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Principle $\mathrm{S}_1(\mathcal{P},\mathcal{R})$: ideals and functions

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Abstract

We investigate ideal versions of Scheepers' $\mathrm{S}_1(\Gamma,\Gamma)$-space, Arkhangel'skiĭ's α_4-space and Scheepers' monotonic sequence selection property, i.e., $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma,\mathcal{J}\text{-}\Gamma)$-space, $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}},\mathcal{J}\text{-}\Gamma_{\mathbf{0}})$-space and $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}},\mathcal{J}\text{-}\Gamma_{\mathbf{0}})$-space, respectively. We show that cardinal invariant $\lambda(\mathcal{I},\mathcal{J})$ introduced in [42] is their common critical cardinality and we study this combinatorial characteristic in its own. For instance, we show that

$$\min\{\mathtt{cov}^*(\mathcal{I}),\mathfrak{b}\}\leq\lambda(\mathcal{I},\mathcal{J})\leq\mathfrak{b}_{\mathcal{J}}$$

and consequently if $\mathtt{cov}^*(\mathcal{I})\geq\mathfrak{b}$ and $\mathcal{J}$ has the Baire property then $\lambda(\mathcal{I},\mathcal{J})=\mathfrak{b}$. Moreover, we show that control sequence $\langle 2^{-n}:\ n\in\omega\rangle$ in the definition of $(\mathcal{I},\mathrm{s}\mathcal{J})$wQN-space in [6] is not essential.

Keywords: ideal convergence, quasi-normal convergence, selection principle
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1. Introduction

All topological spaces are assumed to be infinite and Hausdorff. By a function, if not stated otherwise, we mean real-valued function. Basic set-theoretical and topological terminology follows mainly [5] and [17]. The paper [27] is a long survey on ideals. The terminology will be recalled later.

L. Bukovský, P. Das and J.Š. [6] introduced an ideal version of Arkhangel'skiĭ's α_4-space [1], denoted $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}},\mathcal{J}\text{-}\Gamma_{\mathbf{0}})$ in this paper and $(\mathcal{I},\mathcal{J}\text{-}\alpha_4)$ in [6]. Let X be a topological space, $\mathcal{I}$, $\mathcal{J}$ being ideals on ω. $\mathrm{C}_p(X)$, the space of all continuous functions on X, has **the property** $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}},\mathcal{J}\text{-}\Gamma_{\mathbf{0}})$ if:

for any sequence $\langle\langle f_{n,m}:\ m\in\omega\rangle:\ n\in\omega\rangle$ of sequences of continuous real-valued functions such that $f_{n,m}\xrightarrow{\mathcal{I}}\mathbf{0}$ for each n, there exists a sequence $\langle m_n:\ n\in\omega\rangle$ such that $f_{n,m_n}\xrightarrow{\mathcal{J}}\mathbf{0}$.

If X is a γ-set then $\mathrm{C}_p(X)$ has $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}},\mathcal{J}\text{-}\Gamma_{\mathbf{0}})$ by [6]. Considering the inclusion of the ideal Fin in every ideal $\mathcal{I},\mathcal{J}$ and several other facts discussed later, one obtains the following basic diagram of relations for $\mathrm{C}_p(X)$.

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$$\begin{array}{ccccccc}
\mathrm{S}_1(\Gamma_{\mathbf{0}},\Gamma_{\mathbf{0}}) & \longrightarrow & \mathrm{S}_1(\Gamma_{\mathbf{0}},\mathcal{J}\text{-}\Gamma_{\mathbf{0}}) & \longrightarrow & \mathrm{S}_1(\Gamma_{\mathbf{0}},\Omega_{\mathbf{0}}) & \longrightarrow & \mathrm{Ind}_{\mathbb{Z}}(X)=0 \\
\uparrow & & \uparrow & & & & \\
\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}},\Gamma_{\mathbf{0}}) & \longrightarrow & \mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}},\mathcal{J}\text{-}\Gamma_{\mathbf{0}}) & & & & \\
\uparrow & & & & & & \\
\text{Fréchet} & & & & & &
\end{array}$$

Diagram 1: Selection principles for functions.

D. Fremlin and M. Scheepers [23, 38, 39] characterized wQN-space by this property, i.e., any topological space X is a wQN-space if and only if $\mathrm{C}_p(X)$ has a property $\mathrm{S}_1(\Gamma_{\mathbf{0}},\Gamma_{\mathbf{0}})$. Moreover, L. Bukovský and J. Haleš [8] found its covering characterization. In [6], Theorems 4.1 and 6.1 are ideal versions of Fremlin–Scheepers' and Bukovský–Haleš' characterizations. Let us summarize the results here, the applied notation will be explained later.[3]

Theorem 1.1 (L. Bukovský–P. Das–J.Š.). *If X is a normal topological space then the following are equivalent. Moreover, the equivalence* (a) $\equiv$ (b) *holds for arbitrary topological space X.*

(a) $\mathrm{C}_p(X)$ *has* $\left[{\mathcal{I}\text{-}\Gamma_{\mathbf{0}} \atop s\mathcal{J}\mathrm{QN}_{\mathbf{0}}}\right]$.
(b) $\mathrm{C}_p(X)$ *is an* $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}},\mathcal{J}\text{-}\Gamma_{\mathbf{0}})$*-space.*
(c) X *is an* $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{sh},\mathcal{J}\text{-}\Gamma)$*-space.*

The main objective of the present paper is to investigate property $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}},\mathcal{J}\text{-}\Gamma_{\mathbf{0}})$ as well as property $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}},\mathcal{J}\text{-}\Gamma_{\mathbf{0}})$ which we introduce as ideal version of Scheepers' monotonic sequence selection property [37]. Moreover, we shall investigate covering property $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma,\mathcal{J}\text{-}\Gamma)$ naturally connected to both of them.

