

Modelling HM Processes in Porous Media

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1 Continuum approach to porous media

[BB90, Chs. 1, 2], [LS98, Chs. 1, 2], [dB00, Ch. 5], [KGSW12, Sec. 2.2]

A porous medium is formed by a *solid matrix* and a *void space* (pore space), which is occupied by one or more fluids. A *constituent* (or a *phase*) is a part of the porous medium that is separated from other such parts by sharp interfaces (e.g., a solid, water, air).

At the *microscopic level*, state variables that describe the behaviour of a particular constituent are defined only within subdomains occupied by the constituent. To treat the porous medium with the methods of continuum mechanics, one introduces a *macroscopic level*, where the variables and quantities are defined at *every point* in the porous medium domain.

There are two major approaches that can be used to pass to macroscopic quantities from the microscopic ones: the volume fraction concept and homogenisation. We shall introduce only the first one briefly here.

In the *volume fraction concept*, each point of a control space of a porous medium is considered to be a centroid of a so-called representative elementary volume or average volume element dv . In addition, it is assumed that each volume element is composed of microscopic volume elements dv_m of real constituents (see Figure 1). Let π denote individual constituents of the porous material. In particular, $\pi = s$ marks a solid whereas $\pi = f$ is used for fluids. The partial volume element dv^π is the volume of constituent π within dv . Similarly we write da^π for the part occupied by constituent π of the area da of the volume element, and da_m for microscopic area elements.

To describe the microstructure of a porous medium in the macroscopic manner, neglecting the real topology of the pore structure and the exact location of the individual constituents, one defines

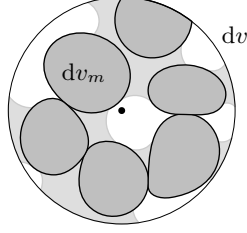


Figure 1: Average volume element dv of a porous medium consisting of three constituents.

the following variables for measuring local fractions of the constituents:

$$\begin{aligned} \eta_\pi &:= \frac{dv^\pi}{dv} \text{ — the volume fraction of constituent } \pi \\ \phi &:= \frac{dv - dv^s}{dv} \text{ — the porosity} \\ S_f &:= \frac{dv^f}{dv - dv^s} = \frac{\eta_f}{\phi} \text{ — the saturation of fluid } f \end{aligned}$$

Macroscopic quantities can be derived from microscopic ones by averaging: taking a microscopic variable ξ , one can introduce

$$\begin{aligned} \langle \xi \rangle_\pi &:= \frac{1}{dv} \int_{dv^\pi} \xi dv_m \text{ — the volume phase average} \\ \langle \xi \rangle_\pi^\pi &:= \frac{1}{dv^\pi} \int_{dv^\pi} \xi dv_m \text{ — the volume intrinsic phase average} \\ \bar{\xi}^\pi &:= \frac{\int_{dv^\pi} \rho \xi dv_m}{\int_{dv^\pi} \rho dv_m} \text{ — the mass average} \\ &\quad \rho \text{ — the microscopic mass density} \end{aligned}$$

and for a vectorial or tensorial ξ

$$\begin{aligned} \bar{\bar{\xi}}^\pi &:= \frac{1}{da} \int_{da^\pi} \xi da_m \text{ — the area average} \\ \bar{\xi}_\pi^\pi &:= \frac{1}{da^\pi} \int_{da^\pi} \xi da_m \text{ — the area intrinsic phase average} \end{aligned}$$

From the definition of volume fraction, the volume averages are related to each other by

$$\langle \xi \rangle_\pi = \eta_\pi \langle \xi \rangle_\pi^\pi$$

Moreover

$$\begin{aligned} \text{if the microscopic mass density } \rho \text{ is constant} &\implies \langle \xi \rangle_\pi^\pi = \bar{\xi}^\pi \\ \text{if the microscopic variable } \xi \text{ is constant} &\implies \langle \xi \rangle_\pi^\pi = \xi \end{aligned}$$

Delesse's law: the surface ratio of each constituent must be equal to its volume ratio on each cut surface in an isotropic mixture. \implies The volume and area averages differ only slightly from each other for an isotropic distribution of constituents.

Associating the average volumes dv and the average areas da with each point of the control space in the definitions above, one obtains macroscopic quantities which are defined in the total control space, and which can be interpreted as local statistical averages of values at the underlying microscale. Within this context, the porous material under consideration is theoretically substituted by a model where each constituent is "smeared" over the control space, and it occupies the

total volume simultaneously with the other constituents. One then speaks of overlapping partial continua.

Two strategies are used to arrive at a description of the mechanical and thermodynamic behaviour of these substitute continua: *Mixture theory* treats the porous medium as a mixture of all constituents directly from a macromechanics viewpoint. On the other hand in *averaging theories*, averaging is used for introducing the macroscopic description from the microscopic one. Especially, macroscopic quantities such as the velocity, external body force, internal energy or external supply of heat are obtained through volume or mass averaging, whereas the macroscopic mass densities are volume-averaged ones. Area averaging is then employed to derive macroscopic quantities such as the stress tensor or heat flux.

In each approach of porous media theory in general, a variety of simplifying assumptions and constraints is introduced, e.g., that the pores and mass of all constituents are statistically uniformly or even periodically distributed. Nevertheless, we shall deal directly with a common macroscopic description with macroscopic quantities, and we shall tacitly suppose that it is relevant for all the physical phenomena involved in the intended applications. In particular, the complete governing equations of the models in the following sections are derived from macroscopic balance equations, which are closed by a sufficient number of constitutive relationships.

Eventually, let us emphasise that rigorous constitutive relations require a consistent theory established on mechanical and thermodynamic principles such as the material objectivity or the entropy principle. However, most of the constitutive models presented below result from experiments and they can be regarded as mere approximations of models developed from fundamentals of mechanics and thermodynamics. Hence one has to be aware of their limited applicability to various problems!

2 Saturated flow

[BC10, Chs. 4, 5]

The entire void space is occupied by water (index w).

Balance equations

Mass balance equation for water

$$\frac{\partial(\phi\rho_w)}{\partial t} = -\operatorname{div}(\rho_w\mathbf{q}_w) + Q_w \quad (2.1)$$

ρ_w — the *mass density* of water t — the *time*

$\mathbf{q}_w \equiv \phi\mathbf{v}_w$ — the *water specific discharge* \mathbf{v}_w — the *water velocity*

Q_w — an external *source* (or a *sink* if negative) of water mass

(= added mass of water per unit volume of porous medium, per unit time)

Introducing

\mathbf{v}_s — the *solid velocity*

$\mathbf{v}_{rw} \equiv \mathbf{v}_w - \mathbf{v}_s$ — the *water relative velocity*

$\mathbf{q}_{rw} \equiv \phi\mathbf{v}_{rw}$ — the *water specific discharge relative to the solid (Darcy velocity)*

(= water volume passing through a unit area of porous medium per unit time)

