. Indeed, fix $\varepsilon > 0$ with $\rho_K(X, Y) < \varepsilon$ and use Lemma 45 applied to countable dense sequences $(x_i)_i$ and $(y_i)_i$ of the spheres S_X and S_Y respectively. The bijection π from Lemma 45 induces a bijection $\phi : M_X \rightarrow M_Y$ defined as follows:

$$\begin{aligned} \phi(x_i) &= y_{\pi(i)}, \\ \phi(p_{F,j}) &= q_{\pi[F], \pi(j)}. \end{aligned}$$

We claim that for every $x, y \in M_X$ we have $|d_X(x, y) - d_Y(\phi(x), \phi(y))| < 4\varepsilon$, i.e. $M_X \simeq_{4\varepsilon} M_Y$. We consider several cases:

Case 1. $(x, y) = (x_i, x_j)$ for some $i, j \in \mathbb{N}$: then we have

$$|d_X(x_i, x_j) - d_Y(y_{\pi(i)}, y_{\pi(j)})| = |\|x_i - x_j\|_X - \|y_{\pi(i)} - y_{\pi(j)}\|_Y| < 4\varepsilon.$$

Case 2. $x = x_i, y = p_{F,k}$ for some $i \in \mathbb{N}, F \subseteq \mathbb{N}, k \in F$: then we have

$$|d_X(x_i, p_{F,k}) - d_Y(y_{\pi(i)}, q_{\pi[F], \pi(k)})| = |\|x_i - x_k\|_X - \|y_{\pi(i)} - y_{\pi(k)}\|_Y| < 4\varepsilon.$$

Case 3. $x = p_{F,j}, y = p_{F,k}$ for some $F \subseteq \mathbb{N}, j \neq k \in F$: then we have

$$\begin{aligned} |d_X(p_{F,j}, p_{F,k}) - d_Y(q_{\pi[F], \pi(j)}, q_{\pi[F], \pi(k)})| &= \frac{|\|\sum_{i \in F} x_i\|_X - \|\sum_{i \in F} y_{\pi(i)}\|_Y|}{|F|} \\ &< \frac{2|F|\varepsilon}{|F|} = 2\varepsilon. \end{aligned}$$

Case 4. $x = p_{F,i}, y = p_{G,j}$, for $F \neq G \subseteq \mathbb{N}, i \in F, j \in G$: then we have

$$|d_X(p_{F,i}, p_{G,j}) - d_Y(q_{\pi[F], \pi(i)}, q_{\pi[G], \pi(j)})| = 20 - 20 = 0.$$

Hence, $d_X \simeq_{4\varepsilon} d_Y$ and, by Lemma 12, we get $\rho_{GH}(M_X, M_Y) \leq 2\varepsilon$ which proves the claim.

Conversely, suppose now that $\rho_{GH}(M_X, M_Y) < \varepsilon$, where $\varepsilon \in (0, 1)$. By Fact 10 there exists a correspondence $\mathcal{R} \subseteq M_X \times M_Y$ such that for every $x, y \in M_X$ and $x', y' \in M_Y$, if $x\mathcal{R}x'$ and $y\mathcal{R}y'$, then $|d_X(x, y) - d_Y(x', y')| < 2\varepsilon$. Pick some $i \neq j \in \mathbb{N}$, a finite subset $F \subseteq \mathbb{N}, k \neq k' \in F$. Set $u_1 = x_i$,

$u_2 = x_j$, $u_3 = p_{F,k}$, $u_4 = p_{F,k'}$. We find elements $v_1, \dots, v_4 \in M_Y$ such that $u_i \mathcal{R} v_i$ for $i \leq 4$. We get the following observations:

- Since $d_X(u_1, u_2) \in [0, 2]$ we get that $d_Y(v_1, v_2) \in [0, 4]$, so we deduce that for every $n \in \mathbb{N}$ and every $y \in M_Y$ such that $x_n \mathcal{R} y$ we have $y = y_m$ for some $m \in \mathbb{N}$. Conversely, for every $n \in \mathbb{N}$ and every $x \in M_X$ such that $x \mathcal{R} y_n$ we have $x = x_m$ for some $m \in \mathbb{N}$.
- Since $d_X(u_3, u_4) \in [15, 16]$ we get that $d_Y(v_3, v_4) \in [13, 18]$. So we deduce that for every finite subsets $G, G' \subseteq \mathbb{N}$ and $l \in G$, $l' \in G'$, and every $y, y' \in M_Y$ such that $p_{G,l} \mathcal{R} y$ and $p_{G',l'} \mathcal{R} y'$ there are finite subsets $H, H' \subseteq \mathbb{N}$ and $h \in H$, $h' \in H'$ such that $y = p_{H,h}$, $y' = p_{H',h'}$ and $G = G'$ if and only if $H = H'$.

To summarize, \mathcal{R} induces a bijection ϕ between $M_X \setminus (x_i)_i$ and $M_Y \setminus (y_i)_i$. Moreover, for every finite $F \subseteq \mathbb{N}$ there is a unique finite set, which we shall denote by $\varphi(F)$, such that ϕ is a bijection between $\{p_{F,i} : i \in F\}$ and $\{q_{\varphi(F),j} : j \in \varphi(F)\}$. For every $i \in F$, by $\varphi_F(i)$ we shall denote the element $i' \in \varphi(F)$ such that $q_{\varphi(F),i'} = \phi(p_{F,i})$.

On the other hand, \mathcal{R} , when restricted on $D_X \times D_Y$, is a correspondence between D_X and D_Y witnessing that $\rho_{GH}(D_X, D_Y) \leq \varepsilon$. Since $d_X \upharpoonright D_X$ and $d_Y \upharpoonright D_Y$ are perfect metric spaces, by a back-and-forth argument (see e.g. the proof of Lemma 14) we construct a bijection $\phi' \subseteq \mathcal{R}$ between D_X and D_Y such that $|d_X(x_i, x_j) - d_Y(\phi'(x_i), \phi'(x_j))| < 2\varepsilon$. Taking the union of the bijections ϕ and ϕ' we get a bijection, which we shall still denote by ϕ , between M_X and M_Y such that for every $x, y \in M_X$, $|d_X(x, y) - d_Y(\phi(x), \phi(y))| < 2\varepsilon$.