We shall use the following schemas of selection principles. Some of them were introduced in [36]. Let $\mathcal{P}$ and $\mathcal{R}$ be families of sets. Then

- X is an $\mathrm{S}_1(\mathcal{P},\mathcal{R})$-space if for a sequence $\langle\mathcal{U}_n:\ n\in\omega\rangle$ of elements of $\mathcal{P}$ we can select a set $U_n\in\mathcal{U}_n$ for each $n\in\omega$ such that $\langle U_n:\ n\in\omega\rangle$ is a member of $\mathcal{R}$.
- X has $\binom{\mathcal{P}}{\mathcal{R}}$ if for any $P\in\mathcal{P}$ we can select a set $R\in\mathcal{R}$ such that $R\subseteq P$.
- X has $\left[{\mathcal{P} \atop \mathcal{R}}\right]$ or X is a $[\mathcal{P},\mathcal{R}]$-space if for every $\langle p_n:\ n\in\omega\rangle\in\mathcal{P}$ there is $\langle n_m:\ m\in\omega\rangle$ such that $\langle p_{n_m}:\ m\in\omega\rangle\in\mathcal{R}$. We shall use this notation also in cases when $\mathcal{P}$ and $\mathcal{R}$ denote convergences of sequences. Namely, if $\mathcal{P}$ and $\mathcal{R}$ denote convergences then X is a $[\mathcal{P}_p,\mathcal{R}_p]$-space if for every $\langle p_n:\ n\in\omega\rangle$ such that $p_n\xrightarrow{\mathcal{P}}p$ there is $\langle n_m:\ m\in\omega\rangle$ such that $p_{n_m}\xrightarrow{\mathcal{R}}p$.

The last schema is usually used in its special cases and different notation is applied in the literature. To provide quick adaptation to our schema we prepared a table of standard and our corresponding notation through $[\mathcal{P},\mathcal{R}]$-schema in preliminary section.

The paper is divided into 10 sections. Preliminary section contains basic notation on ideals, convergence, families of functions and selection principles. Third section discusses the definition of ideal version of monotonic sequence selection property. One of main results of the paper, the one about control sequence $\langle 2^{-n}:\ n\in\omega\rangle$ in the definition of $(\mathcal{I},\mathrm{s}\mathcal{J})$wQN-space in [6] is presented in Section 4. In the follow up section we turn our attention to covers and associated selection principles. We recall basic terminology on covers only there and present connections of ideal selection principles to standard ones. Preservation properties with respect to standard ideals orderings are presented in Section 6. The main discussion about coverings and functions is completed in Section 7 which contains two propositions resembling Theorem 1.1, but this time one devoted to $\mathrm{S}_1(\Gamma^{\mathrm{m}}_{\mathbf{0}},\mathcal{J}\text{-}\Gamma_{\mathbf{0}})$ and the other one to $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma,\mathcal{J}\text{-}\Gamma)$. The rest of the paper is devoted

[3]We shall often interchange all three notions throughout the paper without any comment.

to critical cardinality of investigated principles. We begin with purely combinatorial properties of cardinal invariant $\lambda(\mathcal{I},\mathcal{J})$ in Section 8 and continue with connections of this invariant to our selection principles. The last section is devoted to the summary of relations of investigated notions and associated consistency analysis, which grows up from critical cardinality description. We provide there a solution to Problem 4.2 in [6].

2. Preliminaries

We attract a special attention of a reader to the fact that we consider only countable covers. Such restriction is sometimes necessary since we deal with ideals on countable set.

By an ideal on M we understand a family $\mathcal{I} \subseteq \mathcal{P}(M)$ that is hereditary, i.e., $B \in \mathcal{I}$ for any $B \subset A \in \mathcal{I}$, closed under finite unions, contains all finite subsets of M and $M \notin \mathcal{I}$. If not stated explicitly, ideal is an ideal on ω. Calligraphic $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$ and $\mathcal{S}$ are used exclusively to denote ideals. For $\mathcal{A} \subseteq \mathcal{P}(M)$ we denote

$$\mathcal{A}^d = \{A \subseteq M;\ M \setminus A \in \mathcal{A}\}.$$

$\mathcal{F} \subseteq \mathcal{P}(M)$ is a filter if $\mathcal{F}^d$ is an ideal. A maximal filter $\mathcal{U} \subseteq \mathcal{P}(M)$ is called an ultrafilter. For an ideal $\mathcal{K} \subseteq \mathcal{P}(M)$ we denote $\mathcal{K}^+ = \mathcal{P}(M) \setminus \mathcal{K}$. One can see that $B \in \mathcal{K}^+$ if and only if $M \setminus B \notin \mathcal{K}^d$.

Let us recall that ideal $\mathcal{I}$ is **tall** if for any $B \in [\omega]^\omega$, there exists an $A \in \mathcal{I}$ such that $|A \cap B| = \omega$. A set B is said to be a **pseudounion** of the family $\mathcal{A}$ if $\omega \setminus B \in [\omega]^\omega$ and $A \subseteq^* B$ for any $A \in \mathcal{A}$. We can also use the dual notion of a pseudounion. An infinite set $B \subseteq \omega$ is a **pseudointersection** of family $\mathcal{A} \subset [\omega]^\omega$ if $B \subseteq^* A$ for any $A \in \mathcal{A}$. Directly from previous definitions it is clear that $\mathcal{I}$ is not tall if and only if $\mathcal{I}$ has a pseudounion if and only if $\mathcal{I}^d$ has a pseudointersection and if and only if $\mathcal{I} \leq_K \mathrm{Fin}$.

A sequence $\langle x_n :\ n \in \omega\rangle$ of elements of a topological space X is **$\mathcal{I}$-convergent to** $x \in X$ if the set $\{n \in \omega :\ x_n \notin U\} \in \mathcal{I}$ for each neighborhood U of x. We write $x_n \xrightarrow{\mathcal{I}} x$. A sequence $\langle f_n :\ n \in \omega\rangle$ of functions on X $\mathcal{I}$-converges to a function f on X (written $f_n \xrightarrow{\mathcal{I}} f$), if $f_n(x) \xrightarrow{\mathcal{I}} f(x)$ for every $x \in X$. According to [14], if there exists a sequence of positive reals $\langle \varepsilon_n :\ n \in \omega\rangle$ such that $\varepsilon_n \xrightarrow{\mathcal{I}} 0$ and $\{n \in \omega :\ |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$ for any $x \in X$, then a sequence $\langle f_n :\ n \in \omega\rangle$ is called **$\mathcal{I}$-quasi-normally convergent** to f on X, denoted $f_n \xrightarrow{\mathcal{I}\mathrm{QN}} f$. A sequence $\langle \varepsilon_n :\ n \in \omega\rangle$ is called the **control sequence**. We say that a sequence $\langle f_n :\ n \in \omega\rangle$ is called **strongly $\mathcal{I}$-quasi-normally convergent** to f on X if $f_n \xrightarrow{\mathcal{I}\mathrm{QN}} f$ with control sequence $\langle 2^{-n} :\ n \in \omega\rangle$, and denoted $f_n \xrightarrow{\mathrm{s}\mathcal{I}\mathrm{QN}} f$.