$\frac{D_s}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}_s \cdot \nabla$ — the *total (or material) time derivative* with respect to the solid phase

one can rewrite (2.1) in the form

$$\begin{aligned}\frac{\partial(\phi\rho_w)}{\partial t} &= -\operatorname{div}(\phi\rho_w(\mathbf{v}_w - \mathbf{v}_s)) - \operatorname{div}(\phi\rho_w\mathbf{v}_s) + Q_w \\ &= -\operatorname{div}(\rho_w\mathbf{q}_{rw}) - \mathbf{v}_s \cdot \nabla(\phi\rho_w) - \phi\rho_w \operatorname{div} \mathbf{v}_s + Q_w \\ \frac{D_s(\phi\rho_w)}{Dt} &= -\operatorname{div}(\rho_w\mathbf{q}_{rw}) - \phi\rho_w \operatorname{div} \mathbf{v}_s + Q_w\end{aligned}\quad (2.2)$$

where we have used

$$\operatorname{div}(\phi\rho_w\mathbf{v}_s) = \nabla(\phi\rho_w) \cdot \mathbf{v}_s + \phi\rho_w \operatorname{div} \mathbf{v}_s$$

Mass balance equation for solid

$$\frac{\partial((1-\phi)\rho_s)}{\partial t} = -\operatorname{div}((1-\phi)\rho_s\mathbf{v}_s) \quad (2.3)$$

ρ_s — the solid mass density

– It is needed in the case of a deformable porous medium ($\mathbf{v}_s \neq \mathbf{0}$, $\partial\phi/\partial t \neq 0$)
Similarly as before, (2.3) can be rewritten in the form

$$\frac{D_s((1-\phi)\rho_s)}{Dt} = -(1-\phi)\rho_s \operatorname{div} \mathbf{v}_s \quad (2.4)$$

Constitutive relationships

Water density

We consider that the water is compressible and its density depends on the pressure:

$$\rho_w = \rho_{wo} e^{c_w(p_w - p_{wo})} \quad (2.5)$$

c_w — the coefficient of water *compressibility*

p_w — the *pressure* in the water (positive for compression)

ρ_{wo} , p_{wo} — initial values of the water density and pressure (steady states at standard conditions)

Assuming that c_w is constant (over a certain range of pressures) yields

$$\frac{D_s\rho_w}{Dt} = \frac{d\rho_w}{dp_w} \frac{D_s p_w}{Dt} = \rho_w c_w \frac{D_s p_w}{Dt} \quad (2.6)$$

Darcy's law

$$\mathbf{q}_{rw} = \frac{\mathbf{k}}{\mu} (-\nabla p_w + \rho_w \mathbf{f}) \quad (2.7)$$

\mathbf{k} — the *permeability* (tensor) of the porous medium

μ — the *dynamic viscosity* of water

\mathbf{f} — a *body force* density per unit mass (usually due to gravity: $\mathbf{f} = -g\nabla z$ with

g — the *gravity acceleration*, z — the *elevation* above some datum level,

∇z = a unit vector directed vertically upward)

- Obtained empirically first, but can be also derived from the **water momentum balance equation** under certain simplifying assumptions.

Solid phase

In the case of $D_s\phi/Dt \neq 0$ (a deformable porous medium), we have to investigate stresses: The *total stress* tensor in a porous medium is given by (the stresses are taken positive for tension unlike the water pressure)

$$\begin{aligned}\boldsymbol{\sigma} &= (1 - \phi)\boldsymbol{\sigma}_s + \phi\boldsymbol{\sigma}_w, & \boldsymbol{\sigma}_w &= -p_w\mathbf{I} + \boldsymbol{\tau}_w \\ \boldsymbol{\sigma}_s, \boldsymbol{\sigma}_w &— \text{the stress tensors in the solid and water} \\ \boldsymbol{\tau}_w &— \text{the } \textit{shear} \text{ (or deviatoric) stress in the water}\end{aligned}$$

Neglecting $\boldsymbol{\tau}_w$, one obtains *approximately*

$$\begin{aligned}\boldsymbol{\sigma} &= (1 - \phi)\boldsymbol{\sigma}_s - \phi p_w\mathbf{I} = (1 - \phi)(\boldsymbol{\sigma}_s + p_w\mathbf{I}) - p_w\mathbf{I} = \boldsymbol{\sigma}'_s - p_w\mathbf{I} \\ \boldsymbol{\sigma}'_s &\equiv (1 - \phi)(\boldsymbol{\sigma}_s + p_w\mathbf{I}) — \text{the } \textit{effective stress} \text{ tensor}\end{aligned}\tag{2.8}$$

In a *granular* porous medium with an incompressible grain material, the pressure in the water (or in the fluids in a multiphase system) that almost completely surrounds each solid grain produces no deformation of the grains and does not contribute to deformation of the solid skeleton. Instead, the deformation occurs mainly by rearrangement of grains due to the forces at the points of contact between the grains. Hence, $\boldsymbol{\sigma}'_s$ is the strain-producing part of the stress.

Assumption 1. The horizontal stresses are negligible, so it suffices to consider the vertical ones only.

In this case, (2.8) reduces to

$$\sigma = \sigma'_s - p_w\tag{2.9}$$

Assumption 2. The deformation of the solid phase is *volume preserving* (not of the solid skeleton — voids may be rearranged).

This can be expressed by

$$\frac{\partial v^s}{\partial \sigma'_s} = 0, \quad v^s — \text{the volume of solid of a given fixed mass } m^s$$

or in terms of the porous medium volume v containing v^s ($v = v^s/(1 - \phi)$)

$$\begin{aligned}\frac{\partial v^s}{\partial \sigma'_s} &= (1 - \phi)\frac{\partial v}{\partial \sigma'_s} + v\frac{\partial(1 - \phi)}{\partial \sigma'_s} = 0 \\ \frac{1}{v}\frac{\partial v}{\partial \sigma'_s} &= \frac{1}{1 - \phi}\frac{\partial \phi}{\partial \sigma'_s}\end{aligned}$$

Now assume, that we deal with **relatively small volume changes** and the solid behaves as an **elastic material**. For this case of vertical stresses only, one can define

$$c \equiv \frac{1}{v}\frac{\partial v}{\partial \sigma'_s} = \frac{1}{1 - \phi}\frac{\partial \phi}{\partial \sigma'_s} — \text{the } \textit{compressibility coefficient} \text{ of the } \textit{solid skeleton}$$

Finally assuming **no change in the total stress**, i.e., $\partial\sigma = 0$, hence $\partial\sigma'_s = \partial p_w$ according to (2.9), we get

$$c = \frac{1}{1 - \phi}\frac{\partial \phi}{\partial p_w}\tag{2.10}$$

$$\frac{D_s\phi}{Dt} = (1 - \phi)c\frac{D_s p_w}{Dt}\tag{2.11}$$

By integrating (2.10) we obtain

$$\phi = 1 + (\phi_o - 1)e^{-c(p_w - p_{w_o})}, \quad \phi_o — \text{an initial porosity}\tag{2.12}$$

Remark 1. The resulting relations for the porosity can be employed also in more general cases, where flow and solid deformation do not occur mainly in the vertical direction. However, this simplified approach is not always sufficient, and three-dimensional solid deformation has to be included in the model generally (poromechanics).

