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k získání vědeckého titulu "doktor věd"
ve skupině věd fyzikálně–matematických

Finite Element Techniques for Convection–Diffusion Problems

Komise pro obhajoby doktorských disertací v oboru
Matematická analýza a příbuzné obory

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Resumé

Konvekce a difúze jsou základní fyzikální mechanismy, které ovlivňují či přímo určují mnoho různých procesů v přírodě, vědě a technice. Matematické modely popisující takovéto procesy jsou obvykle natolik komplikované, že je není možno vyřešit analyticky. Proto je potřeba příslušné neznámé veličiny aproximovat pomocí numerických metod. V typických aplikacích, kde konvekce převažuje nad difúzí, však standardní numerické postupy selhávají, neboť přibližná řešení jsou zkreslena nefyzikálními oscilacemi.

Předložená disertace je věnována vývoji a analýze různých postupů v metodě konečných prvků pro numerické řešení skalární rovnice konvekce–difúze–reakce. Zkoumání numerických metod pro tuto modelovou úlohu je zásadní pro úspěšný vývoj přesných, robustních a efektivních postupů pro numerické řešení komplikovanějších úloh, které se vyskytují v aplikacích, avšak je důležité i samo o sobě. Numerické řešení konvektivně dominantních problémů konvekce–difúze představuje stále výzvu, i přes více než čtyři desetiletí intenzivního výzkumu.

Předložená disertace obsahuje řadu příspěvků k numerickému řešení konvektivně difúzních problémů, které její autor vytvořil během uplynulých dvanácti let. Mezi nejdůležitější výsledky patří: vylepšená varianta Mizukamiho–Hughesovy metody, přehled a systematické srovnání metod potlačujících nefyzikální oscilace podél mezních vrstev, nová definice stabilizačního parametru metody SUPG na výtokových hranicích, obecný rámec pro adaptivní optimalizaci parametrů ve stabilizovaných metodách, nové výsledky o stabilitě diskretizací v metodě konečných prvků, zobecnění metody lokálních projekcí umožňující použití projekčních prostorů definovaných na překrývajících se množinách, analýza nelineárních stabilizací v metodě lokálních projekcí, první analýza schémat s algebraickou korekcí toků, vyvinutí a analýza prvního schématu s algebraickou korekcí toků pro rovnice konvekce–difúze–reakce splňujícího diskrétní princip maxima a zachování linearity na obecných sítích.

Résumé

Convection and diffusion are basic physical mechanisms which influence or even determine many various processes in the nature, science and technology. Mathematical models describing such processes are usually too complicated to be solved analytically. Therefore, it is necessary to approximate the respective unknown quantities by means of numerical methods. However, in typical applications where convection dominates diffusion, standard numerical techniques fail since the approximate solutions are usually polluted by spurious oscillations.

The doctoral thesis is devoted to the development and analysis of various finite element techniques for the numerical solution of the scalar convection–diffusion–reaction equation. Investigations of numerical techniques for this model problem are crucial for a successful development of accurate, robust and efficient approaches for the numerical solution of more complicated problems arising in applications but they are important also at its own. Despite more than four decades of intensive research, the numerical solution of convection-dominated convection–diffusion problems is still a challenge in general.

The doctoral thesis presents several contributions to the numerical solution of convection–diffusion problems made by the author during the past twelve years. The most important ones include: an improved variant of the Mizukami–Hughes method, a review and systematic comparison of spurious oscillations at layers diminishing (SOLD) methods, a new definition of the SUPG stabilization parameter at outflow boundaries, a general framework for an adpative optimization of parameters in stabilized methods, new results on the stability of finite element discretizations, a generalization of the local projection stabilization allowing to use projection spaces defined on overlapping sets, an analysis of nonlinear local projection stabilizations, the first analysis of algebraic flux correction schemes, and the development and analysis of the first flux correction scheme for convection–diffusion–reaction equations satisfying the discrete maximum principle and linearity preservation on general meshes.

1 Introduction

These theses summarize the contents of the doctoral thesis *Finite Element Techniques for Convection–Diffusion Problems* submitted to the Czech Academy of Sciences in partial fulfilment of the requirements for the scientific degree of *Doctor Scientiarum*. The doctoral thesis is a commented collection of 16 selected publications of the author which reflect his research on finite element techniques for convection–diffusion problems during the past twelve years. These publications are listed in Section 10 and they are referred to by [D1], [D2],..., [D16] in these theses.

The distribution of physical quantities in various physical, technical, biological and other processes is driven by basic physical mechanisms which are diffusion, convection, and reaction. Often, the diffusion is very small in comparison with the convection or reaction. This causes that the distribution of the respective quantity comprises so-called layers, which are narrow regions where the quantity changes abruptly. It is well known that standard discretizations then provide approximate solutions polluted by spurious oscillations unless the underlying mesh resolves the layers, see, e.g., the monograph [51]. Consequently, special discretization techniques (so-called stabilized methods) have to be applied which always introduce a certain amount of artificial diffusion that should suppress the spurious oscillations but also typically increases the smearing of the layers. Therefore, it is usually still a challenge to obtain an accurate approximate solution, despite the huge amount of research on appropriate discretizations during the last four decades.

The simplest model for the above-mentioned class of problems is a scalar steady-state convection–diffusion–reaction equation

$$-\varepsilon\,\Delta u + \boldsymbol{b}\cdot\nabla u + c\,u = f \quad \text{in } \Omega\,, \tag{1}$$

where $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, is a bounded domain, $\varepsilon > 0$ is a constant diffusion parameter, $\boldsymbol{b}$ is a convection field, c is a reaction coefficient, and f is a term describing sources and sinks. The unknown function u represents, e.g., the temperature in modeling the energy balance, or the concentration or mass fraction in modeling mass balances. Throughout these theses, we shall suppose that the data satisfy the usual technical assumption

$$\sigma := c - \frac{1}{2}\,\mathrm{div}\,\boldsymbol{b} \geq 0\,. \tag{2}$$

To obtain a well-posed problem, (1) has to be equipped with appropriate boundary conditions on the boundary $\partial\Omega$ of Ω. To simplify the presentation of numerical methods in these theses, we shall consider

$$u = 0 \quad \text{on } \partial\Omega\,. \tag{3}$$

Nevertheless, the properties of the methods will be illustrated by an example with nonhomogeneous Dirichlet boundary conditions. The extension of the methods to this case is straightforward. More general boundary conditions can be found in the publications contained in the doctoral thesis.

To solve the equation (1) numerically, various methods can be applied: the finite difference method, finite volume method, finite element method, discontinuous Galerkin method, or spectral method, to name the most common ones. For each of these methods, many contributions on its application to the numerical solution of (1) can be found in the literature. The doctoral thesis is devoted exclusively to the application of the finite element method which we prefer because of its flexibility in treating complex geometries, easy incorporation of natural boundary conditions and suitability for theoretical investigations due to its functional analytical setting based usually on Hilbert spaces.