We are interested mainly in continuous and positive upper semicontinuous functions[4] on X. The set of all such functions on X is denoted $\mathrm{C}_p(X)$ and $\mathrm{USC}_p(X)$, respectively, and is equipped with inherited topology from Tychonoff product topology of ${}^X\mathbb{R}$, i.e., topology of pointwise convergence. Similarly to M. Scheepers [38] we define

$$\mathcal{I}\text{-}\Gamma_x(X) = \{A \in {}^\omega(X \setminus \{x\}) :\ A \text{ is } \mathcal{I}\text{-convergent to } x\}.$$

$$\Omega_x(X) = \left\{A \in {}^\omega(X \setminus \{x\}) :\ x \in \overline{\{y :\ (\exists n \in \omega)\ A(n) = y\}}\right\}.$$

This paper is interested mainly in $\mathcal{I}\text{-}\Gamma_x(\mathrm{C}_p(X))$, thus we omit $\mathrm{C}_p(X)$ from notation. Let $\mathbf{0}$ denote constant zero-value function on X. Due to homogeneity of $\mathrm{C}_p(X)$ we consider only $\mathcal{I}\text{-}\Gamma_{\mathbf{0}}$. We use $\Gamma_{\mathbf{0}}$ instead of $\mathrm{Fin}\text{-}\Gamma_{\mathbf{0}}$. Hence, we again obtain the definition of $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$ defined explicitly in the Introduction. Finally, a useful observation is that if $\langle f_n :\ n \in \omega\rangle \in \mathcal{I}\text{-}\Gamma_{\mathbf{0}}$ then $\langle f_n :\ n \in \omega\rangle \in \Omega_{\mathbf{0}}$.

One can meet different notation in the literature:

$\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$	$=$	$(\mathcal{I},\mathcal{J}\text{-}\alpha_4)$	[6]	$\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{I}\text{-}\Gamma_{\mathbf{0}})$	$=$	$\mathcal{I}$-SSP	[13]
$f_n \xrightarrow{\mathcal{J}\mathrm{QN}} f$	$=$	$f_n \xrightarrow{(\mathcal{J},\mathcal{J})\text{-e}} f$	[20]	$\mathrm{S}_1(\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{I}\text{-}\Gamma_{\mathbf{0}})$	$=$	$\mathcal{I}$-MSSP	[13]

[4]Function $f\colon X \to \mathbb{R}$ is upper semicontinuous if for every real a the set $\{x \in X :\ f(x) < a\}$ is open.

Schema $\left[\begin{smallmatrix}\mathcal{P}\\ \mathcal{R}\end{smallmatrix}\right]$ also covers many special cases used in literature, namely

$\mathrm{C}_p(X)$ has $\left[\begin{smallmatrix}\mathcal{P}_{\mathbf{0}}\\ \mathcal{R}_{\mathbf{0}}\end{smallmatrix}\right]$	$\Leftrightarrow$	X is a $\mathrm{w}(\mathcal{P},\mathcal{R})$-space	[32]
$\mathrm{C}_p(X)$ has $\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}}\\ \mathcal{J}\text{-}\mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right]$	$\Leftrightarrow$	X is an $(\mathcal{I},\mathcal{J})$wQN-space	[6]
$\mathrm{C}_p(X)$ has $\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}}\\ \mathrm{s}\mathcal{J}\mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right]$	$\Leftrightarrow$	X is an $(\mathcal{I},\mathrm{s}\mathcal{J})$wQN-space	[6]
$\mathrm{C}_p(X)$ has $\left[\begin{smallmatrix}\Gamma_{\mathbf{0}}\\ \mathcal{I}\mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right]$	$\Leftrightarrow$	X is an $\mathcal{I}$wQN-space	[14, 6]
$\mathrm{C}_p(X)$ has $\left[\begin{smallmatrix}\Gamma_{\mathbf{0}}\\ \mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right]$	$\Leftrightarrow$	X is a wQN-space	[9]
$\mathrm{USC}_p(X)$ has $\left[\begin{smallmatrix}\Gamma_{\mathbf{0}}\\ \mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right]$	$\Leftrightarrow$	X is a wQN*-space	[4]

There is no restriction on sequence $\langle n_m:\ m\in\omega\rangle$ in the definition of schema $\left[\begin{smallmatrix}\mathcal{P}\\ \mathcal{R}\end{smallmatrix}\right]$, formally it may be constant. However, in many special cases we can ask additional properties. For example, in the definition of $[\mathcal{P}_{\mathbf{0}},\Gamma_{\mathbf{0}}]$-space $\mathrm{C}_p(X)$ we may ask $\langle n_m:\ m\in\omega\rangle$ to be increasing or in the definition of $[\mathcal{P}_{\mathbf{0}},\mathcal{I}\text{-}\Gamma_{\mathbf{0}}]$-space $\mathrm{C}_p(X)$ we may ask $\langle n_m:\ m\in\omega\rangle$ to be $\mathcal{I}$-diverging to $+\infty$. Similarly for $[\mathcal{P},\mathcal{I}\text{-}\Gamma]$-space.

Finally, if $\mathcal{F}\subseteq\mathcal{P}(\omega)$ is a filter then we shall write simple $\mathcal{F}$ instead of $\mathcal{F}^d$ in all previously recalled and following notation whenever it does not lead to confusion. E.g., we shall write $f_n \xrightarrow{\mathcal{F}\mathrm{QN}} f$ instead of $f_n \xrightarrow{\mathcal{F}^{\mathrm{d}}\mathrm{QN}} f$.

3. Monotone convergence

In the present section we are interested in a monotone version of property $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}},\mathcal{J}\text{-}\Gamma_{\mathbf{0}})$, inspired by the monotonic sequence selection property introduced by M. Scheepers [37]. We say that a sequence $\langle f_n:\ n\in\omega\rangle$ is **monotone sequence** if for any $n\in\omega$ and $x\in X$ we have $f_n(x)\geq f_{n+1}(x)$. We set

$$\Gamma^m_{\mathbf{0}}=\{A\in{}^{\omega}(\mathrm{C}_p(X)\setminus\{\mathbf{0}\}):\ A\text{ is monotone and convergent to }\mathbf{0}\}.$$

$\mathrm{S}_1(\Gamma^{\mathrm{m}}_{\mathbf{0}},\mathrm{Fin}\text{-}\Gamma_{\mathbf{0}})$, denoted $\mathrm{S}_1(\Gamma^{\mathrm{m}}_{\mathbf{0}},\Gamma_{\mathbf{0}})$ in the following, is the monotonic sequence selection property, shortly MSSP. Thus

$$\mathrm{S}_1(\Gamma^{\mathrm{m}}_{\mathbf{0}},\Gamma_{\mathbf{0}})\to\mathrm{S}_1(\Gamma^{\mathrm{m}}_{\mathbf{0}},\mathcal{J}\text{-}\Gamma_{\mathbf{0}}).$$

Property $\mathrm{S}_1(\Gamma^{\mathrm{m}}_{\mathbf{0}},\mathcal{J}\text{-}\Gamma_{\mathbf{0}})$ was considered by D. Chandra [13] under notation $\mathcal{J}$-MSSP. Since any monotone sequence on a compact topological space convergent to $\mathbf{0}$ is uniformly convergent, one can easily see by additivity properties of MSSP that any σ-compact space has $\mathrm{S}_1(\Gamma^{\mathrm{m}}_{\mathbf{0}},\Gamma_{\mathbf{0}})$ and thus $\mathrm{S}_1(\Gamma^{\mathrm{m}}_{\mathbf{0}},\mathcal{J}\text{-}\Gamma_{\mathbf{0}})$.