Flow equations

The balance laws are now transformed and the constitutive equations applied for obtaining complete flow equations: Dividing the solid mass balance equation (2.4) by $(1 - \phi)\rho_s$ yields

$$\frac{1}{1 - \phi} \frac{D_s(1 - \phi)}{Dt} = -\frac{1}{\rho_s} \frac{D_s\rho_s}{Dt} - \operatorname{div} \mathbf{v}_s \quad (2.13)$$

(the left-hand side may be interpreted as the relative rate of expansion of the volume occupied by the solid phase)

Under **Assumption 2**, $D_s\rho_s/Dt = 0$ (not $\operatorname{div} \mathbf{v}_s = 0$ at the macroscopic level!), and (2.13) reduces to

$$\frac{1}{1 - \phi} \frac{D_s\phi}{Dt} = -\frac{1}{1 - \phi} \frac{D_s(1 - \phi)}{Dt} = \operatorname{div} \mathbf{v}_s \quad (2.14)$$

Consequently one can develop the total time derivative in the water mass balance equation (2.2) and eliminate $\operatorname{div} \mathbf{v}_s$:

$$\phi \frac{D_s\rho_w}{Dt} + \rho_w \frac{D_s\phi}{Dt} = -\operatorname{div}(\rho_w \mathbf{q}_{rw}) - \phi\rho_w \operatorname{div} \mathbf{v}_s + Q_w \quad (2.15)$$

$$\phi \frac{D_s\rho_w}{Dt} + \rho_w \frac{1}{1 - \phi} \frac{D_s\phi}{Dt} = -\operatorname{div}(\rho_w \mathbf{q}_{rw}) + Q_w \quad (2.16)$$

Expressing the material derivatives of ρ_w and ϕ by (2.6) and (2.11) one obtains

$$\begin{aligned} \phi \frac{D_s\rho_w}{Dt} + \rho_w \frac{1}{1 - \phi} \frac{D_s\phi}{Dt} &= \phi\rho_w c_w \frac{D_s p_w}{Dt} + \rho_w c \frac{D_s p_w}{Dt} = -\operatorname{div}(\rho_w \mathbf{q}_{rw}) + Q_w \\ c_{ms} \frac{D_s p_w}{Dt} &= -\operatorname{div}(\rho_w \mathbf{q}_{rw}) + Q_w \end{aligned} \quad (2.17)$$

$$c_{ms} \equiv \rho_w(\phi c_w + c) \text{ --- the specific mass storativity} \quad (2.18)$$

Inserting Darcy's law (2.7) into (2.17) we get

$$c_{ms} \frac{D_s p_w}{Dt} = -\operatorname{div} \left(\rho_w \frac{\mathbf{k}}{\mu} (-\nabla p_w + \rho_w \mathbf{f}) \right) + Q_w \quad (2.19)$$

Assumption 3. Local spatial variations of p_w are much smaller than the temporal ones, or the solid velocity \mathbf{v}_s is slow:

$$\left| \frac{\partial p_w}{\partial t} \right| \gg |\mathbf{v}_s \cdot \nabla p_w|$$

Then (2.19) reduces *approximately* to

$$\boxed{c_{ms} \frac{\partial p_w}{\partial t} = -\operatorname{div} \left(\rho_w \frac{\mathbf{k}}{\mu} (-\nabla p_w + \rho_w \mathbf{f}) \right) + Q_w} \quad (2.20)$$

In addition, the following assumption may also be justified in practice:

Assumption 4. The temporal rate of density change at a point is much larger than the spatial one:

$$\left| \frac{\partial \rho_w}{\partial t} \right| \gg |\mathbf{v}_{rw} \cdot \nabla \rho_w|$$

Then

$$\left| \phi \frac{D_s \rho_w}{Dt} \right| \gg |\mathbf{q}_{rw} \cdot \nabla \rho_w|$$

and (2.16), (2.17) and (2.20) may be reduced *approximately* to

$$\begin{aligned}\phi \frac{D_s \rho_w}{Dt} + \rho_w \frac{1}{1-\phi} \frac{D_s \phi}{Dt} &= -\rho_w \operatorname{div} \mathbf{q}_{rw} + Q_w \\ c_s \frac{D_s p_w}{Dt} &= -\operatorname{div} \mathbf{q}_{rw} + \frac{Q_w}{\rho_w}, \quad c_s \equiv \phi c_w + c \\ c_s \frac{\partial p_w}{\partial t} &= -\operatorname{div} \left(\frac{\mathbf{k}}{\mu} (-\nabla p_w + \rho_w \mathbf{f}) \right) + \frac{Q_w}{\rho_w}\end{aligned}$$

Remark 2. (i) The last two equations are derived under the assumption that the water density behaves approximately constant, except in the expression for c_s , where one takes into account water compressibility.

(ii) Deformation of the porous medium is not really considered in the above derivation either. It is assumed that its effect can be incorporated in the coefficient of specific storativity, as well.

Remark 3. **Assume** that one can take

$$\operatorname{div}(\rho_w \mathbf{q}_w) = \operatorname{div}(\rho_w \mathbf{q}_{rw}) \quad (2.21)$$

which can be justified in one of the following cases:

- the solid velocity is negligible with respect to the water one:

$$|\mathbf{v}_w \cdot \nabla(\phi \rho_w)| \gg |\mathbf{v}_s \cdot \nabla(\phi \rho_w)| \quad \text{and} \quad |\operatorname{div} \mathbf{v}_w| \gg |\operatorname{div} \mathbf{v}_s|$$

- the solid matrix is stationary or the coordinate system moves with the solid phase, thus $\mathbf{v}_s = \mathbf{0}$.

Then the Darcy law (2.7) can be introduced directly into the water mass balance equation (2.1), and one gets

$$\begin{aligned}\frac{\partial(\phi \rho_w)}{\partial t} &= \operatorname{div} \left(\rho_w \frac{\mathbf{k}}{\mu} (\nabla p_w + \rho_w g \nabla z) \right) + Q_w \\ \frac{\partial(\phi \rho_w)}{\partial t} &= \phi \frac{\partial \rho_w}{\partial t} + \rho_w \frac{\partial \phi}{\partial t} = \left(\phi \frac{\partial \rho_w}{\partial p_w} + \rho_w \frac{\partial \phi}{\partial p_w} \right) \frac{\partial p_w}{\partial t} \stackrel{(2.5),(2.10)}{=} c_{ms}^* \frac{\partial p_w}{\partial t} \\ c_{ms}^* &\equiv \rho_w (\phi c_w + (1-\phi)c) \quad (\text{under } \mathbf{Assumptions 1, 2})\end{aligned} \quad (2.22)$$

$$c_{ms}^* \frac{\partial p_w}{\partial t} = -\operatorname{div} \left(\rho_w \frac{\mathbf{k}}{\mu} (-\nabla p_w + \rho_w \mathbf{f}) \right) + Q_w$$

where c_{ms}^* is another form for the specific storativity.