Pick now an arbitrary finite $F \subseteq \mathbb{N}$. We want to estimate the expression

$$\left| \left\| \sum_{k \in F} x_k \right\|_X - \left\| \sum_{k \in F} \phi(x_k) \right\|_Y \right|.$$

We may suppose F contains at least two elements. Take any $i \neq i' \in F$ and set $G = \varphi(F)$ and $j = \varphi_F(i)$, $j' = \varphi_F(i')$. Since we have $d_X(p_{F,i}, p_{F,i'}) = 15 + \left\| \sum_{k \in F} x_k \right\|_X / |F|$ and $d_Y(q_{G,j}, q_{G,j'}) = 15 + \left\| \sum_{k \in G} y_k \right\|_Y / |G|$, and moreover $|d_X(p_{F,i}, p_{F,i'}) - d_Y(q_{G,j}, q_{G,j'})| < 2\varepsilon$ we get

$$\left| \left\| \sum_{k \in F} x_k \right\|_X - \left\| \sum_{k \in G} y_k \right\|_Y \right| < 2\varepsilon |F|.$$

So we try to estimate

$$\left| \left\| \sum_{k \in G} y_k \right\|_Y - \left\| \sum_{k \in F} \phi(x_k) \right\|_Y \right|.$$

Pick any $k \in F$ and let $k' = \varphi_F(k) \in G$. We have $d_X(x_k, p_{F,k}) = 10$ and $d_Y(\phi(x_k), q_{G,k'}) = 10 + \|\phi(x_k) - y_{k'}\|_Y$. Therefore, since

$$|d_X(x_k, p_{F,k}) - d_Y(\phi(x_k), q_{G,k'})| = |d_X(x_k, p_{F,k}) - d_Y(\phi(x_k), \phi(p_{F,k}))| < 2\varepsilon,$$

we get that $\|\phi(x_k) - y_{k'}\|_Y < 2\varepsilon$. This implies that

$$\left| \left\| \sum_{k \in G} y_k \right\|_Y - \left\| \sum_{k \in F} \phi(x_k) \right\|_Y \right| < 2\varepsilon |F|,$$

which in turn implies that

$$\left| \left\| \sum_{k \in F} x_k \right\|_X - \left\| \sum_{k \in F} \phi(x_k) \right\|_Y \right| < 4\varepsilon|F|.$$

Note that the last inequality in particular implies that ϕ is almost symmetric. Pick any $x \in D_X$. Since also $-x \in D_X$, the previous inequality implies

$$\left| \|x - x\|_X - \|\phi(x) + \phi(-x)\|_Y \right| = \|-\phi(x) - \phi(-x)\|_Y < 8\varepsilon. \quad (10)$$

We set $E = \{e \in X : e = qx, q \in \mathbb{Q}^+ \cup \{0\}, x \in D_X\}$ and $F = \{f \in Y : f = qy, q \in \mathbb{Q}^+ \cup \{0\}, y \in D_Y\}$. Clearly E and F are \mathbb{Q} -homogeneous dense subsets of X and Y respectively. We define a correspondence $\mathcal{R}_0 \subseteq E \times F$, in fact a bijection, such that $e\mathcal{R}_0f$ if and only if there are $x \in D_X$ and $q \in \mathbb{Q}^+$ such that $e = qx$, $f = q\phi(x)$. So for every pair (e, f) such that $e\mathcal{R}_0f$ we have $\|e\|_X = \|f\|_Y$. We now claim that \mathcal{R}_0 is such that

$$\left| \left\| \sum_{i \leq n} u_i \right\|_X - \left\| \sum_{i \leq n} u'_i \right\|_Y \right| \leq 8\varepsilon \left(\sum_{i \leq n} \|u_i\|_X \right) \quad (11)$$

for all $(u_i)_{i \leq n} \subseteq E$ and $(u'_i)_{i \leq n} \subseteq F$, where for all $i \leq n$ we have $u_i\mathcal{R}_0u'_i$.

Fix such a sequence $(u_i)_{i \leq n} \subseteq E$. The corresponding sequence $(u'_i)_{i \leq n}$ is then determined uniquely. First we claim that without loss of generality we may suppose that $(u_i)_{i \leq n} \subseteq D_X$. Indeed, by the homogeneity of the inequality above, we may assume that each u_i is a positive integer multiple of some $x \in D_X$. Since we allow repetitions in the sequence $(u_i)_{i \leq n}$, each element of the form kx , where $k \in \mathbb{N}$ and $x \in D_X$ can be replaced by k -many repetitions of the element x .

Next we show how we may approximate the sequence $(u_i)_{i \leq n}$, in which we allow repetitions, by a sequence $(a_i)_{i \leq n} \subseteq D_X$ in which we do not allow repetitions. For each $i \leq n$, choose some $a_i \in D_X$ such that $\|a_i - u_i\|_X < \varepsilon$. Let $(a'_i)_{i \leq n} \subseteq D_Y$ be the elements such that for all $i \leq n$ we have $a_i\mathcal{R}_0a'_i$. Since by the assumption we have $|\|u_i - a_i\|_X - \|u'_i - a'_i\|_Y| < 2\varepsilon$, we get $\|u'_i - a'_i\|_Y < 3\varepsilon$. Notice that for such sequences we get, by the computations above,

$$\left| \left\| \sum_{i \leq n} a_i \right\|_X - \left\| \sum_{i \leq n} a'_i \right\|_Y \right| \leq 4\varepsilon n.$$

This, together with the inequalities $\|u_i - a_i\|_X < \varepsilon$ and $\|u'_i - a'_i\|_Y < 3\varepsilon$ implies that

$$\left| \left\| \sum_{i \leq n} u_i \right\|_X - \left\| \sum_{i \leq n} u'_i \right\|_Y \right| \leq 8\varepsilon n,$$

which proves the inequality (11).

Before we are in the position to apply Lemma 17 we need to guarantee that \mathcal{R}_0 is \mathbb{Q} -homogeneous. Note that so far it is only closed under multiplication by positive rationals. This will be fixed in the last step.