We try to define the ideal modification of monotone sequence. We say that a sequence $\langle f_n:\ n\in\omega\rangle$ is **$\mathcal{I}$-monotone sequence** if $\{n:\ f_n\not\leq f_m\}\in\mathcal{I}$ for every $m\in\omega$. Members of a sequence which is $\mathcal{I}$-converging to $\mathbf{0}$ and $\mathcal{I}$-monotone are non-negative functions. We say almost monotone sequence instead of Fin-monotone sequence. Almost monotone sequences are bounded from above by monotone ones, as Lemma 3.1 states.

Lemma 3.1. *Let X be a set, $\mathcal{E}$ being a family of functions closed under pointwise maximum. For an almost monotone sequence $\langle f_n:\ n\in\omega\rangle$ such that $f_n\to\mathbf{0}$ and $f_n\in\mathcal{E}$ there is a monotone sequence $\langle g_n:\ n\in\omega\rangle$ such that $g_n\to\mathbf{0}$, $g_n\in\mathcal{E}$ and $f_n\leq g_n$ for any $n\in\omega$.*

Proof. Let $\langle f_n:\ n\in\omega\rangle$ be an almost monotone sequence. We define sets A_n and sequence $\langle n_k:\ k\in\omega\rangle$ such that $n_0=0$ and $A_0=\{m:\ f_m\not\leq f_0\}\cup\{0\}$ and then for $k\geq 1$ we take $n_k=\max A_{k-1}+1$

$$A_k=\{m:\ f_m\not\leq f_{n_k},\ m>n_k\}\cup\{n_k\}.$$

Now we can define functions g_n such that for any $k\in\omega$ and for each $i\in n_{k+1}\setminus n_k$,

$$g_i(x)=\max\{f_m(x):\ m\in n_{k+1}\setminus n_k\}.$$

Sequence $\langle g_n : n \in \omega\rangle$ is monotone since $g_n \le f_{n_{k-1}} \le g_m$ for $m \in n_{k+1} \setminus n_k$, $n \in n_k \setminus n_{k-1}$ and $k > 0$. Moreover, it is also convergent to zero which can be seen by the first part of the latter couple of inequalities. □

For a definition of two parameter property $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$ we propose:

$$\mathcal{I}\text{-}\Gamma^{m}_{\mathbf{0}} = \{A \in {}^{\omega}(\mathrm{C}_p(X) \setminus \{\mathbf{0}\}) : \ A \text{ is } \mathcal{I}\text{-monotone and } \mathcal{I}\text{-convergent to } \mathbf{0}\}\,.$$

Thus we have

$$\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}}) \to \mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}}) \tag{3.1}$$

and by [6] we obtain that if X is a γ-set then $\mathrm{C}_p(X)$ has $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$.

Due to following lemma we shall use $\mathrm{S}_1(\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$ instead of $\mathrm{S}_1(\mathrm{Fin}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$. We shall use similar convention for $[\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathrm{s}\mathcal{J}\mathrm{QN}_{\mathbf{0}}]$-space, an ideal version of $[\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathrm{QN}_{\mathbf{0}}]$-space introduced and investigated in [10].

Lemma 3.2. *Let X be a topological space.*

(1) $\mathrm{C}_p(X)$ *has the property* $\mathrm{S}_1(\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$ *if and only if* $\mathrm{C}_p(X)$ *has the property* $\mathrm{S}_1(\mathrm{Fin}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$.

(2) $\mathrm{C}_p(X)$ *has the property* $\left[{\Gamma^{\mathrm{m}}_{\mathbf{0}} \atop \mathcal{J}\mathrm{QN}_{\mathbf{0}}}\right]$ *if and only if* $\mathrm{C}_p(X)$ *has the property* $\left[{\mathrm{Fin}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}} \atop \mathcal{J}\mathrm{QN}_{\mathbf{0}}}\right]$.

(3) $\mathrm{C}_p(X)$ *has the property* $\left[{\Gamma^{\mathrm{m}}_{\mathbf{0}} \atop s\mathcal{J}\mathrm{QN}_{\mathbf{0}}}\right]$ *if and only if* $\mathrm{C}_p(X)$ *has the property* $\left[{\mathrm{Fin}\text{-}\Gamma^{m}_{\mathbf{0}} \atop s\mathcal{J}\mathrm{QN}_{\mathbf{0}}}\right]$.

Proof. We prove only non-trivial implication of part (1). Parts (2) and (3) are shown similarly.

Let us assume that $\mathrm{C}_p(X)$ has the property $\mathrm{S}_1(\mathrm{Fin}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$. Consider a sequence of almost monotone sequences $\langle\langle f_{n,m} : m \in \omega\rangle : n \in \omega\rangle$ and $f_{n,m} \to \mathbf{0}$ for all $n \in \omega$. By Lemma 3.1 there are monotone sequences $\langle g_{n,m} : n \in \omega\rangle$ such that $f_{n,m} \le g_{n,m}$ for all $n, m \in \omega$ and $g_{n,m} \to \mathbf{0}$.

Since $\mathrm{C}_p(X)$ has the property $\mathrm{S}_1(\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$ there is sequence $\langle m_n : n \in \omega\rangle$ such that $g_{n,m_n} \xrightarrow{\mathcal{J}} \mathbf{0}$. Since $f_{n,m_n} \le g_{n,m_n}$ for all $n \in \omega$ we obtain $f_{n,m_n} \xrightarrow{\mathcal{J}} \mathbf{0}$, so $\mathrm{C}_p(X)$ has the property $\mathrm{S}_1(\mathrm{Fin}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$. □

We shall show in Corollary 6.3 that property $\mathrm{S}_1(\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$ is equivalent to $\mathrm{S}_1(\Gamma^{\mathrm{m}}_{\mathbf{0}}, \Gamma_{\mathbf{0}})$ for a certain class of ideals.