Eventually in the case of **Assumption 4** one obtains *approximately*

$$c_s^* \frac{\partial p_w}{\partial t} = -\operatorname{div} \left(\frac{\mathbf{k}}{\mu} (-\nabla p_w + \rho_w \mathbf{f}) \right) + \frac{Q_w}{\rho_w}, \quad c_s^* \equiv \phi c_w + (1-\phi)c$$

Moreover taking into account the physical interpretation of the left-hand side of (2.22), one can see that c_{ms}^* is the *mass of water released from (or added to) storage in a unit volume of a deformable porous medium per unit decline (or rise) in water pressure*. Hence the term specific storativity. In addition under **Assumption 3** one has *approximately*

$$c_{ms} \frac{\partial p_w}{\partial t} = c_{ms} \frac{D_s p_w}{Dt} \stackrel{(2.17)}{=} -\operatorname{div}(\rho_w \mathbf{q}_{rw}) + Q_w$$

and in consequence of (2.21)

$$c_{ms} \frac{\partial p_w}{\partial t} = -\operatorname{div}(\rho_w \mathbf{q}_w) + Q_w \stackrel{(2.1)}{=} \frac{\partial(\phi \rho_w)}{\partial t}$$

Accordingly, the previous verbal definition of the specific storativity can be applied to c_{ms} as well.

3 Unsaturated flow

[BC10, Ch. 6]

Let the void space be partly filled by water and partly by air (index a) — an unsaturated zone.

Balance equations

Mass balance equation for water

$$\frac{\partial(\phi S_w \rho_w)}{\partial t} = -\operatorname{div}(\rho_w \mathbf{q}_w) + Q_w \quad (3.1)$$

$\mathbf{q}_w \equiv \phi S_w \mathbf{v}_w$ — the *specific discharge* of water

- No exchange of mass between water and air.

Introducing

$$\mathbf{q}_{rw} \equiv \phi S_w \mathbf{v}_{rw} = \phi S_w (\mathbf{v}_w - \mathbf{v}_s) \text{ — the water } \textit{specific discharge relative to the solid (Darcy velocity)}$$

one can transform (3.1) into

$$\begin{aligned} \frac{\partial(\phi S_w \rho_w)}{\partial t} &= -\operatorname{div}(\phi S_w \rho_w (\mathbf{v}_w - \mathbf{v}_s)) - \operatorname{div}(\phi S_w \rho_w \mathbf{v}_s) + Q_w \\ &= -\operatorname{div}(\rho_w \mathbf{q}_{rw}) - \mathbf{v}_s \cdot \nabla(\phi S_w \rho_w) - \phi S_w \rho_w \operatorname{div} \mathbf{v}_s + Q_w \\ \frac{D_s(\phi S_w \rho_w)}{Dt} &= -\operatorname{div}(\rho_w \mathbf{q}_{rw}) - \phi S_w \rho_w \operatorname{div} \mathbf{v}_s + Q_w \end{aligned} \quad (3.2)$$

Mass balance equation for solid

— in the form (2.4) or (2.13)

Constitutive relationships

Retention curve

Introduce the *capillary pressure* (also called *matric suction*)

$$p_c \equiv p_a - p_w \quad (3.3)$$

p_a — the air pressure

and the *suction head*

$$\psi \equiv \frac{p_c}{g\rho_w} \quad (3.4)$$

(this one should be employed only when ρ_w is constant)

There is a relationship between S_w and p_c or ψ — water *retention curves* $S_w = S_w(p_c)$ or $S_w = S_w(\psi)$.

- Retention curves usually depend on the direction and history of drainage and wetting (*hysteresis*). However, we **assume** that these dependencies may be *neglected*.

Then

$$\begin{aligned} \frac{D_s S_w}{Dt} &= \frac{dS_w}{dp_c} \frac{D_s p_c}{Dt} = \frac{dS_w}{dp_c} \left(\frac{D_s p_a}{Dt} - \frac{D_s p_w}{Dt} \right) = \frac{c_m}{\phi} \left(\frac{D_s p_w}{Dt} - \frac{D_s p_a}{Dt} \right) \\ c_m &= c_m(S_w) \equiv -\phi \frac{dS_w}{dp_c} \text{ — the } \textit{water (moisture) capacity} \end{aligned}$$

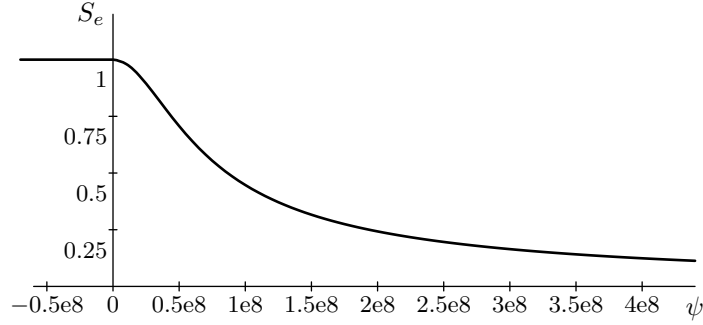


Figure 2: The function $\psi \mapsto S_e(\psi)$ given by (3.5) with $c_{vg} = 2e-8$, $n = 2$ and $m = 1/2$.

An instance of a retention curve proposed in [vG80] (Figure 2):

$$S_e \equiv \frac{S_w - S_r}{S_s - S_r} = S_e(\psi) = \begin{cases} (1 + (c_{vg}\psi)^n)^{-m} & \text{if } \psi \geq 0, \\ 1 & \text{if } \psi < 0 \end{cases} \quad (3.5)$$

S_e — the *effective water saturation*

S_r — the *residual water saturation* (after drying)

S_s — the level of full saturation

c_{vg} , n , $m > 0$ — coefficients

Assumption 5 (stationary air). The resistance to flow in the air phase is everywhere negligible, so that the air is at a constant (hydrostatic) atmospheric pressure, taken as the (reference) zero datum: $p_a = 0$.

Then (3.3) and (3.4) reduce to

$$p_c = -p_w, \quad \psi = -\frac{p_w}{g\rho_w}$$

and water retention curves lead to relationships $S_w = S_w(p_w)$ and

$$\frac{D_s S_w}{Dt} = \frac{dS_w}{dp_w} \frac{D_s p_w}{Dt} = \frac{c_m}{\phi} \frac{D_s p_w}{Dt}, \quad c_m = \phi \frac{dS_w}{dp_w} \quad (3.6)$$

Water density

– the relations (2.5), (2.6)

Darcy's law for water

$$\mathbf{q}_{rw} = \frac{\mathbf{k}}{\mu} (-\nabla p_w + \rho_w \mathbf{f}) \quad (3.7)$$

$\mathbf{k} = \mathbf{k}(S_w)$ — the *effective permeability* (tensor) of the porous medium to water

- It can be derived from the **momentum balance equation for water** under certain simplifying assumptions.