Set $\bar{\mathcal{R}} = \mathcal{R}_0 \cup -\mathcal{R}_0$, where $-\mathcal{R}_0 = \{(x, y) : (-x)\mathcal{R}_0(-y)\}$. Now $\bar{\mathcal{R}}$ is clearly \mathbb{Q} -homogeneous. Pick now an arbitrary sequence $(u_i)_{i \leq n} \subseteq E$ and the sequence $(u'_i)_{i \leq n} \subseteq F$ such that for all $i \leq n$ we have $u_i\bar{\mathcal{R}}u'_i$. For each $i \leq n$, pick $u''_i \in F$ such that $u_i\mathcal{R}_0u''_i$. Either $u''_i = u'_i$, or by (10) we get

$\|u_i'' - u_i'\|_Y \leq 8\varepsilon\|u_i\|_X$. From these inequalities and from (11), which gives us

$$\left| \left\| \sum_{i \leq n} u_i \right\|_X - \left\| \sum_{i \leq n} u_i'' \right\|_Y \right| \leq 8\varepsilon \left(\sum_{i \leq n} \|u_i\|_X \right),$$

we get the estimate

$$\left| \left\| \sum_{i \leq n} u_i \right\|_X - \left\| \sum_{i \leq n} u_i' \right\|_Y \right| \leq 16\varepsilon \left(\sum_{i \leq n} \|u_i\|_X \right).$$

The application of Lemma 17 then gives us that $\rho_K(X, Y) < 17\varepsilon$.

It remains to see that it is possible to find an injective and Borel map $f : \mathcal{B} \rightarrow \mathcal{M}$ such that $f(X)$ is isometric to (M_X, d_X) for every $X \in \mathcal{B}$. Each Banach space X is coded as a norm $\|\cdot\|_X \in \mathcal{B}$ which is defined on a countable infinite-dimensional vector space over \mathbb{Q} denoted by V . First we need to select in a Borel way a countable dense symmetric subset of S_X . Pick $D \subseteq V \setminus \{0\}$ such that D contains exactly one element of $\{tv : t > 0\}$ for every $v \in V \setminus \{0\}$. Fix some bijection $g : D \rightarrow \mathbb{N}$ and define a metric d'_X on \mathbb{N} as follows: for $n, m \in \mathbb{N}$ we set

$$d'_X(n, m) = \left\| \frac{g^{-1}(n)}{\|g^{-1}(n)\|_X} - \frac{g^{-1}(m)}{\|g^{-1}(m)\|_X} \right\|_X.$$

This corresponds to selecting a countable dense symmetric subset of S_X with the metric inherited from $\|\cdot\|_X$. Clearly, the assignment $\|\cdot\|_X \rightarrow d'_X$ is injective and Borel. Then we only add to \mathbb{N} a fixed countable set $\{p_{F,k} : F \in [\mathbb{N}]^{<\omega} \setminus \{\emptyset\}, k \in F\}$ and define the metric d_X on the union of these two countable sets using the norm $\|\cdot\|_X$. Finally, we reenumerate this countable set so that d_X is defined on \mathbb{N} , and so belongs to \mathcal{M} . That is clearly one-to-one and Borel. \square

4. REDUCTIONS FROM PSEUDOMETRICS ON \mathcal{M} TO PSEUDOMETRICS ON \mathcal{B}

This section is devoted to the proof of the following result.

Theorem 46. *There is an injective Borel-uniformly continuous reduction from ρ_{GH} on $\mathcal{M}_{1/2}^1$ to each of the distances $\rho_K, \rho_{BM}, \rho_L, \rho_U, \rho_N, \rho_{GH}^B$ on \mathcal{B} .*

Moreover, for the distances ρ_K and ρ_{BM} , the reduction is not only Borel-uniformly continuous, but also Borel-Lipschitz.

The definition of our reduction is based on a simple geometric idea of renorming of the Hilbert space ℓ_2 . However, the proof that the idea works is technical and splits into many steps.

Let us denote by e_n the sequence in ℓ_2 that has 1 at the n -th place and 0 elsewhere. Let us moreover denote

$$e_{n,m} = \frac{1}{\sqrt{2}}(e_n + e_m), \quad \{n, m\} \in [\mathbb{N}]^2.$$

The following fact can be verified by a simple computation.

Lemma 47. For $\{n, m\}, \{n', m'\} \in [\mathbb{N}]^2$, we have

$$\begin{aligned} \|e_{n,m} - e_{n',m'}\|_{\ell_2} &= 1 && \text{if } |\{n, m\} \cap \{n', m'\}| = 1, \\ \|e_{n,m} + e_{n',m'}\|_{\ell_2} &= \sqrt{3} && \text{if } |\{n, m\} \cap \{n', m'\}| = 1, \\ \|e_{n,m} \pm e_{n',m'}\|_{\ell_2} &= \sqrt{2} && \text{if } \{n, m\}, \{n', m'\} \text{ are disjoint.} \end{aligned}$$

Hence, the set of all vectors $\pm e_{n,m}$ is 1-separated.

Let us fix numbers α and δ such that

$$1 < \alpha < \alpha + \delta \leq \frac{200}{199}.$$

For every $f : [\mathbb{N}]^2 \rightarrow [0, 1]$, we define an equivalent norm $\|\cdot\|_f$ on ℓ_2 by

$$\|x\|_f = \sup \left(\{\|x\|_{\ell_2}\} \cup \left\{ \frac{1}{\sqrt{2}} \cdot (\alpha + \delta \cdot f(n, m)) \cdot |x_n + x_m| : n \neq m \right\} \right)$$

for $x = (x_n)_{n=1}^\infty \in \ell_2$. This is an equivalent norm indeed, as $\frac{1}{\sqrt{2}} \cdot |x_n + x_m| = |\langle x, e_{n,m} \rangle| \leq \|x\|_{\ell_2}$, and consequently

$$\|x\|_{\ell_2} \leq \|x\|_f \leq \frac{200}{199} \|x\|_{\ell_2}, \quad x \in \ell_2.$$

Let us define

$$P_{n,m} = \left\{ x \in \ell_2 : \|x\|_{\ell_2} \leq \frac{1}{\sqrt{2}} \cdot (\alpha + \delta \cdot f(n, m)) \cdot (x_n + x_m) \right\}.$$

It follows from the following lemma that any non-zero $x \in \ell_2$ belongs to at most one set $\pm P_{n,m}$.