4. Control sequences

By [6], the control sequence $\langle 2^{-n} : n \in \omega\rangle$ in the definition of $[\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathrm{s}\mathcal{J}\mathrm{QN}_{\mathbf{0}}]$-space may be replaced by any sequence $\langle \varepsilon_n : n \in \omega\rangle$ of positive reals such that $\sum_{n=0}^{\infty} \varepsilon_n < \infty$. One can easily deduce that in the definition of $[\Gamma_{\mathbf{0}}, \mathrm{QN}_{\mathbf{0}}]$-space an arbitrary control sequence (converging to zero) may be asked. More generally, such argumentation leads to

$$\mathrm{C}_p(X) \text{ has } \left[{\mathcal{I}\text{-}\Gamma_{\mathbf{0}} \atop \mathrm{QN}_{\mathbf{0}}}\right] \text{ if and only if } \mathrm{C}_p(X) \text{ has } \left[{\mathcal{I}\text{-}\Gamma_{\mathbf{0}} \atop \mathrm{sQN}_{\mathbf{0}}}\right].$$

Although the same is true for $[\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathrm{s}\mathcal{J}\mathrm{QN}_{\mathbf{0}}]$-space, we devote most of this section to the proof of this fact. Moreover, if $\mathcal{J}$ is a P-ideal then the control sequence can be even $\mathcal{J}$-convergent to zero, see the proof of Corollary 3.5 in [20]. We begin with a crucial lemma.

Firstly, we need the following property of a family $\mathcal{E}$ of functions on X:[5]

$$\begin{gathered} \mathcal{E} \text{ is closed under taking normally convergent series of functions from } \mathcal{E} \\ \text{and} \\ \text{if } f \in \mathcal{E}, c_1, c_2 > 0 \text{ then } \min\{c_1, |c_2 f|\} \in \mathcal{E}. \end{gathered} \tag{4.1}$$

By $\mathcal{E}$ we have in mind mainly families of all continuous, Borel, non-negative upper or lower semicontinuous functions. All these families are closed under uniformly convergent series.

[5] A series $\sum_{n=0}^{\infty} f_n$ is normally convergent on X if the series $\sum_{n=0}^{\infty} \sup\{|f_n(x)| : x \in X\}$ is convergent. Normally convergent series is uniformly convergent.

Lemma 4.1. *Let $\langle \varepsilon_n : n \in \omega \rangle$, $\langle \delta_n : n \in \omega \rangle$ be sequences of positive reals in $(0,1]$ such that $\varepsilon_n \to 0$, $\langle \langle f_{n,m} : m \in \omega \rangle : n \in \omega \rangle$ being a sequence of sequences of functions on X. Then there is a sequence $\langle g_m : m \in \omega \rangle$ of functions with values in $[0,2]$ such that the following holds.*

(1) *If $f_{n,m} \in \mathcal{E}$ for all $n \in \omega$ then $g_m \in \mathcal{E}$, assuming $\mathcal{E}$ satisfies (4.1).*
(2) *If $f_{n,m} \xrightarrow{\mathcal{I}} \mathbf{0}$ for each $n \in \omega$ then $g_m \xrightarrow{\mathcal{I}} \mathbf{0}$.*
(3) *If $\langle f_{n,m} : m \in \omega \rangle \in \Gamma^m_{\mathbf{0}}$ for each $n \in \omega$ then $\langle g_m : m \in \omega \rangle \in \Gamma^m_{\mathbf{0}}$.*
(4) *If $k_n = \max \{ i : 2^{-i} \geq \varepsilon_n \}$ then for any $x \in X$ and $m \in \omega$ we have*

$$\text{if } g_m(x) < \varepsilon_n \text{ then } |f_{k_n,m}(x)| < \frac{\delta_n}{2^{k_n}} \leq \delta_n. \tag{4.2}$$

Proof. Let $\langle \langle f_{n,m} : m \in \omega \rangle : n \in \omega \rangle$ be a sequence of sequences of functions on X. One can easily see that k_n is well-defined and moreover the set $\{j : k_j = i\}$ is finite. Let

$$\theta_i = \begin{cases} \min \{\delta_j : k_j = i\} & (\exists n \in \omega)\ k_n = i \\ 1 & \text{otherwise.} \end{cases}$$

Now we define functions $g_m \colon X \to \mathbb{R}$ by

$$g_m(x) = \sum_{i=0}^{\infty} \min \left\{ 2^{-i}, \left| \frac{1}{\theta_i} f_{i,m}(x) \right| \right\}$$

for all $x \in X$.

(1) follows directly by the assumption (4.1) on family $\mathcal{E}$.

(2) Let us suppose that $f_{n,m} \xrightarrow{\mathcal{I}} \mathbf{0}$ for each $n \in \omega$. Let $x \in X$ and $\varepsilon > 0$. Then there exists n_0 such that $\sum_{n \geq n_0} 2^{-n} < \frac{\varepsilon}{2}$. If $n < n_0$ the set $B_n = \left\{ m : |\frac{1}{\theta_n} f_{n,m}(x)| \geq \frac{\varepsilon}{2n_0} \right\} \in \mathcal{I}$. In addition, for $m \notin \bigcup_{n<n_0} B_n$ we have

$$g_m(x) < \sum_{n<n_0} \frac{\varepsilon}{2n_0} + \sum_{n \geq n_0} 2^{-n} < \varepsilon$$

Thus $g_m \xrightarrow{\mathcal{I}} \mathbf{0}$.

(3) If $f_{i,m} \geq f_{i,m+1} \geq \mathbf{0}$ then $\min\{2^{-i}, |\theta_i^{-1} f_{i,m}(x)|\} \geq \min\{2^{-i}, |\theta_i^{-1} f_{i,m+1}(x)|\}$ and after summing up through all i's we obtain $g_m \geq g_{m+1}$.

(4) Let $x \in X$ and $m \in \omega$ be such that $g_m(x) < \varepsilon_n$. Thus $g_m(x) < 2^{-k_n}$ and for each $j \in \omega$ we get:

$$2^{-j} < 2^{-k_n} \text{ or } \left| \frac{1}{\theta_j} f_{j,m}(x) \right| < 2^{-k_n}.$$

However, especially for $j = k_n$ we have $|\frac{1}{\theta_{k_n}} f_{k_n,m}(x)| < 2^{-k_n}$. Thus $|f_{k_n,m}(x)| < \frac{\delta_n}{2^{k_n}}$. □

We are now ready to prove the main theorem of this section.