It should be emphasized that the model (1) as a stand-alone equation is considered because, on the one hand, it comprises the effects of diffusion, convection, and reaction and, on the other hand, it simplifies the analysis of numerical techniques for its solution. Nevertheless, also for this simplest available model, there are many discretizations the analysis of which still remains an open problem. In applications, equations of type (1) are often a part of complex systems of equations. For example, they may be coupled with the Navier–Stokes equations describing the convection $\boldsymbol{b}$ which is, in turn, influenced by the temperature or concentrations determined by equations of type (1).

This work is mainly devoted to studies of numerical techniques for the equation (1) in the convection-dominated regime characterized by the conditions $\varepsilon \ll \|\boldsymbol{b}\|_{L^\infty}$ and $\|c\|_{L^\infty} \lesssim \|\boldsymbol{b}\|_{L^\infty}$, which is the case usually encountered in applications. As already mentioned, the main feature of solutions in this regime is the appearance of layers, i.e., narrow regions with large gradients of the solution. Then the standard Galerkin finite element method applied to (1) (see Section 2), which corresponds to central finite differencing for constant data and suitable meshes, leads to heavily oscillating solutions unless the layers are resolved by the respective mesh. Therefore, much research has been devoted to the development of numerical methods using anisotropic layer-adapted meshes. Such meshes can be defined either a priori (see, e.g., [44, 51]) or a posteriori by means of adaptive techniques (see, e.g., [1, 52]). Nevertheless, since the layer width is proportional to $\sqrt{\varepsilon}$ or even ε (depending on the type of layer), the geometric resolution of the layers is often not feasible due to high memory and computational time requirements. Therefore, it is important to develop numerical methods providing sufficiently accurate results also on meshes which are coarse in comparison with the width of the layers. This is the main aim of this work.

To suppress the oscillations present in Galerkin solutions obtained on coarse meshes, various stabilized methods have been developed, see, e.g., [51, 50, 31] for reviews. The stabilizing effect of these approaches can be characterized by the artificial diffusion they add to the underlying Galerkin discretization. To diminish the spurious oscillations to a sufficient extent, the artificial diffusion has to be sufficiently large. However, to avoid an excessive smearing of the layers, the artificial diffusion is not allowed to be too large. Consequently, the design of a proper stabilization is very difficult. Despite more than four decades of research, there is so far no efficient discretization for (1) available which would produce accurate numerical solutions (in particular, with sharp layers at correct positions) without unphysical features (e.g., negative concentrations). This statement is supported by theoretical and numerical studies in, e.g., [3, 29, 30] and [D3,D4].

2 Galerkin method and SUPG method

The Galerkin finite element discretization of the problem (1), (3) defines an approximate solution u_h from a finite element space V_h approximating the Sobolev space $H_0^1(\Omega)$ as the solution of the variational problem

$$a(u_h, v_h) = (f, v_h) \qquad \forall\, v_h \in V_h\,, \tag{4}$$

where

$$a(u_h, v_h) = \varepsilon\,(\nabla u_h, \nabla v_h) + (\boldsymbol{b}\cdot\nabla u_h, v_h) + (c\,u_h, v_h)$$

and $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$ or $L^2(\Omega)^d$. As we already mentioned in the preceding section, this discretization is not appropriate in the convection-dominated regime and has to be stabilized.

One of the most successful linear stabilizations is the streamline upwind Petrov–Galerkin (SUPG) finite element method [25, 12] which consistently introduces artificial diffusion along streamlines. It combines good stability properties with a high accuracy away from layers. Because this method will be frequently discussed throughout these theses, it will be now formulated for the equation (1) in detail.

The SUPG method adds a weighted residual of (1) to the Galerkin method and defines the approximate solution $u_h \in V_h$ by

$$a(u_h, v_h) + \sum_{T\in\mathscr{T}_h} (-\varepsilon\,\Delta u_h + \boldsymbol{b}\cdot\nabla u_h + c\,u_h - f, \tau\,\boldsymbol{b}\cdot\nabla v_h)_T = (f, v_h) \quad \forall\, v_h \in V_h\,, \tag{5}$$

where $\mathscr{T}_h$ is a triangulation of Ω used for defining the finite element space V_h, τ is a nonnegative stabilization parameter (typically constant on each $T \in \mathscr{T}_h$), and $(\cdot,\cdot)_T$ denotes the inner product in $L^2(T)$ or $L^2(T)^d$. The additional term is written as a sum of local contributions since the operator Δ usually cannot be applied to u_h globally. The parameter τ determines the amount of the artificial diffusion added by the SUPG method to the Galerkin discretization. For linear or bilinear finite elements, it is often defined, on any element $T \in \mathscr{T}_h$, by the formula

$$\tau|_T = \frac{h_T}{2\,|\boldsymbol{b}|}\left(\coth Pe_T - \frac{1}{Pe_T}\right) \quad \text{with} \quad Pe_T = \frac{|\boldsymbol{b}|\,h_T}{2\,\varepsilon}\,, \tag{6}$$

which originates from the one-dimensional case. The notation Pe_T is used for the Péclet number, which determines whether the problem is

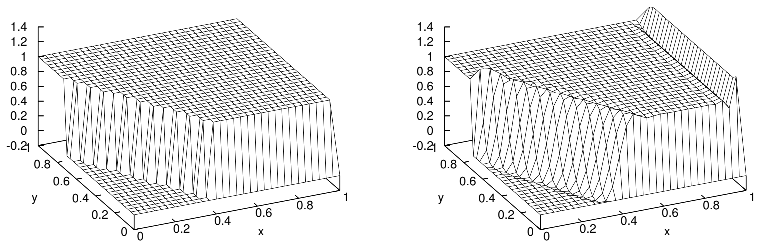

Figure 1: Exact solution of Example 1 (left) and its SUPG approximation (right) (Fig. 5 from [D4]).

locally convection-dominated or diffusion-dominated, and h_T is the element diameter in the direction of the convection vector $\boldsymbol{b}$. We refer to [D3] for various justifications of this formula and for a precise definition of h_T. If τ satisfies suitable assumptions, one can prove the stability and an error estimate for (5) with respect to the norm

$$(7) \qquad \|v\|_{SUPG} = \left(\varepsilon\,|v|_{1,\Omega}^2 + \|\sigma^{1/2}\,v\|_{0,\Omega}^2 + \|\tau^{1/2}\,\boldsymbol{b}\cdot\nabla v\|_{0,\Omega}^2\right)^{1/2},$$

where we use σ from (2), $|\cdot|_{1,\Omega}$ is the usual seminorm in $H_0^1(\Omega)$ and $\|\cdot\|_{0,\Omega}$ is the norm in $L^2(\Omega)$. The SUPG method represents a significant improvement in comparison with the Galerkin method, nevertheless, since it is not a monotone method, it may compute solutions suffering from spurious oscillations in layer regions. Let us demonstrate it by means of the following classical test problem.

Example 1 The equation (1) is considered with $\Omega = (0,1)^2$, $\varepsilon = 10^{-8}$, $\boldsymbol{b} = (\cos(-\pi/3), \sin(-\pi/3))^T$, $c = 0$, $f = 0$, and the boundary condition $u = u_b$ on $\partial\Omega$ with

$$u_b(x,y) = \begin{cases} 0 & \text{for } x = 1 \text{ or } y \le 0.7, \\ 1 & \text{else.} \end{cases}$$

The discontinuity in the boundary condition at the point $(0, 0.7)$ is propagated by the convection into the interior of the computational domain, which creates an interior layer. Moreover, the boundary condition

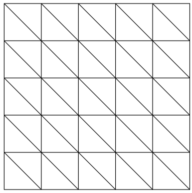

Figure 2: Type of the meshes used in the most computations (Fig. 3a from [D1]).