Lemma 48. Let us denote $h = \alpha + \delta \cdot f(n, m)$. If $\|x\|_{\ell_2} = 1$, then

$$x \in P_{n,m} \quad \Leftrightarrow \quad \|x - e_{n,m}\|_{\ell_2} \leq \sqrt{\frac{2(h-1)}{h}}.$$

In particular,

$$x \in P_{n,m} \quad \Rightarrow \quad \|x - e_{n,m}\|_{\ell_2} \leq \frac{1}{10}.$$

Proof. We compute

$$\begin{aligned} \|x - e_{n,m}\|_{\ell_2} \leq \sqrt{\frac{2(h-1)}{h}} &\Leftrightarrow \|x\|_{\ell_2}^2 - 2\langle x, e_{n,m} \rangle + \|e_{n,m}\|_{\ell_2}^2 \leq \frac{2(h-1)}{h} \\ &\Leftrightarrow 2\langle x, e_{n,m} \rangle \geq \frac{2}{h} \\ &\Leftrightarrow \frac{1}{\sqrt{2}} \cdot h \cdot (x_n + x_m) \geq 1 \\ &\Leftrightarrow x \in P_{n,m}. \end{aligned}$$

Finally, since $h \leq \alpha + \delta \leq \frac{200}{199}$, we have $\sqrt{\frac{2(h-1)}{h}} \leq \frac{1}{10}$. \square

Our proof of Theorem 46 is based on the following technical lemma.

Lemma 49. Let $f, g : [\mathbb{N}]^2 \rightarrow [0, 1]$. If $\rho_K((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)) < \eta$ for some η satisfying $0 < \eta < \frac{1}{100}$, $\eta < \frac{1}{10} \cdot \frac{\sqrt{\alpha^2-1}}{\alpha}$ and $\eta \leq \frac{1}{2}(1 - \frac{1}{\alpha})$, then

$$\exists \pi \in S_\infty \forall \{n, m\} \in [\mathbb{N}]^2 : |g(\pi(n), \pi(m)) - f(n, m)| < \frac{3}{\delta} \cdot \eta.$$

Due to the assumption of the lemma, we can pick a Banach space Z and isometries $I : (\ell_2, \|\cdot\|_f) \rightarrow Z$ and $J : (\ell_2, \|\cdot\|_g) \rightarrow Z$ such that

$$\rho_H^Z(I(B_{(\ell_2, \|\cdot\|_f)}), J(B_{(\ell_2, \|\cdot\|_g)})) < \eta.$$

We need to prove the following claim first.

Claim 50. *For every $\{n, m\} \in [\mathbb{N}]^2$, there are $\{k, l\} \in [\mathbb{N}]^2$ and $s \in \{-1, 1\}$ such that*

$$\|Ie_{n,m} - s \cdot Je_{k,l}\|_Z < \frac{1}{7}$$

and, moreover,

$$f(n, m) - g(k, l) < \frac{3}{\delta} \cdot \eta.$$

Proof. We prove the claim in eight steps. Let us fix $\{n, m\} \in [\mathbb{N}]^2$ and keep the notation $h = \alpha + \delta \cdot f(n, m)$ throughout the proof. Analogously as above, we define

$$Q_{k,l} = \left\{ x \in \ell_2 : \|x\|_{\ell_2} \leq \frac{1}{\sqrt{2}} \cdot (\alpha + \delta \cdot g(k, l)) \cdot (x_k + x_l) \right\}.$$

1. step: We show that there is x orthogonal to $e_{n,m}$ such that $\|x\|_{\ell_2} = 1$ and $\|Ix - \pm Je_{k,l}\|_Z \geq \frac{1}{2}$ for all $k \neq l$. This is an easy consequence of the fact that the vectors $\pm Je_{k,l}$ are 1-separated (which follows from Lemma 47 and from $\|\cdot\|_g \geq \|\cdot\|_{\ell_2}$). Indeed, let $E \subseteq \ell_2$ be a two-dimensional subspace orthogonal to $e_{n,m}$. Let us pick $x \in S_E$. If $\|Ix - \pm Je_{k,l}\|_Z \geq \frac{1}{2}$ for all k, l , we are done. In the opposite case, there is a point $w = Je_{k,l}$ or $w = -Je_{k,l}$ for which $\|Ix - w\|_Z < \frac{1}{2}$. Since $I(S_E)$ is a closed curve in Z with diameter at least 2, we can find $x' \in S_E$ such that $\|Ix' - w\|_Z = \frac{1}{2}$. Then x' works, as the distance of Ix' to other vectors is at least $1 - \frac{1}{2}$ by the triangle inequality.

2. step: We denote

$$p_+ = \frac{1}{h}e_{n,m} + \frac{\sqrt{h^2 - 1}}{h}x, \quad p_- = \frac{1}{h}e_{n,m} - \frac{\sqrt{h^2 - 1}}{h}x.$$

It is easy to see that

$$\|p_+\|_f = \|p_-\|_f = 1.$$

Let us choose q_+ and q_- with $\|q_+\|_g \leq 1, \|q_-\|_g \leq 1$, satisfying

$$\|Ip_+ - Jq_+\|_Z < \eta, \quad \|Ip_- - Jq_-\|_Z < \eta.$$

3. step: We show that

$$\frac{\|q_+ + q_-\|_g}{2} > 1 - \eta.$$

Since $\|(Ip_+ + Ip_-)/2 - (Jq_+ + Jq_-)/2\|_Z < \eta$, we have

$$\frac{\|q_+ + q_-\|_g}{2} = \left\| \frac{1}{2}(Jq_+ + Jq_-) \right\|_Z > \left\| \frac{1}{2}(Ip_+ + Ip_-) \right\|_Z - \eta = \left\| \frac{1}{h}e_{n,m} \right\|_f - \eta.$$

As $\|e_{n,m}\|_f = h$, the desired inequality follows.