We assume

$$\mathbf{k}(S_w) = \mathbf{k}_s k_r(S_w) \quad (3.8)$$

\mathbf{k}_s — the permeability of the porous medium at full saturation
(also called *intrinsic permeability*)

$k_r(S_w)$ — the *relative permeability* to water

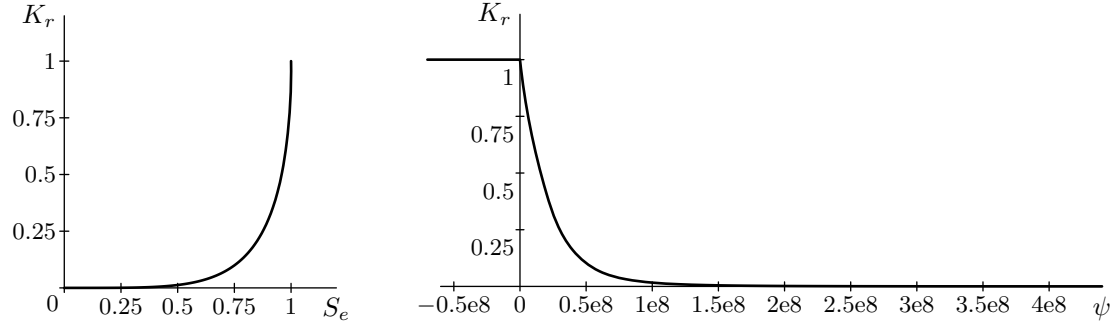


Figure 3: The functions $S_e \mapsto K_r(S_e)$ (left) and $\psi \mapsto K_r(S_e(\psi))$ (right) given by (3.9) and (3.10), respectively, with $c_{vg} = 2e-8$, $n = 2$ and $m = 1/2$.

or in terms of *hydraulic conductivities*

$$\mathbf{K}(S_w) \equiv \frac{\mathbf{k}(S_w)\rho_w g}{\mu} = \mathbf{K}_s K_r(S_w)$$

A relationship suggested in [vG80] for S_e given by (3.5) with $m = 1 - 1/n \in (0, 1)$ (Figure 3):

$$K_r(S_w) = K_r(S_e) = S_e^{1/2} (1 - (1 - S_e^{1/m})^m)^2 \quad (3.9)$$

$$\stackrel{(3.5)}{=} \begin{cases} \frac{[1 - (c_{vg}\psi)^n (1 + (c_{vg}\psi)^n)^{-m}]^2}{(1 + (c_{vg}\psi)^n)^{m/2}} & \text{if } \psi \geq 0, \\ 1 & \text{if } \psi < 0 \end{cases} \quad (3.10)$$

Solid phase

We extend the introduction of the *effective stress* $\boldsymbol{\sigma}'_s$ from saturated flow: **Neglecting** the shear stress in the fluids in the unsaturated zone, one can express the *total stress* tensor $\boldsymbol{\sigma}$ as

$$\boldsymbol{\sigma} = (1 - \phi)\boldsymbol{\sigma}_s - \phi p_v \mathbf{I} = (1 - \phi)(\boldsymbol{\sigma}_s + p_v \mathbf{I}) - p_v \mathbf{I} = \boldsymbol{\sigma}'_s - p_v \mathbf{I} \quad (3.11)$$

$$\boldsymbol{\sigma}'_s \equiv (1 - \phi)(\boldsymbol{\sigma}_s + p_v \mathbf{I})$$

p_v — an *average pore pressure* of the fluids in the void space

Consequently under **Assumptions 1** and **2**, the relations (2.10)–(2.12) for the porosity can be extended to

$$c = \frac{1}{1 - \phi} \frac{\partial \phi}{\partial p_v}$$

$$\frac{D_s \phi}{Dt} = (1 - \phi)c \frac{D_s p_v}{Dt} \quad (3.12)$$

$$\phi = 1 + (\phi_o - 1)e^{-c(p_v - p_{vo})}, \quad p_{vo} \text{ — an initial average fluid pressure} \quad (3.13)$$

For example, one can use the relationship

$$p_v = \chi(S_w)p_w + (1 - \chi(S_w))p_a$$

where *Bishop's coefficient* χ is some differentiable function with $\chi(1) = 1$ (e.g., $\chi(S_w) = S_w$, which corresponds to the mean pressure $S_w p_w + S_a p_a$).

Then under **Assumption 5**

$$p_v = \chi(S_w)p_w \quad (3.14)$$

$$\frac{D_s p_v}{Dt} = \chi(S_w) \frac{D_s p_w}{Dt} + p_w \chi'(S_w) \frac{D_s S_w}{Dt} \quad (3.15)$$

$$\stackrel{(3.6)}{=} \left(\chi(S_w) + p_w \chi'(S_w) \frac{c_m}{\phi} \right) \frac{D_s p_w}{Dt} \quad (3.16)$$

Flow equations

Due to the solid mass balance equation (2.13) reduced to (2.14) under **Assumption 2**, the water mass balance equation (3.2) becomes

$$\begin{aligned}\phi S_w \frac{D_s \rho_w}{Dt} + S_w \rho_w \frac{D_s \phi}{Dt} + \phi \rho_w \frac{D_s S_w}{Dt} &= -\operatorname{div}(\rho_w \mathbf{q}_{rw}) - \phi S_w \rho_w \operatorname{div} \mathbf{v}_s + Q_w \quad (3.17) \\ \phi S_w \frac{D_s \rho_w}{Dt} + S_w \rho_w \frac{1}{1-\phi} \frac{D_s \phi}{Dt} + \phi \rho_w \frac{D_s S_w}{Dt} &= -\operatorname{div}(\rho_w \mathbf{q}_{rw}) + Q_w\end{aligned}$$

Introduction of (2.6), (3.12) with (3.16), and (3.6) for the material derivatives of ρ_w , ϕ and S_w gives

$$\begin{aligned}\phi S_w \rho_w c_w \frac{D_s p_w}{Dt} + S_w \rho_w c \left(\chi(S_w) + p_w \chi'(S_w) \frac{c_m}{\phi} \right) \frac{D_s p_w}{Dt} + \rho_w c_m \frac{D_s p_w}{Dt} &= -\operatorname{div}(\rho_w \mathbf{q}_{rw}) + Q_w \\ c_{ms} \frac{D_s p_w}{Dt} &= -\operatorname{div}(\rho_w \mathbf{q}_{rw}) + Q_w \\ c_{ms} &\equiv \rho_w \left(\phi S_w c_w + S_w c \left(\chi(S_w) + p_w \chi'(S_w) \frac{c_m}{\phi} \right) + c_m \right)\end{aligned} \quad (3.18)$$

where we have extended the definition (2.18) of the specific mass storativity c_{ms} (with $S_w = 1$, or $S_w = S_s$? and $c_m := 0$ for the saturated zone).

Inserting Darcy's law (3.7) yields

$$c_{ms} \frac{D_s p_w}{Dt} = -\operatorname{div} \left(\rho_w \frac{\mathbf{k}(S_w)}{\mu} (-\nabla p_w + \rho_w \mathbf{f}) \right) + Q_w$$

and under **Assumption 3** *approximately*

$$\boxed{c_{ms} \frac{\partial p_w}{\partial t} = -\operatorname{div} \left(\rho_w \frac{\mathbf{k}(S_w)}{\mu} (-\nabla p_w + \rho_w \mathbf{f}) \right) + Q_w} \quad (3.19)$$

In the case of **Assumption 4**, (3.19) reduces further *approximately* to

$$c_s \frac{\partial p_w}{\partial t} = -\operatorname{div} \left(\frac{\mathbf{k}(S_w)}{\mu} (-\nabla p_w + \rho_w \mathbf{f}) \right) + \frac{Q_w}{\rho_w}, \quad c_s \equiv \phi S_w c_w + S_w c \left(\chi(S_w) + p_w \chi'(S_w) \frac{c_m}{\phi} \right) + c_m$$

4 Saturated flow and deformation of the porous medium

[LS98, Ch. 2]

Balance equations

Mass balance equation for water

– in the form (2.2) or (2.15)

Mass balance equation for solid

– in the form (2.4) or (2.13)

Equilibrium equation

By summing up the equilibrium equations for water and solid one obtains

$$\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{f} = \mathbf{0} \quad (4.1)$$

$$\rho = (1 - \phi) \rho_s + \phi \rho_w \text{ — the average density of the multiphase medium} \quad (4.2)$$

Constitutive relationships

Water density

– the relations (2.5), (2.6)

Darcy's law for water

– in the form (2.7)

Stresses

Assumption 6. The solid phase can be restricted to the *small-strain* framework.