Theorem 4.2. *Let X be a topological space, $\mathcal{E}$ being a family of functions satisfying (4.1).*

(1) *The following are equivalent.*
   (a) *$\mathcal{E}$ has $\left[{\mathcal{I}\text{-}\Gamma_{\mathbf{0}} \atop \mathrm{s}\mathcal{J}\mathrm{QN}_{\mathbf{0}}} \right]$.*
   (b) *There is a sequence $\langle \varepsilon_n : n \in \omega \rangle$ of positive reals such that $\varepsilon_n \to 0$ and for any sequence $\langle f_n : n \in \omega \rangle$ of functions from $\mathcal{E}$ $\mathcal{I}$-converging to zero there exists a sequence $\langle m_n : n \in \omega \rangle$ such that $f_{m_n} \xrightarrow{\mathcal{J}\mathrm{QN}} \mathbf{0}$ with control sequence $\langle \varepsilon_n : n \in \omega \rangle$.*
   (c) *For every sequence $\langle \varepsilon_n : n \in \omega \rangle$ of positive reals such that $\varepsilon_n \to 0$ and for any sequence $\langle f_n : n \in \omega \rangle$ of functions from $\mathcal{E}$ $\mathcal{I}$-converging to zero there exists a sequence $\langle m_n : n \in \omega \rangle$ such that $f_{m_n} \xrightarrow{\mathcal{J}\mathrm{QN}} \mathbf{0}$ with control sequence $\langle \varepsilon_n : n \in \omega \rangle$.*

(2) *The following are equivalent.*
   (a) $\mathrm{C}_p(X)$ *has* $\left[{\Gamma^{\mathrm{m}}_{\mathbf{0}} \atop s\mathcal{J}\mathrm{QN}_{\mathbf{0}}}\right]$.
   (b) *There is a sequence* $\langle \varepsilon_n : n \in \omega\rangle$ *of positive reals such that* $\varepsilon_n \to 0$ *and for any monotone sequence* $\langle f_n : n \in \omega\rangle$ *of continuous functions converging to zero there exists a sequence* $\langle m_n : n \in \omega\rangle$ *such that* $f_{m_n} \xrightarrow{\mathcal{J}\mathrm{QN}} \mathbf{0}$ *with control sequence* $\langle \varepsilon_n : n \in \omega\rangle$.
   (c) *For every sequence* $\langle \varepsilon_n : n \in \omega\rangle$ *of positive reals such that* $\varepsilon_n \to 0$ *and for any monotone sequence* $\langle f_n : n \in \omega\rangle$ *of continuous functions converging to zero there exists a sequence* $\langle m_n : n \in \omega\rangle$ *such that* $f_{m_n} \xrightarrow{\mathcal{J}\mathrm{QN}} \mathbf{0}$ *with control sequence* $\langle \varepsilon_n : n \in \omega\rangle$.

Proof. Let us consider part (1). We shall show (b) → (c). The implications (a) → (b), (c) → (a) are trivial. The proof of monotone version (2) is similar.

Let $\langle \varepsilon_n : n \in \omega\rangle$ be a control sequence from part (b) and $\langle \delta_n : n \in \omega\rangle$ be a sequence of positive numbers such that $\delta_n \to 0$. Moreover, we may assume that $\varepsilon_n \leq 1$. Let $\langle f_m : m \in \omega\rangle$ be a sequence of functions from $\mathcal{E}$ such that $f_m \xrightarrow{\mathcal{I}} \mathbf{0}$. Set $\langle k_n : n \in \omega\rangle$ as in Lemma 4.1 and $f_{n,m} = f_m$ for all $n \in \omega$. Thus by Lemma 4.1 there exists sequence of functions $\langle g_m : m \in \omega\rangle$ from $\mathcal{E}$ on X such that $g_m \xrightarrow{\mathcal{I}} \mathbf{0}$ and property (4.2) holds.

Since $\mathcal{E}$ is an $[\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathrm{s}\mathcal{J}\mathrm{QN}_{\mathbf{0}}]$-space with control sequence $\langle \varepsilon_n : n \in \omega\rangle$ there exists sequence $\langle m_n : n \in \omega\rangle$ such that $\{n : g_{m_n}(x) \geq \varepsilon_n\} \in \mathcal{J}$.

Let $x \in X$. Consider any $n \in \omega$ such that $g_{m_n}(x) < \varepsilon_n$. By Lemma 4.1 we obtain $|f_{m_n}(x)| = |f_{k_n,m_n}(x)| < \delta_n$. Thus $\{n : |f_{m_n}(x)| \geq \delta_n\} \subseteq \{n : g_{m_n}(x) \geq \varepsilon_n\} \in \mathcal{J}$. □

If $\mathrm{C}_p(X)$ is an $[\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathrm{s}\mathcal{J}\mathrm{QN}_{\mathbf{0}}]$-space then for any sequence $\langle f_n : n \in \omega\rangle$ of continuous functions on X $\mathcal{I}$-converging to zero there exists a sequence $\langle m_n : n \in \omega\rangle$ such that $f_{m_n} \xrightarrow{(\mathcal{J},\mathrm{Fin})\text{-e}} \mathbf{0}$ in the sense of R. Filipów and M. Staniszewski [20].

When we consider only discrete space D then $\mathrm{C}_p(D)$ is the family of all functions on D. In this sense, the results in [21, 41] can be viewed as closely related to the investigation of $[\mathcal{P}, \mathcal{R}]$-space. Namely, R. Filipów and M. Staniszewski [21] investigate an $[\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, (\mathcal{I}, \mathcal{J})\text{-e}]$-space while M. Staniszewski [41] and M. Repický [32] are interested in a more general situation of an $[\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, (\mathcal{J}, \mathcal{K})\text{-e}]$-space. In fact, an $[\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, (\mathcal{J}, \mathrm{Fin})\text{-e}]$-space is the most interesting case for this paper since

$$\text{if } \mathrm{C}_p(X) \text{ is an } [\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathrm{s}\mathcal{J}\mathrm{QN}_{\mathbf{0}}]\text{-space then } \mathrm{C}_p(X) \text{ is an } [\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, (\mathcal{J}, \mathrm{Fin})\text{-e}]\text{-space.}$$

However, we do not know whether these notions are distinct. Let us consider a sequence $\langle f_n : n \in \omega\rangle$ of continuous functions $\mathcal{I}$-converging to $\mathbf{0}$. By Theorem 4.2, if we prescribe any converging control sequence and $\mathrm{C}_p(X)$ is an $[\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathrm{s}\mathcal{J}\mathrm{QN}_{\mathbf{0}}]$-space then there is $\langle n_m : m \in \omega\rangle$ such that $f_{n_m} \xrightarrow{\mathcal{J}\mathrm{QN}} \mathbf{0}$ with the prescribed control. In contrary, if $\mathrm{C}_p(X)$ is an $[\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, (\mathcal{J}, \mathrm{Fin})\text{-e}]$-space then there is a control and $\langle n_m : m \in \omega\rangle$ such that $f_{n_m} \xrightarrow{\mathcal{J}\mathrm{QN}} \mathbf{0}$ with such control.