4. step: We show that

$$\|q_+ - q_-\|_g = \|q_+ - q_-\|_{\ell_2},$$

as $q_+ - q_-$ does not belong to any $\pm Q_{k,l}$. Let us denote

$$u = \frac{h}{2\sqrt{h^2-1}} \cdot (q_+ - q_-) \quad \text{and} \quad z = \frac{1}{\|u\|_{\ell_2}} \cdot u.$$

Then

$$\begin{aligned} \|Ju - Ix\|_Z &= \frac{h}{2\sqrt{h^2-1}} \cdot \|(Jq_+ - Jq_-) - (Ip_+ - Ip_-)\|_Z \\ &< \frac{h}{2\sqrt{h^2-1}} \cdot 2\eta = \frac{h}{\sqrt{h^2-1}} \cdot \eta \leq \frac{\alpha}{\sqrt{\alpha^2-1}} \cdot \eta < \frac{1}{10} \end{aligned}$$

by an assumption of Lemma 49. By the choice of x (1. step), we obtain for all $k \neq l$ that

$$\|u - \pm e_{k,l}\|_g = \|Ju - \pm Je_{k,l}\|_Z > \|Ix - \pm Je_{k,l}\|_Z - \frac{1}{10} \geq \frac{1}{2} - \frac{1}{10} = \frac{2}{5}.$$

Also, it is easy to check that

$$\| \|u\|_{\ell_2} - 1 \| < \frac{1}{199} + \frac{1}{10}.$$

Indeed, we have $\|u\|_{\ell_2} \leq \|u\|_g < \|x\|_f + \frac{1}{10} \leq \frac{200}{199} \cdot \|x\|_{\ell_2} + \frac{1}{10} = 1 + \frac{1}{199} + \frac{1}{10}$ and $\|u\|_{\ell_2} \geq \frac{199}{200} \cdot \|u\|_g > \frac{199}{200} \cdot (\|x\|_f - \frac{1}{10}) \geq \frac{199}{200} \cdot (\|x\|_{\ell_2} - \frac{1}{10}) = \frac{199}{200} \cdot (1 - \frac{1}{10}) > 1 - (\frac{1}{199} + \frac{1}{10})$.

Since $z = u - (\|u\|_{\ell_2} - 1)z$ and $\|z\|_{\ell_2} = 1$, we obtain for all $k \neq l$ that

$$\|z - \pm e_{k,l}\|_{\ell_2} \geq \|u - \pm e_{k,l}\|_{\ell_2} - \| \|u\|_{\ell_2} - 1 \| \|z\|_{\ell_2} > \frac{199}{200} \cdot \frac{2}{5} - \left(\frac{1}{199} + \frac{1}{10} \right) > \frac{1}{10}.$$

By Lemma 48, z does not belong to $\pm Q_{k,l}$. The same holds for $q_+ - q_-$, as it is a multiple of z .

5. step: We show that

$$\frac{\|q_+ + q_-\|_{\ell_2}}{2} < \frac{1}{h} + \eta.$$

By the parallelogram law,

$$\|q_+ + q_-\|_{\ell_2}^2 + \|q_+ - q_-\|_{\ell_2}^2 = 2\|q_+\|_{\ell_2}^2 + 2\|q_-\|_{\ell_2}^2 \leq 2\|q_+\|_g^2 + 2\|q_-\|_g^2 \leq 4.$$

Using the conclusion of the previous step,

$$\|q_+ - q_-\|_{\ell_2} = \|q_+ - q_-\|_g > \|p_+ - p_-\|_f - 2\eta \geq \|p_+ - p_-\|_{\ell_2} - 2\eta = 2 \cdot \frac{\sqrt{h^2-1}}{h} - 2\eta,$$

$$\|q_+ - q_-\|_{\ell_2}^2 > 4 \cdot \frac{h^2-1}{h^2} - 8 \cdot \frac{\sqrt{h^2-1}}{h} \cdot \eta + 4\eta^2 > 4 \cdot \frac{h^2-1}{h^2} - 8 \cdot \frac{1}{h} \cdot \eta$$

and

$$\|q_+ + q_-\|_{\ell_2}^2 < 4 - 4 \cdot \frac{h^2-1}{h^2} + 8 \cdot \frac{1}{h} \cdot \eta < \frac{4}{h^2} + 8 \cdot \frac{1}{h} \cdot \eta + 4\eta^2 = 4 \cdot \left(\frac{1}{h} + \eta \right)^2.$$

The desired inequality follows.

6. step: We realize that $q_+ + q_-$ belongs to some $Q_{k,l}$ or $-Q_{k,l}$. In the opposite case, we obtain $\|q_+ + q_-\|_g = \|q_+ + q_-\|_{\ell_2}$ and

$$1 - \eta < \frac{\|q_+ + q_-\|_g}{2} = \frac{\|q_+ + q_-\|_{\ell_2}}{2} < \frac{1}{h} + \eta \leq \frac{1}{\alpha} + \eta,$$

which is disabled by an assumption of Lemma 49.

7. step: Let $q_+ + q_-$ belong to $s \cdot Q_{k,l}$, where $s \in \{-1, 1\}$. We show that

$$\|Ie_{n,m} - s \cdot Je_{k,l}\|_Z < \frac{1}{7}$$

for such k, l and s . If we denote

$$a = \frac{1}{\|q_+ + q_-\|_{\ell_2}} \cdot (q_+ + q_-),$$

then a belongs to $s \cdot Q_{k,l}$ as well. Lemma 48 provides

$$\|a - s \cdot e_{k,l}\|_{\ell_2} \leq \frac{1}{10},$$

and so

$$\|Ja - s \cdot Je_{k,l}\|_Z = \|a - s \cdot e_{k,l}\|_g \leq \frac{200}{199} \cdot \frac{1}{10}.$$

Since $\|(Ip_+ + Ip_-)/2 - (Jq_+ + Jq_-)/2\|_Z < \eta < \frac{1}{100}$, we obtain

$$\left\| \frac{1}{h} Ie_{n,m} - \frac{\|q_+ + q_-\|_{\ell_2}}{2} Ja \right\|_Z < \frac{1}{100}$$

and

$$\|Ie_{n,m} - Ja\|_Z < \frac{1}{100} + \left(1 - \frac{1}{h}\right) \|Ie_{n,m}\|_Z + \left(1 - \frac{\|q_+ + q_-\|_{\ell_2}}{2}\right) \|Ja\|_Z.$$