The *effective stress* $\boldsymbol{\sigma}'_s$ is given by (2.8). A modified effective stress is introduced for a *compressible grain* material:

The water pressure p_w induces an equal stress distribution in the solid phase. The ensuing deformation is a purely volumetric strain. Denoting it ε_v^p , one has

$$\varepsilon_v^p = -\frac{p_w}{K_s}, \quad K_s = \left(\frac{1}{\rho_s} \frac{\partial \rho_s}{\partial p_w} \right)^{-1} \text{ — the bulk modulus of the solid phase (grains)}$$

or in an incremental and tensorial form

$$d\varepsilon_v^p = \frac{1}{3} \mathbf{I} d\varepsilon_v^p = -\mathbf{I} \frac{dp_w}{3K_s} \quad (4.3)$$

The effective stress $\boldsymbol{\sigma}'_s$ causes all other relevant deformations of the solid skeleton. The constitutive relationship may be written as

$$d\boldsymbol{\sigma}'_s = \mathbf{D}(d\varepsilon - d\varepsilon_v^p - d\varepsilon^0) \quad (4.4)$$

\mathbf{D} — a fourth-order *tangent constitutive tensor* for the solid skeleton

$\varepsilon \equiv \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top)$ — the *linear strain tensor* of the skeleton

\mathbf{u} — the *displacement* of the skeleton

ε^0 — all other strains in the skeleton not directly associated with stress changes

The *modified effective stress* tensor $\boldsymbol{\sigma}''_s$ takes the form (cf. (2.8))

$$\boldsymbol{\sigma}''_s = \boldsymbol{\sigma} + \alpha p_w \mathbf{I}, \quad \alpha \text{ — the Biot-Willis coefficient} \quad (4.5)$$

so that

$$d\boldsymbol{\sigma}''_s = \mathbf{D}(d\varepsilon - d\varepsilon^0) \quad (4.6)$$

One needs a constitutive equation of the solid phase for determination of α : By substituting (4.6) and (4.3) into (4.4) one obtains

$$\begin{aligned} d\boldsymbol{\sigma}'_s &= d\boldsymbol{\sigma}''_s + \mathbf{D}\mathbf{I} \frac{dp_w}{3K_s} \\ d\boldsymbol{\sigma} + \alpha \mathbf{I} dp_w &= d\boldsymbol{\sigma}''_s = d\boldsymbol{\sigma}'_s - \mathbf{D}\mathbf{I} \frac{dp_w}{3K_s} \stackrel{(3.11)}{=} d\boldsymbol{\sigma} + \mathbf{I} dp_w - \mathbf{D}\mathbf{I} \frac{dp_w}{3K_s} \\ \alpha \mathbf{I} dp_w &= \mathbf{I} dp_w - \mathbf{D}\mathbf{I} \frac{dp_w}{3K_s} \end{aligned} \quad (4.7)$$

Assumption 7. Let the solid matrix be elastic and isotropic.

In this case

$$Dd\varepsilon = \lambda(\text{tr } d\varepsilon)\mathbf{I} + 2\mu d\varepsilon \quad (4.8)$$

λ, μ — the *Lamé coefficients* of the *solid skeleton* (porous medium)

$$D\mathbf{I} = 3K\mathbf{I}$$

$$K = (3\lambda + 2\mu)/3 \text{ — the } \textit{bulk modulus} \text{ of the } \textit{solid skeleton} \quad (4.9)$$

Hence (4.7) yields

$$\begin{aligned} \alpha \mathbf{I} dp_w &= \mathbf{I} dp_w - \frac{K}{K_s} \mathbf{I} dp_w \\ \alpha &= 1 - \frac{K}{K_s} \end{aligned} \quad (4.10)$$

For an *incompressible* grain material $1/K_s = 0$, $\alpha = 1$.

Solid density

When considering the solid phase as compressible, a relationship for the total time derivative of the solid density ρ_s is needed. **Assuming** $\rho_s = \rho_s(p_w, \text{tr } \boldsymbol{\sigma}'_s)$ leads to

$$\begin{aligned} \frac{1}{\rho_s} \frac{D_s \rho_s}{Dt} &= \frac{1}{\rho_s} \frac{\partial \rho_s}{\partial p_w} \frac{D_s p_w}{Dt} + \frac{1}{\rho_s} \frac{\partial \rho_s}{\partial (\text{tr } \boldsymbol{\sigma}'_s)} \frac{D_s (\text{tr } \boldsymbol{\sigma}'_s)}{Dt} \\ &= \frac{1}{K_s} \frac{D_s p_w}{Dt} - \frac{1}{3(1-\phi)K_s} \frac{D_s (\text{tr } \boldsymbol{\sigma}'_s)}{Dt} \end{aligned}$$

Using the **constitutive relationship** for $\text{tr } \boldsymbol{\sigma}'_s$ (cf. (4.4) with (4.8), (4.9), (4.3) and $d\varepsilon^0$ omitted – under Assumptions 6 and 7)

$$\begin{aligned} \frac{D_s (\text{tr } \boldsymbol{\sigma}'_s)}{Dt} &= 3K \left(\frac{D_s \varepsilon_v}{Dt} + \frac{1}{K_s} \frac{D_s p_w}{Dt} \right) \\ \varepsilon_v &= \text{div } \mathbf{u} \text{ — the } \textit{volumetric strain} \text{ of the } \textit{solid skeleton} \end{aligned}$$

one gets

$$\begin{aligned} \frac{1}{\rho_s} \frac{D_s \rho_s}{Dt} &= \left(\frac{1}{K_s} - \frac{1}{(1-\phi)K_s} \frac{K}{K_s} \right) \frac{D_s p_w}{Dt} - \frac{K}{(1-\phi)K_s} \frac{D_s \varepsilon_v}{Dt} \\ &\stackrel{(4.10)}{=} \frac{1}{1-\phi} \left((\alpha - \phi) \frac{1}{K_s} \frac{D_s p_w}{Dt} - (1-\alpha) \frac{D_s \varepsilon_v}{Dt} \right) \\ &\stackrel{(4.10)}{=} \frac{1}{1-\phi} \left((\alpha - \phi) \frac{1-\alpha}{K} \frac{D_s p_w}{Dt} - (1-\alpha) \frac{D_s \varepsilon_v}{Dt} \right) \end{aligned} \quad (4.11)$$