5. Coverings

Let us recall that we deal only with countable covers. Let X be a topological space. A sequence $\langle U_n : n \in \omega\rangle$ of subsets of X is called an **$\mathcal{I}$-γ-cover**, if for every n, $U_n \neq X$ and for every $x \in X$, the set $\{n \in \omega : x \notin U_n\} \in \mathcal{I}$, see [15]. A sequence $\langle U_n : n \in \omega\rangle$ of subsets of X is called an **ω-cover**, if for every n, $U_n \neq X$ and for every finite $F \subseteq X$ there is n such that $F \subseteq U_n$. The symbol **$\mathcal{I}$-Γ** denotes the family of all open $\mathcal{I}$-γ-covers of X. As usually, a γ-cover is a Fin-γ-cover and Γ = Fin-Γ.[6] Ω is the family of all open ω-covers.

[6] A γ-cover is standardly a family, see [6].

A cover $\langle V_n : n \in \omega \rangle$ is called a **shrinking** of the cover $\langle U_n : n \in \omega \rangle$ if $V_n \subseteq U_n$ for all $n \in \omega$. Then an $\mathcal{I}$-γ-cover $\langle U_n : n \in \omega \rangle$ is **shrinkable** if there exists a closed $\mathcal{I}$-γ-cover that is a shrinking of $\langle U_n : n \in \omega \rangle$. The family of all open shrinkable $\mathcal{I}$-γ-covers is denoted by $\mathcal{I}\text{-}\Gamma^{sh}$.

As a corollary of Theorem 1.1 from Introduction, L. Bukovský, P. Das and J. Š. obtained the ideal version of Scheepers' result [39].

$$\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma) \to \mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{sh}, \mathcal{J}\text{-}\Gamma) \Leftrightarrow \mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}}) \to \mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}}). \tag{5.1}$$

Note that by [6], any γ-set is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-space. Moreover, we add a simple observation. It follows by the fact that any γ-cover is an $\mathcal{I}$-γ-cover and any $\mathcal{J}$-γ-cover is an ω-cover. Similarly, set of all members of $\mathcal{J}$-convergent sequence with limit $\mathbf{0}$ contains $\mathbf{0}$ in a closure.

Observation 5.1.

(1) *If X is an $\mathrm{S}_1(\Gamma, \mathcal{J}\text{-}\Gamma)$-space then X is an $\mathrm{S}_1(\Gamma, \Omega)$-space.*

(2) *If $\mathrm{C}_p(X)$ is an $\mathrm{S}_1(\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$-space then $\mathrm{C}_p(X)$ is an $\mathrm{S}_1(\Gamma_{\mathbf{0}}, \Omega_{\mathbf{0}})$-space.*

The notion of an $\mathrm{S}_1(\Gamma_{\mathbf{0}}, \Omega_{\mathbf{0}})$-space was introduced by M. Scheepers [37] as the weak sequence selection property. M. Sakai [33] showed that any normal topological space X such that $\mathrm{C}_p(X)$ is an $\mathrm{S}_1(\Gamma_{\mathbf{0}}, \Omega_{\mathbf{0}})$-space is zero-dimensional.

Thus the relations of an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-space, its special cases for $\mathcal{I} = \mathrm{Fin}$ or $\mathcal{J} = \mathrm{Fin}$ and classical covering properties $\mathrm{S}_1(\Omega, \Gamma)$, $\mathrm{S}_1(\Gamma, \Omega)$ are as follows.

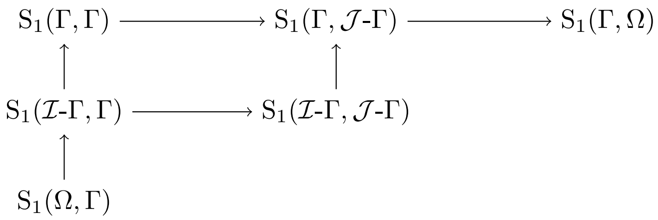

Diagram 2: Covering selection principles.

Proposition 5.2. *Let X be a topological space. Then*

(1) *X is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma)$-space if and only if X has $\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma\\ \Gamma\end{smallmatrix}\right]$ and $\mathrm{S}_1(\Gamma, \Gamma)$.*

(2) *$\mathrm{C}_p(X)$ is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$-space if and only if $\mathrm{C}_p(X)$ has $\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}}\\ \Gamma_{\mathbf{0}}\end{smallmatrix}\right]$ and $\mathrm{S}_1(\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$.*

Proof. We show only the equivalence (1), the equivalence (2) is proved similarly. Moreover, the implication $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma) \to \mathrm{S}_1(\Gamma, \Gamma)$ is trivial.

Let $\mathcal{U}$ be an $\mathcal{I}$-γ-cover, by $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma)$ for sequence $\langle \mathcal{U} : n \in \omega \rangle$ we obtain a γ-cover $\langle U_n : n \in \omega \rangle$ such that $U_n \in \mathcal{U}$. Thus the implication $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma) \to \left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma\\ \Gamma\end{smallmatrix}\right]$ holds.

To prove the reversed implication let $\langle \mathcal{U}_n : n \in \omega \rangle$ be a sequence of $\mathcal{I}$-γ-covers of X. Since X is an $[\mathcal{I}\text{-}\Gamma, \Gamma]$-space we can choose γ-cover $\langle U_{n,m} : m \in \omega \rangle$ for each n such that $U_{n,m} \in \mathcal{U}_n$. By $\mathrm{S}_1(\Gamma, \Gamma)$ there is a γ-cover $\langle U_{n,m_n} : n \in \omega \rangle$. □

The relation between an $\mathrm{S}_1(\Omega, \Gamma)$-space, $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma)$-space and $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$-space is much deeper.