Since $\|e_{n,m}\|_f = h$, $\|a\|_{\ell_2} = 1$ and

$$\frac{\|q_+ + q_-\|_{\ell_2}}{2} \geq \frac{199}{200} \cdot \frac{\|q_+ + q_-\|_g}{2} > \frac{199}{200} \cdot (1 - \eta) > \frac{199}{200} \cdot \frac{99}{100},$$

we obtain

$$\|Ie_{n,m} - Ja\|_Z < \frac{1}{100} + (h - 1) + \left(1 - \frac{199}{200} \cdot \frac{99}{100}\right) \cdot \frac{200}{199}$$

and

$$\begin{aligned} \|Ie_{n,m} - s \cdot Je_{k,l}\|_Z &\leq \|Ie_{n,m} - Ja\|_Z + \|Ja - s \cdot Je_{k,l}\|_Z \\ &< \frac{1}{100} + \frac{1}{199} + \left(1 - \frac{199}{200} \cdot \frac{99}{100}\right) \cdot \frac{200}{199} + \frac{200}{199} \cdot \frac{1}{10} \\ &< \frac{1}{7}. \end{aligned}$$

8. step: Let $q_+ + q_-$ belong to $Q_{k,l}$ or $-Q_{k,l}$. We show that

$$f(n, m) - g(k, l) < \frac{3}{\delta} \cdot \eta$$

for such k and l . If we denote $h' = \alpha + \delta \cdot g(k, l)$, then the elements of $\pm Q_{k,l}$ fulfill

$$\|w\|_g = \frac{1}{\sqrt{2}} \cdot h' \cdot |w_k + w_l| = h' \cdot |\langle w, e_{k,l} \rangle| \leq h' \cdot \|w\|_{\ell_2}, \quad w \in \pm Q_{k,l}.$$

We obtain

$$1 - \eta < \frac{\|q_+ + q_-\|_g}{2} \leq h' \cdot \frac{\|q_+ + q_-\|_{\ell_2}}{2} < h' \cdot \left(\frac{1}{h} + \eta\right),$$

and so

$$h \cdot (1 - \eta) < h' \cdot (1 + h\eta).$$

It follows that

$$\delta \cdot (f(n, m) - g(k, l)) = h - h' < h\eta \cdot (1 + h') \leq \frac{200}{199} \cdot \left(1 + \frac{200}{199}\right) \cdot \eta < 3\eta,$$

which provides the desired inequality. \square

Proof of Lemma 49. For every $\{n, m\} \in [\mathbb{N}]^2$, Claim 50 provides $\Sigma(n, m) \in [\mathbb{N}]^2$ and $s(n, m) \in \{-1, 1\}$ such that

$$\|Ie_{n,m} - s(n, m) \cdot Je_{\Sigma(n,m)}\|_Z < \frac{1}{7}$$

and, moreover,

$$f(n, m) - g(\Sigma(n, m)) < \frac{3}{\delta} \cdot \eta.$$

Let us make a series of observations concerning $\Sigma(n, m)$ and $s(n, m)$.

(a) If $\{n, m\}$ and $\{n', m'\}$ have exactly one common element, then the same holds for $\Sigma(n, m)$ and $\Sigma(n', m')$. Indeed, using Lemma 47, we can compute

$$\begin{aligned} & \|s(n, m) \cdot e_{\Sigma(n,m)} - s(n', m') \cdot e_{\Sigma(n',m')}\|_{\ell_2} \\ & \leq \|s(n, m) \cdot e_{\Sigma(n,m)} - s(n', m') \cdot e_{\Sigma(n',m')}\|_g \\ & < \|e_{n,m} - e_{n',m'}\|_f + \frac{1}{7} + \frac{1}{7} \leq \frac{200}{199} \cdot 1 + \frac{1}{7} + \frac{1}{7} < \sqrt{2}, \\ & \|s(n, m) \cdot e_{\Sigma(n,m)} - s(n', m') \cdot e_{\Sigma(n',m')}\|_{\ell_2} \\ & \geq \frac{199}{200} \cdot \|s(n, m) \cdot e_{\Sigma(n,m)} - s(n', m') \cdot e_{\Sigma(n',m')}\|_g \\ & > \frac{199}{200} \cdot \left(\|e_{n,m} - e_{n',m'}\|_f - \frac{1}{7} - \frac{1}{7}\right) \geq \frac{199}{200} \cdot \left(1 - \frac{1}{7} - \frac{1}{7}\right) > 0, \end{aligned}$$

and it is sufficient to apply Lemma 47 again (in fact, we obtain also $s(n, m) = s(n', m')$).

(b) If $\{n, m\}$ and $\{n', m'\}$ are disjoint, then the same holds for $\Sigma(n, m)$ and $\Sigma(n', m')$. This can be shown by the same method as above. This time, we have $\|e_{n,m} - e_{n',m'}\|_{\ell_2} = \sqrt{2}$ and

$$\begin{aligned} & \|s(n, m) \cdot e_{\Sigma(n,m)} - s(n', m') \cdot e_{\Sigma(n',m')}\|_{\ell_2} < \frac{200}{199} \cdot \sqrt{2} + \frac{1}{7} + \frac{1}{7} < \sqrt{3}, \\ & \|s(n, m) \cdot e_{\Sigma(n,m)} - s(n', m') \cdot e_{\Sigma(n',m')}\|_{\ell_2} > \frac{199}{200} \cdot \left(\sqrt{2} - \frac{1}{7} - \frac{1}{7}\right) > 1. \end{aligned}$$

(c) For every $n \in \mathbb{N}$, there is $\pi(n) \in \mathbb{N}$ such that $\pi(n) \in \Sigma(n, m)$ for all $m \neq n$. Assume the opposite and pick distinct p, q different from n . By (a), we can denote the elements of $\Sigma(n, p)$ and $\Sigma(n, q)$ by a, b, c in the way that

$$\Sigma(n, p) = \{a, b\}, \quad \Sigma(n, q) = \{a, c\}.$$

By our assumption, there is $m \neq n$ such that a does not belong to $\Sigma(n, m)$. Then the only possibility for $\Sigma(n, m)$ allowed by (a) is

$$\Sigma(n, m) = \{b, c\}.$$

Pick some r different from n, m, p, q . Then there is no possibility for $\Sigma(n, r)$ allowed by (a). Indeed, no set has exactly one common element with all sets $\{a, b\}$, $\{a, c\}$ and $\{b, c\}$.