Moreover, one has for the solid velocity

$$\mathbf{v}_s = \frac{D_s \mathbf{u}}{Dt} \equiv \frac{\partial \mathbf{u}}{\partial t} + (\nabla \mathbf{u}) \mathbf{v}_s$$

Hence under **Assumption 6** one gets *approximately*

$$\begin{aligned} \mathbf{v}_s &\simeq \frac{\partial \mathbf{u}}{\partial t} \\ \text{div } \mathbf{v}_s &\simeq \text{div } \frac{\partial \mathbf{u}}{\partial t} = \frac{\partial \varepsilon_v}{\partial t} \simeq \frac{D_s \varepsilon_v}{Dt} \end{aligned} \quad (4.12)$$

and (4.11) takes the form

$$\frac{1}{\rho_s} \frac{D_s \rho_s}{Dt} = \frac{1}{1-\phi} \left((\alpha - \phi) \frac{1-\alpha}{K} \frac{D_s p_w}{Dt} - (1-\alpha) \text{div } \mathbf{v}_s \right) \quad (4.13)$$

Complete equations

By (4.13) for the material derivative of ρ_s , the solid mass balance equation (2.13) can be further transformed into

$$\frac{D_s \phi}{Dt} = \frac{1 - \phi}{\rho_s} \frac{D_s \rho_s}{Dt} + (1 - \phi) \operatorname{div} \mathbf{v}_s \quad (4.14)$$

$$= \frac{(\alpha - \phi)(1 - \alpha)}{K} \frac{D_s p_w}{Dt} + (\alpha - \phi) \operatorname{div} \mathbf{v}_s \quad (4.15)$$

which inserted into the water mass balance equation (2.15) gives

$$\phi \frac{D_s \rho_w}{Dt} + \rho_w \frac{(\alpha - \phi)(1 - \alpha)}{K} \frac{D_s p_w}{Dt} + \rho_w \alpha \operatorname{div} \mathbf{v}_s = -\operatorname{div}(\rho_w \mathbf{q}_{rw}) + Q_w$$

Introduction of (2.6) for $D_s \rho_w / Dt$, and the Darcy law (2.7) leads to

$$\rho_w \left(\phi c_w + \frac{(\alpha - \phi)(1 - \alpha)}{K} \right) \frac{D_s p_w}{Dt} + \rho_w \alpha \operatorname{div} \mathbf{v}_s = -\operatorname{div} \left(\rho_w \frac{\mathbf{k}}{\mu} (-\nabla p_w + \rho_w \mathbf{f}) \right) + Q_w$$

Under **Assumption 6 (Assumption 3)** this can be *approximately* reduced with the aid of (4.12) for $\operatorname{div} \mathbf{v}_s$ to

$$\boxed{\rho_w \left(\phi c_w + \frac{(\alpha - \phi)(1 - \alpha)}{K} \right) \frac{\partial p_w}{\partial t} + \rho_w \alpha \operatorname{div} \frac{\partial \mathbf{u}}{\partial t} = -\operatorname{div} \left(\rho_w \frac{\mathbf{k}}{\mu} (-\nabla p_w + \rho_w \mathbf{f}) \right) + Q_w} \quad (4.16)$$

Moreover, the equilibrium equation (4.1) with (4.2) and (4.5) results in

$$\boxed{\operatorname{div}(\boldsymbol{\sigma}_s'' - \alpha p_w \mathbf{I}) + ((1 - \phi)\rho_s + \phi\rho_w) \mathbf{f} = \mathbf{0}} \quad (4.17)$$

+ a stress-strain relationship in the form (4.6)

Remark 4. The time scale of the structural response is generally many orders of magnitude faster than the time scale of the flow. When the coupled process is studied on the time scale of the flow, one can therefore assume that the solid reaches a new equilibrium immediately in response to a change in flow conditions, and apply the equilibrium equation (4.17) (with all inertial effects neglected) to the time-dependent flow model (4.16).

Remark 5 (Evaluation of the solid density and porosity). Replacing the total time derivatives with the corresponding partial ones and using (4.12) (both under **Assumption 6**), one can reduce (4.11) and (4.15) *approximately* to

$$\frac{1}{\rho_s} \frac{\partial \rho_s}{\partial t} = \frac{1 - \alpha}{1 - \phi} \left(\frac{\alpha - \phi}{K} \frac{\partial p_w}{\partial t} - \frac{\partial \varepsilon_v}{\partial t} \right) \quad (4.18)$$

$$\frac{\partial \phi}{\partial t} = (\alpha - \phi) \left(\frac{1 - \alpha}{K} \frac{\partial p_w}{\partial t} + \frac{\partial \varepsilon_v}{\partial t} \right) \quad (4.19)$$

Integration of (4.19) gives

$$\phi = \alpha + (\phi_o - \alpha) \exp \left(-\frac{1 - \alpha}{K} (p_w - p_{wo}) - (\varepsilon_v - \varepsilon_{vo}) \right) \quad \varepsilon_{vo} \text{ — an initial volumetric strain}$$

and its first-order Taylor approximation

$$\begin{aligned} \phi &\simeq \alpha + (\phi_o - \alpha) \left(1 - \frac{1 - \alpha}{K} (p_w - p_{wo}) - (\varepsilon_v - \varepsilon_{vo}) \right) \\ &= \phi_o + (\alpha - \phi_o) \left(\frac{1 - \alpha}{K} (p_w - p_{wo}) + (\varepsilon_v - \varepsilon_{vo}) \right) \end{aligned}$$

Similarly from (4.18)

$$\begin{aligned}
\rho_s &= \rho_{so} \exp\left(\int_0^t \frac{1-\alpha}{1-\phi} \left(\frac{\alpha-\phi}{K} \frac{\partial p_w}{\partial t} - \frac{\partial \varepsilon_v}{\partial t}\right)\right) \\
&\simeq \rho_{so} \exp\left(\frac{1-\alpha}{1-\phi_o} \left(\frac{\alpha-\phi_o}{K} (p_w - p_{wo}) - (\varepsilon_v - \varepsilon_{vo})\right)\right) \\
&\simeq \rho_{so} \left(1 + \frac{1-\alpha}{1-\phi_o} \left(\frac{\alpha-\phi_o}{K} (p_w - p_{wo}) - (\varepsilon_v - \varepsilon_{vo})\right)\right) \\
&\quad \rho_{so} \text{ — an initial solid density}
\end{aligned}$$

5 Unsaturated flow and deformation of the porous medium

[LS98, Ch. 2]

Balance equations

Mass balance equation for water

– in the form (3.2) or (3.17)

Mass balance equation for solid

– in the form (2.4) or (4.14)

Equilibrium equation

By summing up the equilibrium equations for water and solid one obtains

$$\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{f} = \mathbf{0} \quad (5.1)$$

$$\rho = (1-\phi)\rho_s + \phi S_w \rho_w \text{ — the average density} \quad (5.2)$$

- No momentum exchange with air.

Constitutive relationships

Retention curve

– a relationship $S_w = S_w(p_w)$, which leads to (3.6) (**Assumption 5**)

Water density

– the relations (2.5), (2.6)

Darcy's law for water

– in the form (3.7) with the effective permeability in the form (3.8)

Stresses

- Under **Assumption 6**.