$u = 1$ is propagated to a boundary part where $u = 0$, which leads to two exponential boundary layers. The exact solution of this problem is shown in Fig. 1 (left). All numerical results presented in Sections 2–6 were computed using continuous piecewise linear functions on a mesh of the type depicted in Fig. 2. The SUPG solution for the stabilization parameter given in (6) can be seen in Fig. 1 (right). One can observe spurious oscillations along the interior layer and one of the boundary layers.

3 Mizukami–Hughes method

The Mizukami–Hughes method is an interesting approach proposed in [49] for a two-dimensional convection–diffusion equation (i.e., (1) with $c = 0$ and $d = 2$) discretized using a finite element space V_h consisting of continuous piecewise linear functions over a triangular mesh. To formulate the method, we denote by $\varphi_1, \dots, \varphi_M$ the standard piecewise linear basis functions of the space V_h. Then the Galerkin discretization (4) can be written in the form

$$\varepsilon\,(\nabla u_h, \nabla\varphi_i) + (\boldsymbol{b}\cdot\nabla u_h, \varphi_i) = (f, \varphi_i)\,, \qquad i = 1, \dots, M\,.$$

The Mizukami–Hughes method replaces the test functions φ_i by functions $\widetilde{\varphi}_i$ obtained by adding suitable constants to φ_i on the triangles forming its support. Then the approximate solution $u_h \in V_h$ is defined by

$$\varepsilon\,(\nabla u_h, \nabla\varphi_i) + (\boldsymbol{b}\cdot\nabla u_h, \widetilde{\varphi}_i) = (f, \widetilde{\varphi}_i)\,, \qquad i = 1, \dots, M\,.$$

Thus, it is a Petrov–Galerkin method like the SUPG method. It is assumed that $\boldsymbol{b}$ is constant on each element of the triangulation; in practice, $\boldsymbol{b}$ is replaced by a piecewise constant approximation.

The idea of the Mizukami–Hughes method is to define the constants in the definition of the test functions $\widetilde{\varphi}_i$ in such a way that the local finite element matrices corresponding to the convective term are of nonnegative type, i.e., their row sums are nonnegative and off-diagonal entries are nonpositive. Whether this is possible depends on the orientation of $\boldsymbol{b}$ with respect to the given element of the triangulation. However, Mizukami and Hughes made the important observation that u still solves the equation (1) if one replaces $\boldsymbol{b}$ by any function $\tilde{\boldsymbol{b}}$ such that $\tilde{\boldsymbol{b}} - \boldsymbol{b}$ is orthogonal to ∇u. This suggests to define the constants in the definition of the functions $\widetilde{\varphi}_i$ in such a way that the local convection matrix is of nonnegative type for $\boldsymbol{b}$ replaced by a suitable function $\tilde{\boldsymbol{b}}$, which is always possible. Since ∇u is not known a priori, one obtains a nonlinear problem where the constants in the definition of $\widetilde{\varphi}_i$ depend on the unknown approximate solution u_h.

The Mizukami–Hughes method is probably the first nonlinear method for (1) satisfying the discrete maximum principle. Like for many other methods proposed later, see, e.g., [14, 15, 4], this property is proved only for weakly acute meshes, i.e., the magnitude of all angles in the triangles of the mesh is less than or equal to $\pi/2$. Nevertheless, it is also possible to derive methods for which the discrete maximum principle holds on arbitrary meshes, see Section 8. The discrete maximum principle is an important property which ensures that no spurious oscillations will appear, not even in the vicinity of sharp layers. In contrast to many methods satisfying the discrete maximum principle, the Mizukami–Hughes method does not lead to a pronounced smearing of layers and it often provides very accurate results.

However, we observed that, in some cases, the Mizukami–Hughes method does not lead to correct solutions. Moreover, sometimes it is very difficult to solve the nonlinear problem with a prescribed accuracy. Therefore, in [D1], we proposed several improvements of the method which correct the mentioned shortcomings and keep its quality in cases in which it works well. This was achieved by a more careful definition of the constants in the test functions $\widetilde{\varphi}_i$. In particular, a continuous dependence of these constants on the orientation of $\boldsymbol{b}$ and ∇u_h was in-

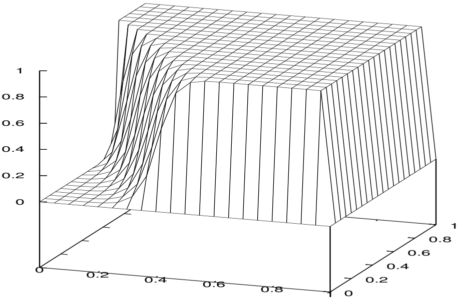

Figure 3: Approximate solution of Example 1 obtained using the improved Mizukami–Hughes method for $\varepsilon = 10^{-7}$ (Fig. 12 from [D1]).

troduced. Moreover, the method was extended to convection–diffusion–reaction equations and to the three-dimensional case. It was shown that the improved method still satisfies the discrete maximum principle and its high accuracy was demonstrated by many numerical results. The superiority of the improved Mizukami–Hughes method to linear upwinding finite element methods satisfying the discrete maximum principle was clearly demonstrated in [32]. The approximate solution of Example 1 obtained using the improved Mizukami–Hughes method on a mesh of the type from Fig. 2 is depicted in Fig. 3. One can observe a perfect approximation of the boundary layers and an acceptable smearing of the interior layer, without any spurious oscillations.

Both the Mizukami–Hughes method in [49] and its improved variant in [D1] were designed for the strongly convection-dominated case $\varepsilon \ll |\boldsymbol{b}|$. In [D2], the method was extended to the whole range of the diffusion parameter and it was proved that the extended method satisfies the discrete maximum principle. The favourable properties of the new method were illustrated by means of numerical experiments.

A drawback of both the original and the improved versions of the Mizukami–Hughes method is that no existence, uniqueness and convergence results are available. Moreover, it seems to be rather difficult to generalize the method to more complicated problems or to other types of finite elements. So far, only a variant for bilinear finite elements is available, see [33].

4 SOLD methods

The SUPG method formulated in Section 2 (like many other approaches adding a linear stabilization term to the Galerkin discretization, see, e.g., [13, 19, 24, 48]) significantly reduces the spurious oscillations present in Galerkin solutions but does not preclude small over- and undershoots in the vicinity of layers. Although the remaining nonphysical oscillations are often small in magnitude, they are not permissible in many applications. An example are chemically reacting flows where it is essential to guarantee that the concentrations of all species are nonnegative. Another example are free-convection computations where temperature oscillations create spurious sources and sinks of momentum that effect the computation of the flow field. The small spurious oscillations may also deteriorate the solution of nonlinear problems, e.g., in two-equations turbulence models or in numerical simulations of compressible flow problems, where the solution may develop discontinuities (shocks) whose poor resolution may effect the global stability of the numerical calculations.