(d) The function π is injective. Indeed, assume that $n \neq m$ and pick distinct p, q different from n and m . Then $\pi(n)$ and $\pi(m)$ belong to the sets $\Sigma(n, p)$ and $\Sigma(m, q)$ that are disjoint by (b).

(e) As π is injective, we have $\Sigma(n, m) = \{\pi(n), \pi(m)\}$ for all $\{n, m\}$, and we can write

$$\|Ie_{n,m} - s(n, m) \cdot Je_{\pi(n), \pi(m)}\|_Z < \frac{1}{7}$$

and

$$f(n, m) - g(\pi(n), \pi(m)) < \frac{3}{\delta} \cdot \eta.$$

(f) Due to the symmetry, there is an injective function $\xi : \mathbb{N} \rightarrow \mathbb{N}$ with the property that

$$\|Je_{k,l} - s'(k, l) \cdot Ie_{\xi(k), \xi(l)}\|_Z < \frac{1}{7}$$

and

$$g(k, l) - f(\xi(k), \xi(l)) < \frac{3}{\delta} \cdot \eta$$

for all $\{k, l\}$ and for a suitable $s'(k, l) \in \{-1, 1\}$.

(g) We have $\pi(\xi(k)) = k$ for every k and, consequently, π is surjective. Let $k \in \mathbb{N}$ be given. For every $l \neq k$, we obtain

$$\begin{aligned} & \|e_{k,l} - s'(k, l)s(\xi(k), \xi(l)) \cdot e_{\pi(\xi(k)), \pi(\xi(l))}\|_g \\ & \leq \|Je_{k,l} - s'(k, l) \cdot Ie_{\xi(k), \xi(l)}\|_Z \\ & \quad + |s'(k, l)| \cdot \|Ie_{\xi(k), \xi(l)} - s(\xi(k), \xi(l)) \cdot Je_{\pi(\xi(k)), \pi(\xi(l))}\|_Z \\ & < \frac{1}{7} + \frac{1}{7}. \end{aligned}$$

Due to Lemma 47, this is possible only if $\{k, l\} = \{\pi(\xi(k)), \pi(\xi(l))\}$ and $s'(k, l)s(\xi(k), \xi(l)) = 1$. If we pick distinct l_1 and l_2 different from k , then $k \in \{\pi(\xi(k)), \pi(\xi(l_1))\} \cap \{\pi(\xi(k)), \pi(\xi(l_2))\} = \{\pi(\xi(k))\}$.

(h) We check that π works. We already know that $\pi \in S_\infty$ and $\pi^{-1} = \xi$. Thus, we obtain

$$g(\pi(n), \pi(m)) - f(n, m) = g(\pi(n), \pi(m)) - f(\xi(\pi(n)), \xi(\pi(m))) < \frac{3}{\delta} \cdot \eta$$

for every $\{n, m\} \in [\mathbb{N}]^2$. Finally, combining this with an above inequality,

$$|g(\pi(n), \pi(m)) - f(n, m)| < \frac{3}{\delta} \cdot \eta,$$

which completes the proof of the lemma. \square

Proof of Theorem 46. During the proof, we make no difference between a metric $f \in \mathcal{M}_{1/2}^1$ and the corresponding function $f : [\mathbb{N}]^2 \rightarrow [1/2, 1]$. For $f \in \mathcal{M}_{1/2}^1$, we can thus consider the norm $\|\cdot\|_f$ defined above. It is clear that there is an injective Borel mapping from $\mathcal{M}_{1/2}^1$ into \mathcal{B} such that the image of f is isometric to $(\ell_2, \|\cdot\|_f)$ (it is sufficient to restrict the norm $\|\cdot\|_f$ to V).

To prove the first part of the theorem, we show a series of inequalities that illustrates that the Gromov-Hausdorff distance of M_f and M_g and all the involved distances between $(\ell_2, \|\cdot\|_f)$ and $(\ell_2, \|\cdot\|_g)$ are uniformly equivalent.

(1) We show that

$$\rho_{BM}((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)) \leq C\rho_{GH}(M_f, M_g)$$

for every $f, g \in \mathcal{M}_{1/2}^1$. If $\rho_{GH}(M_f, M_g) \geq \frac{1}{4}$, then

$$\rho_{BM}((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)) \leq 2 \log\left(\frac{200}{199}\right) \leq 2 \log\left(\frac{200}{199}\right) \cdot 4\rho_{GH}(M_f, M_g).$$

Assuming $\rho_{GH}(M_f, M_g) < \frac{1}{4}$, we pick r with $\rho_{GH}(M_f, M_g) < r < \frac{1}{4}$. Since $f, g \in \mathcal{M}_p$ and $\rho_{GH}(M_f, M_g) < p/2$ for $p = 1/2$, Lemma 13 provides $\pi \in S_\infty$ such that

$$|g(\pi(n), \pi(m)) - f(n, m)| \leq 2r, \quad \{n, m\} \in [\mathbb{N}]^2.$$

Let us consider the isometry $T : \ell_2 \rightarrow \ell_2$ which maps e_n to $e_{\pi(n)}$. For $x \in \ell_2$, we have

$$\begin{aligned} \|Tx\|_g &= \sup\left(\{\|Tx\|_{\ell_2}\} \cup \left\{\frac{1}{\sqrt{2}} \cdot (\alpha + \delta \cdot g(k, l)) \cdot |(Tx)_k + (Tx)_l| : k \neq l\right\}\right) \\ &= \sup\left(\{\|x\|_{\ell_2}\} \cup \left\{\frac{1}{\sqrt{2}} \cdot (\alpha + \delta \cdot g(\pi(n), \pi(m))) \cdot |x_n + x_m| : n \neq m\right\}\right), \end{aligned}$$

and so

$$\begin{aligned} \|\|Tx\|_g - \|x\|_f\| &\leq \sup\left\{\frac{1}{\sqrt{2}} \cdot \delta \cdot |g(\pi(n), \pi(m)) - f(n, m)| \cdot |x_n + x_m| : n \neq m\right\} \\ &\leq \delta \cdot 2r \cdot \|x\|_{\ell_2}. \end{aligned}$$

It follows that $\|Tx\|_g \leq (1 + 2\delta r)\|x\|_f$ and $\|x\|_f \leq (1 + 2\delta r)\|Tx\|_g$. We obtain $\rho_{BM}((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)) \leq 2 \log(1 + 2\delta r) \leq 2 \cdot 2\delta r$. As r could be chosen arbitrarily close to $\rho_{GH}(M_f, M_g)$, we arrive at

$$\rho_{BM}((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)) \leq 4\delta\rho_{GH}(M_f, M_g).$$

Therefore, the choice $C = \max\{8 \log(\frac{200}{199}), 4\delta\}$ works.