The *effective stress* $\boldsymbol{\sigma}'_s$ is given by (3.11) with the relations (3.14) and (3.16) for the average fluid pressure p_v in the void space (under **Assumption 5**). By replacing the water pressure p_w with p_v , one extends the expression for the *modified effective stress* $\boldsymbol{\sigma}''_s$ from saturated flow as

$$\boldsymbol{\sigma}''_s = \boldsymbol{\sigma} + \alpha p_v \mathbf{I} \quad (5.3)$$

so that (4.6) still holds.

Under **Assumption 7**, the *Biot-Willis coefficient* α is given by (4.10) with

$$K_s = \left(\frac{1}{\rho_s} \frac{\partial \rho_s}{\partial p_v}\right)^{-1} \text{ — the bulk modulus of the solid phase (grains)} \quad (5.4)$$

For an *incompressible* grain material $1/K_s = 0$, $\alpha = 1$.

Solid density

When considering the solid phase as compressible, the relationship (4.13) for the total time derivative of the solid density ρ_s can be extended from saturated flow as follows: By **assuming** $\rho_s = \rho_s(p_v, \text{tr } \boldsymbol{\sigma}'_s)$ with

$$\frac{1}{\rho_s} \frac{D_s \rho_s}{Dt} = \frac{1}{K_s} \frac{D_s p_v}{Dt} - \frac{1}{3(1-\phi)K_s} \frac{D_s(\text{tr } \boldsymbol{\sigma}'_s)}{Dt}$$

and using the **constitutive relationship** for $\text{tr } \boldsymbol{\sigma}'_s$

$$\frac{D_s(\text{tr } \boldsymbol{\sigma}'_s)}{Dt} = 3K \left(\frac{D_s \varepsilon_v}{Dt} + \frac{1}{K_s} \frac{D_s p_v}{Dt} \right)$$

one gets

$$\frac{1}{\rho_s} \frac{D_s \rho_s}{Dt} = \frac{1}{1-\phi} \left((\alpha - \phi) \frac{1-\alpha}{K} \frac{D_s p_v}{Dt} - (1-\alpha) \frac{D_s \varepsilon_v}{Dt} \right) \quad (5.5)$$

$$\stackrel{(4.12)}{=} \frac{1}{1-\phi} \left((\alpha - \phi) \frac{1-\alpha}{K} \frac{D_s p_v}{Dt} - (1-\alpha) \text{div } \mathbf{v}_s \right) \quad (5.6)$$

Complete equations

The solid mass balance equation (4.14) with (5.6) for the material derivative of ρ_s becomes

$$\frac{D_s \phi}{Dt} = \frac{(\alpha - \phi)(1-\alpha)}{K} \frac{D_s p_v}{Dt} + (\alpha - \phi) \text{div } \mathbf{v}_s \quad (5.7)$$

which inserted into the water mass balance equation (3.17) gives

$$\phi S_w \frac{D_s \rho_w}{Dt} + S_w \rho_w \frac{(\alpha - \phi)(1-\alpha)}{K} \frac{D_s p_v}{Dt} + \phi \rho_w \frac{D_s S_w}{Dt} + S_w \rho_w \alpha \text{div } \mathbf{v}_s = -\text{div}(\rho_w \mathbf{q}_{rw}) + Q_w$$

Introduction of (2.6), (3.16), (3.6) for the material derivatives of ρ_w , p_v and S_w , and the Darcy law (3.7) leads to

$$\begin{aligned} \rho_w \left(\phi S_w c_w + S_w \frac{(\alpha - \phi)(1-\alpha)}{K} \left(\chi(S_w) + p_w \chi'(S_w) \frac{c_m}{\phi} \right) + c_m \right) \frac{D_s p_w}{Dt} + \rho_w \alpha S_w \text{div } \mathbf{v}_s \\ = -\text{div} \left(\rho_w \frac{\mathbf{k}(S_w)}{\mu} (-\nabla p_w + \rho_w \mathbf{f}) \right) + Q_w \end{aligned}$$

Under **Assumption 6 (Assumption 3)** this can be *approximately* reduced with the aid of (4.12) for $\text{div } \mathbf{v}_s$ to

$$\boxed{\begin{aligned} \rho_w \left(\phi S_w c_w + S_w \frac{(\alpha - \phi)(1-\alpha)}{K} \left(\chi(S_w) + p_w \chi'(S_w) \frac{c_m}{\phi} \right) + c_m \right) \frac{\partial p_w}{\partial t} + \rho_w \alpha S_w \text{div } \frac{\partial \mathbf{u}}{\partial t} \\ = -\text{div} \left(\rho_w \frac{\mathbf{k}(S_w)}{\mu} (-\nabla p_w + \rho_w \mathbf{f}) \right) + Q_w \end{aligned}} \quad (5.8)$$

Moreover, the equilibrium equation (5.1) with (5.2), (5.3) and (3.14) results in

$$\boxed{\begin{aligned} \text{div}(\boldsymbol{\sigma}''_s - \alpha \chi(S_w) p_w \mathbf{I}) + ((1-\phi)\rho_s + \phi S_w \rho_w) \mathbf{f} = \mathbf{0} \\ + \text{a stress-strain relationship in the form (4.6)} \end{aligned}} \quad (5.9)$$

Remark 6 (Evaluation of the solid density and porosity). In an analogous way to Remark 5, one can get *approximately* from (5.5), (5.7) and (4.12) (under **Assumption 6**)

$$\begin{aligned}\frac{1}{\rho_s} \frac{\partial \rho_s}{\partial t} &= \frac{1-\alpha}{1-\phi} \left(\frac{\alpha-\phi}{K} \frac{\partial p_v}{\partial t} - \frac{\partial \varepsilon_v}{\partial t} \right) \\ \frac{\partial \phi}{\partial t} &= (\alpha-\phi) \left(\frac{1-\alpha}{K} \frac{\partial p_v}{\partial t} + \frac{\partial \varepsilon_v}{\partial t} \right)\end{aligned}$$

and consequently

$$\begin{aligned}\phi &= \alpha + (\phi_o - \alpha) \exp\left(-\frac{1-\alpha}{K}(p_v - p_{vo}) - (\varepsilon_v - \varepsilon_{vo})\right) \\ &\simeq \alpha + (\phi_o - \alpha) \left(1 - \frac{1-\alpha}{K}(p_v - p_{vo}) - (\varepsilon_v - \varepsilon_{vo})\right) \\ &= \phi_o + (\alpha - \phi_o) \left(\frac{1-\alpha}{K}(p_v - p_{vo}) + (\varepsilon_v - \varepsilon_{vo})\right) \\ \rho_s &\simeq \rho_{so} \exp\left(\frac{1-\alpha}{1-\phi_o} \left(\frac{\alpha-\phi_o}{K}(p_v - p_{vo}) - (\varepsilon_v - \varepsilon_{vo})\right)\right) \\ &\simeq \rho_{so} \left(1 + \frac{1-\alpha}{1-\phi_o} \left(\frac{\alpha-\phi_o}{K}(p_v - p_{vo}) - (\varepsilon_v - \varepsilon_{vo})\right)\right)\end{aligned}$$

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