The above-mentioned spurious oscillations in SUPG solutions indicate that using the streamlines as upwind direction is not always sufficient. Therefore, as a remedy, various nonlinear terms introducing artificial crosswind diffusion in the neighborhood of layers have been proposed to be added to the SUPG formulation in order to obtain a method which is monotone, at least in some model cases, or which at least reduces the local oscillations. This procedure is often referred to as discontinuity capturing or shock capturing, nevertheless, we prefer to call these methods *spurious oscillations at layers diminishing (SOLD) methods*, which we regard as more apposite.

It may be surprising that nonlinear methods are applied to the numerical solution of the linear equation (1). However, for the limit $\varepsilon = 0$, the famous Godunov theorem [22] states that a linear monotone discretization is at most of first order convergence so that applying linear methods limits the accuracy if one insists on the monotonicity. We are not aware of an analogous mathematical theorem for $\varepsilon > 0$, but numerical experience suggests that the situation is similar for the case of small ε.

A typical SOLD term added to the left-hand side of (5) is of the form

$$(\widetilde{\varepsilon}(u_h)\,\nabla u_h, \nabla v_h) \tag{8}$$

or

$$(\widetilde{\varepsilon}(u_h)\,D\,\nabla u_h, \nabla v_h)\,, \tag{9}$$

where $\widetilde{\varepsilon}(u_h)$ is a nonnegative solution-dependent artificial diffusion parameter and D is the projection onto the line or plane orthogonal to $\boldsymbol{b}$. Thus, the term in (8) introduces an isotropic artificial diffusion whereas the term in (9) adds a crosswind artificial diffusion. An example of $\widetilde{\varepsilon}(u_h)$ is a modification of the artificial diffusion parameter by Codina [18] proposed in [D3], which is given by

$$\widetilde{\varepsilon}(u_h)|_T = \max\left\{0, \eta\,\frac{\mathrm{diam}(T)\,|R_h(u_h)|}{2\,|\nabla u_h|} - \varepsilon\right\} \tag{10}$$

on any element T of the triangulation. Here, $\mathrm{diam}(T)$ is the diameter of T, $\eta > 0$ is a user-chosen parameter (e.g., $\eta = 0.7$ for linear finite elements) and

$$R_h(u_h) = -\varepsilon\,\Delta u_h + \boldsymbol{b}\cdot\nabla u_h + c\,u_h - f \tag{11}$$

is the residual. The result of an application of the SOLD method (9), (10) to Example 1 is shown in Fig. 4. The underlying SUPG method was used with the stabilization parameter defined in (6). One observes that the oscillations of the SUPG solution (cf. Fig. 1) are removed but the layers are slightly smeared. Nevertheless, on different types of meshes than that in Fig. 2, the SOLD method (9), (10) may provide solutions with spurious oscillations, cf., e.g., [D4, Fig. 8b].

The literature on SOLD methods is rather extended but the various numerical tests published in the literature do not allow to draw a clear conclusion concerning their advantages and drawbacks. Therefore, in [D3], we presented a review of the most published SOLD methods, discussed the motivations of their derivation, proposed some alternative choices of parameters and classified them. The review was followed by a numerical comparison of the considered SOLD methods at two test problems whose solutions possess characteristic features of solutions of (1).

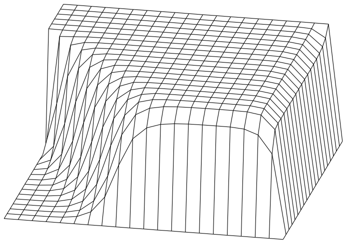

Figure 4: Approximate solution of Example 1 obtained using the SOLD method (9), (10) (Fig. 6.1b from [D7]).

The numerical results gave a first systematic insight into the behaviour of the SOLD methods and showed that the improved Mizukami–Hughes method was always the best method if the nonlinear iterations converged. Among the other SOLD methods, no one could be preferred in all cases but several methods were identified that should not be applied.

The studies in [D3] were followed by a second part published in [D4] where the most promising SOLD methods from the first part were investigated in more detail for linear and bilinear finite elements. Analytical and numerical studies showed that SOLD methods without user-chosen parameters are in general not able to remove the spurious oscillations of the solution obtained with the SUPG discretization. For methods with a free parameter, like the one in (9), (10), values of the parameter could be derived in two examples such that the spurious oscillations were almost removed. It turned out that a spatially constant choice of the parameters was not sufficient in general and that the optimal parameters depended on the data of the problem and on the mesh. In addition, an example was presented for which none of the investigated methods provided a qualitatively correct approximate solution. The iterative solution of the nonlinear discrete problems was also studied. It was shown that the number of iterations or the convergence of the iterative process depend again on the problem, the mesh and the parameters of the SOLD methods. It could be observed that the convergence is often strongly influenced by the choice of an appropriate damping factor and a strategy was proposed for an automatic and dynamic computation of this factor. The studies in this paper revealed that it is in general completely open how to obtain oscillation-free solutions using the considered classes of methods.

The papers [D3,D4] were supplemented by numerical studies for a convection–diffusion problem with a nonconstant convection field whose solution possesses interior layers in [D5]. This setting is closer to problems one encounters in applications than the test problems considered in the two previous publications. The conclusions were similar as in [D4]. Further comparisons of various SOLD methods can be found in [27, 28].

5 Choice of the SUPG stabilization parameter

The studies summarized in Section 4 showed that it is in general not clear how to design SOLD methods which would suppress the spurious oscillations present in SUPG solutions to a sufficient extent (without smearing the layers considerably). One possibility how to circumvent this problem is to try to improve the definition of the SUPG stabilization parameter. The formula (6) leads to nodally exact solutions in the one-dimensional case under simplifying assumptions, see, e.g., [17], but in two and three dimensions it is not optimal in general. The choice of the stabilization parameter at characteristic layers has only a limited influence on the spurious oscillations appearing in these regions (cf., e.g., [46]), but there is a hope of improvement at outflow boundary layers.

One possibility how to define the SUPG stabilization parameter at outflow boundary layers was proposed in [D6] for linear triangular finite elements. To present this definition, let us first denote by $G_h \subset \Omega$ the union of all triangles intersecting the outflow boundary of Ω (i.e., the part of $\partial\Omega$ where the product of $\boldsymbol{b}$ and the outward normal vector to $\partial\Omega$ is positive). Then, by analogy to (6), the parameter τ is defined, on any triangle $T \subset G_h$, by

$$(12) \qquad \tau|_T = \tau_0|_T \left(\coth Pe_T - \frac{1}{Pe_T} \right) \qquad \text{with} \qquad Pe_T = \frac{|\boldsymbol{b}_T|\, h_T}{2\,\varepsilon}\,,$$

where τ_0 is a piecewise constant function satisfying

$$(13) \qquad \int_{G_h} \varphi_i + \tau_0\, \boldsymbol{b} \cdot \nabla \varphi_i \, \mathrm{d}\mathbf{x} = 0, \qquad i = 1, \dots, M\,,$$

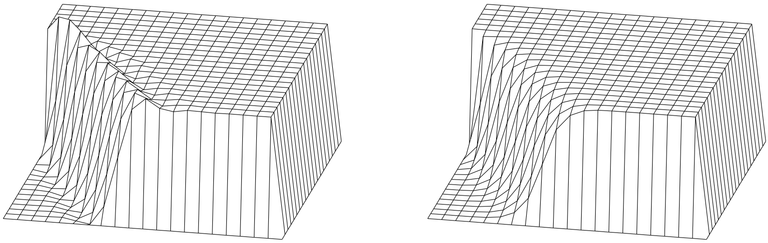

Figure 5: Approximate solution of Example 1 obtained using the SUPG method with τ from (12) (left) and using this approach combined with the SOLD term (9), (10) away from the boundary layers (right) (Figs. 6.1c,d from [D7]).