(2) It is easy to check that

$$\begin{aligned} 2\rho_N((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)) &\leq \rho_U((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)) \\ &\leq 2\rho_L((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)) \leq \rho_{BM}((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)) \end{aligned}$$

for every $f, g \in \mathcal{M}_{1/2}^1$.

(3) By [10, Proposition 2.1], we have

$$\rho_{GH}^{\mathcal{B}}((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)) \leq e^{2\rho_N((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g))} - 1$$

for every $f, g \in \mathcal{M}_{1/2}^1$.

(4) There is a function $\varphi : (0, 1] \rightarrow (0, 1]$ with $\lim_{\varepsilon \rightarrow 0} \varphi(\varepsilon) = 0$ such that

$$\rho_K((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)) \leq C\varphi\left(\rho_{GH}^{\mathcal{B}}((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g))\right)$$

for every $f, g \in \mathcal{M}_{1/2}^1$. Considering any $0 < r < 1$, a function provided by [18, Theorem 3.6] (denoted f there) works. Indeed, if we adopt some notation from [18], then [18, Theorem 3.7] provides

$$\begin{aligned} \rho_K((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)) &\leq C(r) \cdot \kappa_0((\ell_2, \|\cdot\|_f)) \cdot d_r((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)) \\ &\leq C(r) \cdot \frac{200}{199} \cdot \kappa_0(\ell_2) \cdot \varphi\left(\rho_{GH}^{\mathcal{B}}((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g))\right). \end{aligned}$$

(5) We show that

$$\rho_{GH}(M_f, M_g) \leq C \rho_K((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g))$$

for every $f, g \in \mathcal{M}_{1/2}^1$. Let us denote

$$d = \rho_K((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)), \quad \eta_{max} = \min \left\{ \frac{1}{100}, \frac{1}{10} \cdot \frac{\sqrt{\alpha^2 - 1}}{\alpha}, \frac{1}{2} \left(1 - \frac{1}{\alpha} \right) \right\}.$$

If $d \geq \eta_{max}$, then

$$\rho_{GH}(M_f, M_g) \leq 1 \leq \frac{1}{\eta_{max}} \cdot d.$$

Assuming $d < \eta_{max}$, we pick $d < \eta < \eta_{max}$. Then Lemma 49 can be applied, and we obtain $f \simeq_{2\varepsilon} g$ for $\varepsilon = \frac{1}{2} \cdot \frac{3}{\delta} \cdot \eta$. By Lemma 12, we get $\rho_{GH}(M_f, M_g) \leq \frac{1}{2} \cdot \frac{3}{\delta} \cdot \eta$. As η can be chosen arbitrarily close to d , we arrive at

$$\rho_{GH}(M_f, M_g) \leq \frac{1}{2} \cdot \frac{3}{\delta} \cdot d.$$

It follows that the choice $C = \max\{\frac{1}{\eta_{max}}, \frac{3}{2\delta}\}$ works.

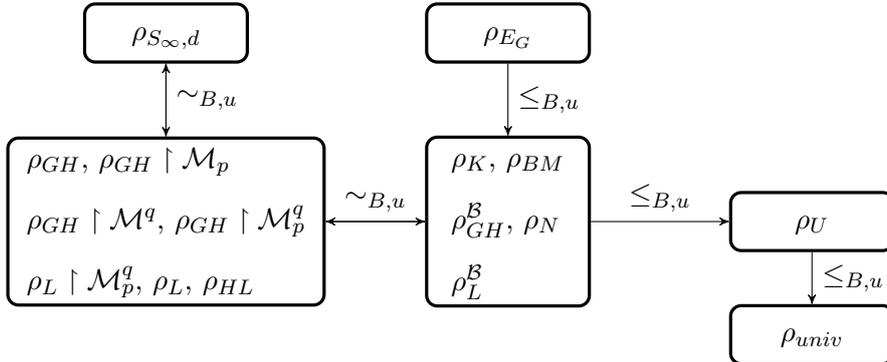
Finally, concerning the moreover part of the theorem, it remains to notice that

$$\rho_K((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)) \leq \rho_{BM}((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g))$$

by [27, Proposition 6.2] (or [10, Proposition 2.1]), and it is sufficient to use the inequalities proven in (1) and (5). \square

5. CONCLUDING REMARKS

The following diagram summarizes the reducibility results we have proved in this paper and includes also the reducibility results proved in [8]. By ρ_{E_G} we denote the pseudometric induced by the universal orbit equivalence relation E_G discussed in [8, Section 5]; by $\rho_{S_\infty, d}$ we denote the CTR orbit pseudometric given in [8, Section 3, Example 2], by ρ_{univ} we mean the universal analytic pseudometric which exists by [8, Theorem 6]; all the remaining pseudometrics are explained in this paper. The reducibilities which are not explicitly mentioned in the diagram are not known to us.



We believe there is enough space for investigating other reductions. The interested reader can find many more distances for which their exact place in the reducibility diagram is not known. This includes the uniform distance ρ_U , or distances that we mentioned in [8, Section 3] but left untouched, such

as the completely bounded Banach-Mazur distance or e.g. the orbit version of the Kadison-Kastler distance.

Acknowledgements

M. Cúth was supported by Charles University Research program No. UNCE/SCI/023 and by the Research grant GAČR 17-04197Y. M. Doucha was supported by the GAČR projects 16-34860L and EXPRO 20-31529X, and RVO: 67985840. O. Kurka was supported by the Research grant GAČR 17-04197Y and by RVO: 67985840.

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