$\boldsymbol{b}_T$ is the mean value of $\boldsymbol{b}$ on T, and φ_i are the same basis functions of V_h as in Section 3. On triangles $T \not\subset G_h$, the parameter τ is defined by (6) with $\boldsymbol{b}$ replaced by $\boldsymbol{b}_T$. It was shown in [D6] that a piecewise constant function τ_0 satisfying (13) exists and an algorithm how to construct it was given. Numerical results in [D6] demonstrate a significant reduction of spurious oscillations in approximate solutions in comparison to the standard choice of τ given by (6) while accuracy away from layers is preserved. For simple model problems, even nodally exact solutions are obtained. Whereas all definitions of stabilization parameters published in the literature so far were based on local information on a given element of the triangulation, the results of [D6] show that this local information is not sufficient for obtaining oscillation-free SUPG solutions in general.

The choice of the SUPG stabilization parameter τ introduced in [D6] was further discussed in [D7]. It was demonstrated that a combination of this choice of τ and the SOLD method (9), (10) provides fairly satisfactory approximations of solutions to (1). The results of [D7] also show that it is essential to define both the parameter τ and the mesh in such a way that the spurious oscillations in the SUPG solution are as small as possible. Otherwise the addition of a SOLD term cannot be expected to lead to an oscillation-free solution. Numerical tests in [D7] illustrate how small modifications of the mesh may significantly improve the quality of SUPG solutions.

Fig. 5 (left) shows that, along the boundary layers, the new definition of the SUPG stabilization parameter formulated in (12), (13) completely removes the oscillations present in the SUPG solution for τ given by (6) and does not introduce any smearing of these layers. Of course, the oscillations along the interior layer still persist. They can be removed by adding the SOLD term (9), (10) away from the boundary layers, see Fig. 5 (right).

6 Adaptive optimization of stabilization parameters

The above discussion revealed that a basic problem of most of the stabilized methods is the design of appropriate stabilization parameters which would lead to sufficiently small nonphysical oscillations without compromising accuracy. As it follows from the publications discussed in Section 4, 'optimal' parameters depend on the data of the problem and the used mesh in a complicated way so that, in general, one cannot expect to be able to define them a priori. Therefore, in [D8], we proposed to compute the stabilization parameters a posteriori by minimizing a target functional characterizing the quality of the approximate solution. This is a nonlinear constraint optimization problem that has to be solved iteratively. A key component of this approach consists in the efficient computation of the Fréchet derivative of the functional with respect to the stabilization parameter. This was achieved by utilizing an adjoint problem with an appropriate right-hand side, which led to a new general framework for the optimization of parameters in stabilized methods for convection–diffusion equations. Benefits of this approach were demonstrated on its application to the optimization of a piecewise constant parameter τ in the SUPG method.

The above-mentioned target functional can be defined, e.g., by

$$I_h(u_h) = \|R_h(u_h)\|^2_{0,\Omega\setminus B_h} + \|\phi(|D\,\nabla u_h)|)\|_{L^1(\Omega\setminus B_h)}\,, \tag{14}$$

where $R_h(u_h)$ is the residual defined in (11), D is the projection onto the line or plane orthogonal to $\boldsymbol{b}$ used in (9), B_h is the union of all elements of the triangulation $\mathscr{T}_h$ intersecting the boundary of Ω, and

$$\phi(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 1, \\ 0.5\,(5\,x^2 - 3\,x^3) & \text{if } x < 1. \end{cases}$$

The set B_h is excluded from the computation of the integral terms in (14) since the contributions from this set would dominate the value of I_h and prevent the method from reducing spurious oscillations outside B_h sufficiently. Note that the error in B_h is large even for a nodally exact solution since the layers are not resolved by the mesh in the applications we have in mind. Thus, a significant reduction of the error in B_h is not possible. The second term in (14) controls the crosswind derivative of u_h and its minimization should lead to suppressing spurious oscillations in the crosswind direction which can be observed, e.g., in Fig. 5 (left). It is defined as the integral of $\sqrt{|D\,\nabla u_h|}$ with a regularization of the square root near the origin. A motivation for this definition can be found in [D9]. The target functional defined in (14) gave the best results in [D8].

In [D9], the methodology proposed in [D8] was applied to the optimization of the parameters in a SOLD method. Since one of the most promising approaches among the SOLD methods seems to be the modified method of Codina (9), (10), we considered the SUPG method enriched by the crosswind artificial diffusion term (9) with

$$\widetilde{\varepsilon}(u_h)|_T = \eta\,\frac{\mathrm{diam}(T)\,|R_h(u_h)|}{2\,|\nabla u_h|} \qquad \forall\, T \in \mathscr{T}_h\,.$$

Both the parameters τ and η were optimized as piecewise constant functions. In this way very accurate numerical results with steep layers and negligible spurious oscillations could be obtained. The only drawback of this approach is the increased computational cost connected with the solution of the optimization problem.

Two results of the parameter optimization for Example 1 are shown in Fig. 6. One observes that the optimization of the SUPG parameter leads to a perfect approximation of the boundary layers but the spurious oscillations along the interior layer are not removed completely. This can be accomplished by adding the SOLD term (9), (10) and optimizing the parameter η simultaneously with the parameter τ, see Fig. 6 (right). One can observe that no additional smearing of the layers appears.

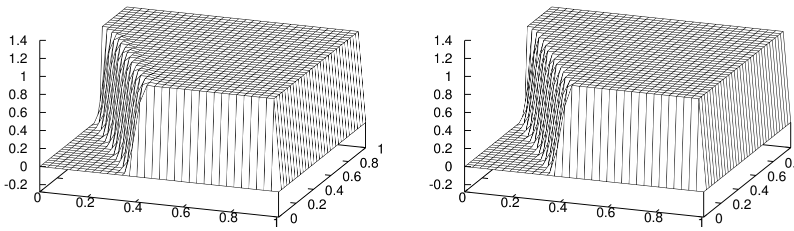

Figure 6: Results of the parameter optimization for Example 1: SUPG method (left) and the SOLD method (9), (10) (right) (Fig. 1 from [D9]).

7 Local projection stabilization

The enhanced stability of the SUPG method (5) in comparison with the Galerkin method (4) originates from the term $(\boldsymbol{b}\cdot\nabla u_h, \tau\,\boldsymbol{b}\cdot\nabla v_h)$. For several reasons, which will be mentioned below, it would be convenient to consider only this term instead of the whole weighted residual stabilization term in (5). Then, however, the resulting method would not be consistent and the accuracy of the method would considerably deteriorate. A possible remedy is to consider only a small-scale part of $\boldsymbol{b}\cdot\nabla u_h$ defined using local projections into large-scale spaces. If the local projection spaces are chosen appropriately, the stability of the SUPG method is preserved without compromising the accuracy.

The local projection stabilization (LPS) was originally proposed in [6] as a technique for stabilizing discretizations of the Stokes problem in which both the pressure and the velocity components are approximated using the same finite element space. Later, the local projection method was extended to stabilization of convection dominated problems [7] and applied to various types of incompressible flow problems (see the review article [11]) and to convection–diffusion–reaction problems, see, e.g., [20, 35, 48]. To define a local projection stabilization of the Galerkin discretization (4), one introduces a second division $\mathscr{M}_h$ of Ω which typically consists of macroelements, i.e., unions of elements of $\mathscr{T}_h$. For each $M\in\mathscr{M}_h$, one introduces a finite dimensional space $D_M\subset L^2(M)$ and defines an orthogonal L^2 projection π_M of $L^2(M)$ onto D_M. It is assumed that there is a positive constant β independent of h such that

$$\sup_{v \in V_M} \frac{(v,q)_M}{\|v\|_{0,M}} \geq \beta \, \|q\|_{0,M} \qquad \forall \; q \in D_M, \; M \in \mathscr{M}_h \,, \tag{15}$$

where $V_M = \{v \in V_h;\, v = 0 \text{ in } \Omega \setminus M\}$. This inf–sup condition is crucial for proving both optimal error estimates and improved stability results, cf. [47, 35] and [D10,D12]. Finally, it is convenient to introduce a constant approximation $\boldsymbol{b}_M$ of $\boldsymbol{b}$ on each set M. Then, denoting by $\kappa_M := id - \pi_M$ the so-called fluctuation operator (where id is the identity operator on $L^2(M)$), the local projection discretization of (1), (3) defines an approximate solution $u_h \in V_h$ satisfying

$$a(u_h, v_h) + s_h(u_h, v_h) = (f, v_h) \quad \forall \; v_h \in V_h$$

with

$$s_h(u_h, v_h) = \sum_{M \in \mathscr{M}_h} \tau_M \left(\kappa_M(\boldsymbol{b}_M \cdot \nabla u_h), \kappa_M(\boldsymbol{b}_M \cdot \nabla v_h) \right)_M \,, \tag{16}$$

where τ_M is a nonnegative stabilization parameter. It is also possible to use the full gradient instead of $\boldsymbol{b}_M \cdot \nabla$ in the stabilization term, i.e.,

$$s_h(u_h, v_h) = \sum_{M \in \mathscr{M}_h} \tau_M \left(\kappa_M \nabla u_h, \kappa_M \nabla v_h \right)_M \,, \tag{17}$$

where κ_M is applied to the vector-valued functions componentwise. The parameter τ_M in (16) can be defined analogously as in the SUPG method (cf. (6)); the parameter in (17) should be additionally multiplied by $\|\boldsymbol{b}\|^2_{L^\infty(M)}$. Let us mention that a standard choice is to use $\boldsymbol{b}$ instead of $\boldsymbol{b}_M$ in (16). However, we demonstrated in [35] that then it is generally not possible to obtain optimal convergence results if τ_M scales with respect to the data like in (6).

The advantage of the LPS method compared to the SUPG method is that it does not require the costly computation of second order derivatives and can be easily applied to non-steady problems. Moreover, when applied to systems of partial differential equations, it is possible to avoid undesirable couplings between various components of the solution. A further advantage of these techniques is that they are symmetric. Therefore, if they are applied to optimization problems, the operations 'discretization' and 'optimization' commute [8, 10].

The action of the operator π_M onto a function can be interpreted as extracting its large-scale part. Then the fluctuation operator κ_M provides the small-scale part (fluctuations around the large-scale part). The LPS method can be also interpreted as a variational multiscale method where the influence of the unresolved scales is modeled by the stabilization term determined by the small scales.

A natural norm for the LPS method is given by

$$\|v\|_{LPS} = \left(\varepsilon\, |v|_{1,\Omega}^2 + \|\sigma^{1/2}\, v\|_{0,\Omega}^2 + s_h(v,v)\right)^{1/2}, \tag{18}$$

which is clearly weaker than the SUPG norm defined in (7). For a long time, it was not clear whether the LPS method is less stable than the SUPG method. The first contribution to clarifying this question was made in [D10], where it was shown that the LPS method is stable in the sense of an inf–sup condition with respect to the norm

$$|||v||| = \left(\|v\|_{LPS}^2 + \sum_{M\in\mathscr{M}_h} \delta_M\, \|\Pi_M(\boldsymbol{b}\cdot\nabla v)\|_{0,M}^2\right)^{1/2}, \tag{19}$$

where Π_M is the orthogonal L^2 projection of $L^2(M)$ onto V_M and δ_M is defined analogously as the SUPG parameter in (6). It was proved that, under certain simplifying assumptions, this norm can be bounded from below by a norm analogous to the SUPG norm, which implies, roughly speaking, that the LPS method is as stable as the SUPG method. For the stabilization term (16), the norm $|||\cdot|||$ could be bounded by an analogue of the SUPG norm also from above. The stability of the LPS method with respect to the norm (19) holds true also for $\tau_M = 0$, i.e., the results of [D10] show that the Galerkin finite element method (4) is more stable than usually believed. It was demonstrated in [D10] that this result implies that certain types of oscillating solutions are not allowed by the Galerkin method; basically, only a small-scale part of the Galerkin solution has to be stabilized – and this is exactly what the LPS method does.

Originally, the LPS method was designed as a two-level approach where the mesh $\mathscr{T}_h$ was obtained by a refinement of a triangulation

$\mathscr{M}_h$ of Ω. A crucial property of these refinements is that they always create an additional vertex in the interior of any refined element of $\mathscr{M}_h$. Later, in [47], the one-level approach was introduced where $\mathscr{M}_h = \mathscr{T}_h$ and the validity of the inf–sup condition (15) was assured by defining V_h as a finite element space enriched using higher-order polynomial bubble functions. In [D11], a critical comparison of the two approaches, both computational and analytical, was given, which showed that there are no convincing arguments for preferring one of these approaches.

A drawback of both variants of the LPS method is that they require more degrees of freedom than the SUPG method since the finite element space is either defined on a refined mesh or enriched by additional functions. Therefore, in [36] and [D12], we introduced a generalization of the LPS method which avoids these drawbacks by allowing to use overlapping macroelements. The error analysis for this generalized LPS method with respect to the norm (18) was presented in [36] for both stabilization terms (16) and (17). In [D12], the results of [D10] were improved in the sense that the stability of the LPS method defined using (16) with respect to the SUPG norm was shown without any simplifying assumptions. Another stability result with respect to the SUPG norm was established in [34] by defining the local projection operators using a weighted L^2 inner product.

Like the SUPG method, the LPS does not remove the spurious oscillations present in Galerkin solutions completely and some of them still remain in the vicinity of layers. Therefore, in [D13], we combined the LPS method defined using (16) with the SOLD term

$$\sum_{M\in\mathscr{M}_h} (\widetilde{\varepsilon}_M(u_h)\,\kappa_M(D_M\nabla u_h), \kappa_M(D_M\nabla v_h))_M \;,$$

where

$$\widetilde{\varepsilon}_M(u_h) = \eta\, h_M\, |\boldsymbol{b}_M|\, |\kappa_M(D_M\nabla u_h)| \tag{20}$$

or

$$\widetilde{\varepsilon}_M(u_h) = \eta\, h_M\, |\boldsymbol{b}_M|\, \frac{h_M^{d/2}\, |\kappa_M(D_M\nabla u_h)|}{|u_h|_{1,M}}\,, \tag{21}$$

h_M is the diameter of M, $\eta > 0$ is a constant user-chosen parameter, and $D_M : \mathbb{R}^d \to \mathbb{R}^d$ is the projection onto the line or plane orthogonal to the vector $\boldsymbol{b}_M$ (cf. (9), (10)). In this paper, also the transient convection–diffusion–reaction equation

$$u_t - \varepsilon\,\Delta u + \boldsymbol{b}\cdot\nabla u + c\,u = f \quad \text{in } (0,T]\times\Omega$$

equipped with initial and boundary conditions was considered. The data $\boldsymbol{b}$, c, and f were assumed to vary on the time interval $[0,T]$. A one-step θ-scheme was applied as temporal discretization whereas the discretization with respect to the space variables was performed as in the steady-state case. For both the steady-state and transient cases, the solvability, uniqueness (for the variant (20) or a sufficiently small time step) and error estimates were proved. In the transient case, both the fully nonlinear scheme and a linearized variant were considered. Promising numerical results were also obtained for $\widetilde{\varepsilon}_M(u_h)$ defined by replacing the fraction in (21) by its square. The corresponding analysis for the steady-state and transient cases was performed in [5] and [37], respectively.

8 Algebraic flux correction

As we have discussed in the previous sections, most of the methods developed for the numerical solution of convection-dominated problems either do not suppress spurious oscillations in layer regions sufficiently, or introduce too much artificial diffusion and lead to a pronounced smearing of layers. However, there is one class of methods that seems not to suffer from these two deficiencies: the algebraic flux correction (AFC) schemes. These schemes are designed to satisfy the discrete maximum principle by construction (so that spurious oscillations cannot appear) and provide sharp approximations of layers, cf. the numerical results in, e.g., [3, 23, 30, 40]. Like many of the schemes discussed above, the AFC schemes are nonlinear. A drawback of these schemes is that they have been applied successfully only for lowest order finite elements, which limits the accuracy of the computed solutions.

The basic philosophy of flux correction schemes was formulated already in the 1970s in [9, 53]. Later, the idea was applied in the finite element context, e.g., in [2, 45]. In the last fifteen years, these methods have been further intensively developed by Dmitri Kuzmin and his coworkers, see, e.g., [38, 39, 40, 41, 43]. Despite the attractiveness of

AFC schemes, there was no rigorous numerical analysis for this class of methods for a long time. To the best of our knowledge, our results in [D14,D15,D16] represent the first contributions in this direction.

In contrast to the methods discussed in the preceding sections, which are all based on variational formulations, the idea of the AFC schemes is to modify the algebraic system corresponding to a discrete problem. As this underlying discrete problem, we use the Galerkin discretization (4) with a finite element space V_h consisting of continuous piecewise linear functions with respect to a simplicial triangulation of Ω and assume that $\operatorname{div} \boldsymbol{b} = 0$ and $c \geq 0$. We shall formulate the AFC scheme in a form which can be used also with nonhomogeneous Dirichlet boundary conditions for u. To this end, we denote by $x_1, \ldots, x_M$ the interior vertices of $\mathscr{T}_h$ and by $x_{M+1}, \ldots, x_N$ the vertices of $\mathscr{T}_h$ lying on $\partial\Omega$. Then, a continuous piecewise linear approximate solution u_h can be represented by the vector $U \equiv (u_1, \ldots, u_N)$ of its values at the vertices $x_1, \ldots, x_N$, and the Galerkin discretization (4) can be equivalently written as a linear system

$$\sum_{j=1}^{N} a_{ij}\, u_j = f_i\,, \qquad i = 1, \ldots, M\,, \tag{22}$$

where the values $u_{M+1}, \ldots, u_N$ are determined by the Dirichlet boundary condition on $\partial\Omega$; in our case, they all vanish. Now, the matrix of (22) is extended to a matrix $(a_{ij})_{i,j=1}^N$ (typically, one uses the finite element matrix corresponding to the equation (1) with homogeneous Neumann boundary conditions) and one defines a symmetric artificial diffusion matrix $(d_{ij})_{i,j=1}^N$ with the entries

$$d_{ij} = d_{ji} = -\max\{a_{ij}, 0, a_{ji}\} \qquad \forall\; i \neq j\,, \qquad\qquad d_{ii} = -\sum_{j \neq i} d_{ij}\,.$$

Using this artificial diffusion matrix, the linear system (22) is rewritten in a form with a M-matrix on the left-hand side and a sum of antidiffusive fluxes on the right-hand side. Those of these fluxes that are responsible for a violation of the discrete maximum principle are limited using solution-dependent correction factors. In this way, the linear system (22) is replaced by the nonlinear problem

$$(23)\qquad \sum_{j=1}^{N} a_{ij}\, u_j + \sum_{j=1}^{N} (1-\alpha_{ij}(U))\, d_{ij}\, (u_j - u_i) = f_i\,, \quad i = 1,\dots,M\,,$$

with $\alpha_{ij}(U) \in [0,1]$, $i,j = 1,\dots,N$. The limiter functions α_{ij} are to be chosen in such a way that the AFC scheme (23) satisfies the discrete maximum principle.

In [D14], the AFC scheme (23) was investigated in the one-dimensional case for a limiter defined in [39]. In contrast to the common application of AFC schemes, it was not assumed that $\alpha_{ij} = \alpha_{ji}$, which may cause a lack of conservation. It was proved that the scheme satisfies a discrete maximum principle if a solution exists. However, examples were constructed which show that this scheme does not necessarily have a solution. A modification of the scheme was proposed for which the existence of a solution and a weak variant of the discrete maximum principle were proved.

In [D15], the AFC scheme (23) with limiters satisfying the symmetry condition $\alpha_{ij} = \alpha_{ji}$ was analyzed for general linear boundary value problems in any space dimension. Under a continuity assumption on the limiters, the existence of a solution was proved. As a consequence, the unique solvability of the linearized problem (23) (i.e., with α_{ij} independent of U) was obtained, which is useful for computing the solution of (23) numerically using a fixed-point iteration. Furthermore, the AFC scheme was formulated in a variational form and an abstract error estimate was derived. As usual for stabilized methods, the norm for which the error estimate is given contains a contribution from the flux correction term in (23). Then the abstract theory was applied to a discretization of the convection–diffusion–reaction equation (1) and an error estimate was derived. Numerical results in [D15] show that, under the minimal assumptions on the limiters used in the analysis, the derived error estimate is sharp. Finally, for the limiter of [39], the AFC scheme (23) was proved to satisfy the discrete maximum principle on Delaunay meshes.

The limiter of [39] investigated in [D14,D15] can be regarded as a standard limiter for steady-state problems. However, apart from the fact that is does not guarantee the discrete maximum principle on gen-

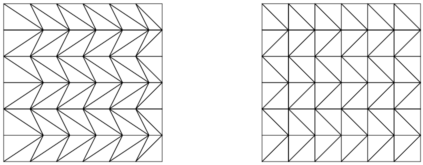

Figure 7: Distorted mesh used in the simulations of [D16] (left) and starting point for its construction (right) (Fig. 4 from [D16]).

eral meshes, its further drawback is that it is not linearity preserving in general. This property demands that the AFC term vanishes if the solution is a polynomial of degree 1 (at least locally). This restriction, which can be interpreted as a weak consistency requirement, is believed to lead to improved accuracy in regions where the solution is smooth. In fact, in previous works, linearity preservation was linked to good convergence properties for diffusion problems (see, e.g., [26, 42]). In addition, it has been observed in different works (see, e.g., [16] and, especially, the introduction in [21] for a discussion) that linearity preservation improves the quality of the approximate solution on distorted meshes.

The above considerations were a motivation for our recent publication [D16]. Here we specified rather weak assumptions on the limiters that are sufficient for proving the discrete maximum principle. Then a limiter was designed that fulfills these assumptions by modifying the algorithm proposed in [40]. The linearity preservation was assured by introducing an explicit geometric information about the mesh into the definition of the limiter. Numerical studies in [D16] support the analytical results and indicate that the linearity preservation is important for an optimal convergence of the AFC scheme. To the best of our knowledge, the method presented in [D16] is the first AFC scheme for a convection–diffusion–reaction equation that satisfies both the discrete maximum principle and linearity preservation on general simplicial meshes.

Numerical results in [D16] were computed on distorted meshes. They were constructed starting from Delaunay meshes of the type depicted in Fig. 7 (right) by shifting interior nodes to the right by half of the horizontal mesh width on each even horizontal mesh line. This leads to meshes of the type shown in Fig. 7 (left). For most of the diagonal edges,

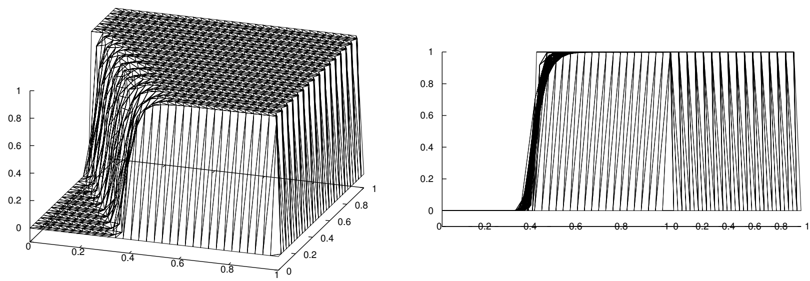

Figure 8: Two views of the approximate solution of Example 1 computed using the AFC scheme (23) with the limiter introduced in [D16] (Figs. 7 and 8 from [D16]).

the sum of the two angles opposite the edge is greater than $5\pi/4$ and hence the mesh is not of Delaunay type. Two views of the approximate solution of Example 1 computed using the AFC scheme (23) with the limiter introduced in [D16] on a mesh of the type shown in Fig. 7 (left) are depicted in Fig. 8. One can observe that the method provides very sharp approximations of all the layers and no spurious oscillations appear.

9 Future work

The doctoral thesis shows that there are still many open questions and a wide potential for improvement in the field of discretization techniques for convection–diffusion problems. In particular, the algebraic flux correction seems to be a promising approach which deserves deeper investigations and we plan to continue our research in this area in the near future. For example, it would be interesting to analyze the time-dependent case or to extend the analysis to anisotropic meshes, to derive a posteriori error estimates and to develop adaptive techniques, or to improve the efficiency of the solution of the nonlinear algebraic systems.

10 List of publications included in the doctoral thesis

The doctoral thesis is a collection of publications listed in this section. They are ordered in the same way as in the doctoral thesis.

[D1] P. Knobloch: Improvements of the Mizukami–Hughes method for convection–diffusion equations, *Computer Methods in Applied Mechanics and Engineering* 196 (1-3): 579–594, 2006.

[D2] P. Knobloch: Numerical solution of convection–diffusion equations using a nonlinear method of upwind type, *Journal of Scientific Computing* 43 (3): 454–470, 2010.

[D3] V. John, P. Knobloch: On spurious oscillations at layers diminishing (SOLD) methods for convection–diffusion equations: Part I – A review, *Computer Methods in Applied Mechanics and Engineering* 196 (17-20): 2197–2215, 2007.

[D4] V. John, P. Knobloch: On spurious oscillations at layers diminishing (SOLD) methods for convection–diffusion equations: Part II – Analysis for P_1 and Q_1 finite elements, *Computer Methods in Applied Mechanics and Engineering* 197 (21-24): 1997–2014, 2008.

[D5] V. John, P. Knobloch: On the performance of SOLD methods for convection–diffusion problems with interior layers, *International Journal of Computing Science and Mathematics* 1 (2-4): 245–258, 2007.

[D6] P. Knobloch: On the choice of the SUPG parameter at outflow boundary layers, *Advances in Computational Mathematics* 31 (4): 369–389, 2009.

[D7] P. Knobloch: On the definition of the SUPG parameter, *Electronic Transactions on Numerical Analysis* 32: 76–89, 2008.

[D8] V. John, P. Knobloch, S.B. Savescu: A posteriori optimization of parameters in stabilized methods for convection–diffusion problems – Part I, *Computer Methods in Applied Mechanics and Engineering* 200 (41-44): 2916–2929, 2011.

[D9] V. John, P. Knobloch: Adaptive computation of parameters in stabilized methods for convection–diffusion problems, in: A. Cangiani, R.L. Davidchack, E.H. Georgoulis, A. Gorban, J. Levesley, M.V. Tretyakov (eds.), Numerical Mathematics and Advanced Applications 2011, Proceedings of ENUMATH 2011, Springer-Verlag, Berlin, 2013, pp. 275–283.

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[D11] P. Knobloch, G. Lube: Local projection stabilization for advection–diffusion–reaction problems: One-level vs. two-level approach, *Applied Numerical Mathematics* 59 (12): 2891–2907, 2009.

[D12] P. Knobloch: A generalization of the local projection stabilization for convection–diffusion–reaction equations, *SIAM Journal on Numerical Analysis* 48 (2): 659–680, 2010.

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[D14] G.R. Barrenechea, V. John, P. Knobloch: Some analytical results for an algebraic flux correction scheme for a steady convection–diffusion equation in one dimension, *IMA Journal of Numerical Analysis* 35 (4): 1729–1756, 2015.

[D15] G.R. Barrenechea, V. John, P. Knobloch: Analysis of algebraic flux correction schemes, *SIAM Journal on Numerical Analysis* 54 (4): 2427–2451, 2016.

[D16] G.R. Barrenechea, V. John, P. Knobloch: An algebraic flux correction scheme satisfying the discrete maximum principle and linearity preservation on general meshes, *Mathematical Models and Methods in Applied Sciences* 27 (3): 2017, doi:10.1142/S0218202517500087, in press.

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