Lemma 5.3. (1) *For any countable ω-cover $\mathcal{U}$ of X and its bijective enumeration $\langle U_n : n \in \omega \rangle$ there is an ideal $\mathcal{I}$ such that $\langle U_n : n \in \omega \rangle$ is an $\mathcal{I}$-γ-cover.*

(2) *For any countable family of functions $\mathcal{E}$ on X such that $\mathbf{0} \in \overline{\mathcal{E} \setminus \{\mathbf{0}\}}$ and its bijective enumeration $\langle f_n : n \in \omega \rangle$ there is an ideal $\mathcal{I}$ such that $f_n \xrightarrow{\mathcal{I}} \mathbf{0}$.*

Proof. (1) Let $\mathcal{U}$ be a countable ω-cover and $\langle U_n : n \in \omega \rangle$ its bijective enumeration. We set $a(F) = \{n \in \omega : F \subseteq U_n\}$ for every $F \in [X]^{<\omega}$. Since $\mathcal{U}$ is an ω-cover, each $a(F)$ is infinite. Moreover, $a(F_1) \cap a(F_2) = a(F_1 \cup F_2) \in [\omega]^\omega$. Thus the family $\{a(F) : F \in [X]^{<\omega}\}$ has finite intersection property and there is a filter $\mathcal{F} \supseteq \{a(F) : F \in [X]^{<\omega}\}$. Finally, $\langle U_n : n \in \omega \rangle$ is an $\mathcal{F}$-γ-cover[7].

(2) Consider a sequence $\langle f_n : n \in \omega \rangle$ of functions from the statement (2). We denote $a_n(F) = \{m : 2^{-m} + |f_m(x)| < 2^{-n} \text{ for each } x \in F\}$ for every $F \in [X]^{<\omega}$. Since $\mathbf{0}$ is in the closure of the set $\{2^{-n} + |f_n| : n \in \omega\}$ we have $a_n(F) \in [\omega]^\omega$ for each $n \in \omega$ and each $F \in [X]^{<\omega}$. Furthermore, $a_n(F_1) \cap a_k(F_2) \supseteq a_n(F_1) \cap a_n(F_2) = a_n(F_1 \cup F_2)$ for any $n \geq k$ and any $F_1, F_2 \in [X]^{<\omega}$. Thus similarly as in previous case there is a filter $\mathcal{F} \supseteq \{a_n(F) : F \in [X]^{<\omega}, n \in \omega\}$ and $f_n \xrightarrow{\mathcal{F}} \mathbf{0}$.[8] □

We shall use the fact that

$$X \text{ has } \binom{\Omega}{\Gamma} \Leftrightarrow X \text{ has } \begin{bmatrix}\Omega\\ \Gamma\end{bmatrix} \qquad \text{and} \qquad \mathrm{C}_p(X) \text{ has } \binom{\Omega_{\mathbf{0}}}{\Gamma_{\mathbf{0}}} \Leftrightarrow \mathrm{C}_p(X) \text{ has } \begin{bmatrix}\Omega_{\mathbf{0}}\\ \Gamma_{\mathbf{0}}\end{bmatrix}.$$

Let us recall a folklore result by J. Gerlits and Zs. Nagy [25] for a Tychonoff space X:[9]

$$X \text{ has } \binom{\Omega}{\Gamma} \Leftrightarrow X \text{ has } \mathrm{S}_1(\Omega,\Gamma) \Leftrightarrow \mathrm{C}_p(X) \text{ has } \binom{\Omega_{\mathbf{0}}}{\Gamma_{\mathbf{0}}} \Leftrightarrow \mathrm{C}_p(X) \text{ has } \mathrm{S}_1(\Omega_{\mathbf{0}},\Gamma_{\mathbf{0}}).$$

Theorem 5.4. *Let X be a Tychonoff topological space. The following statements are equivalent.*[10]

(a) *X is an $\mathrm{S}_1(\Omega,\Gamma)$-space.*
(b) *X is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma,\Gamma)$-space for every ideal $\mathcal{I}$.*
(c) *$\mathrm{C}_p(X)$ is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}},\Gamma_{\mathbf{0}})$-space for every ideal $\mathcal{I}$.*
(d) *X has $\begin{bmatrix}\mathcal{I}\text{-}\Gamma\\ \Gamma\end{bmatrix}$ for every ideal $\mathcal{I}$.*
(e) *$\mathrm{C}_p(X)$ has $\begin{bmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}}\\ \Gamma_{\mathbf{0}}\end{bmatrix}$ for every ideal $\mathcal{I}$.*

Proof. We prove only (d) → (a) and (e) → (a). Implications (a) → (b) and (b) → (c) are proven in [6]. Implications (b) → (d) and (c) → (e) were shown in Proposition 5.2.

To prove implication (d) → (a), let $\langle U_n : n \in \omega \rangle$ be an ω-cover of X. By part (1) of Lemma 5.3 there is an ideal $\mathcal{I}$ such that $\langle U_n : n \in \omega \rangle$ is also $\mathcal{I}$-γ-cover of X. Since X is an $[\mathcal{I}\text{-}\Gamma,\Gamma]$-space there exists $\langle n_m : m \in \omega \rangle$ such that $\langle U_{n_m} : m \in \omega \rangle$ is a γ-cover of X.

Proof of the implication (e) → (a) is based on Lemma 5.3 as well. Moreover, it is enough to show that $\mathrm{C}_p(X)$ is an $[\Omega_{\mathbf{0}},\Gamma_{\mathbf{0}}]$-space instead of statement (a). Let $\langle f_n : n \in \omega \rangle$ be a sequence of functions (enumerated bijectively) such that $\langle f_n : n \in \omega \rangle \in \Omega_{\mathbf{0}}$. By Lemma 5.3 there is ideal $\mathcal{I}$ such that $f_n \xrightarrow{\mathcal{I}} \mathbf{0}$. Since $\mathrm{C}_p(X)$ is an $[\mathcal{I}\text{-}\Gamma_{\mathbf{0}},\Gamma_{\mathbf{0}}]$-space there exists a sequence $\langle n_m : m \in \omega \rangle$ such that $f_{n_m} \to \mathbf{0}$. □

6. Ideal orderings

The purpose of this section is to describe preservation of investigated properties with respect to standard ideal orderings, see Proposition 6.2. Let us begin with recalling the orderings. Let M_1, M_2 be infinite sets, $\mathcal{K}_1 \subseteq \mathcal{P}(M_1)$, $\mathcal{K}_2 \subseteq \mathcal{P}(M_2)$. If $\varphi : M_2 \to M_1$, the image of $\mathcal{K}_2$ is the family

$$\varphi^{\to}(\mathcal{K}_2) = \left\{A \subseteq M_1 : \varphi^{-1}(A) \in \mathcal{K}_2\right\}.$$

If $\mathcal{K}_2$ is an ideal on M_2 then $\varphi^{\to}(\mathcal{K}_2)$ is closed under subsets and finite unions and $M_1 \notin \varphi^{\to}(\mathcal{K}_2)$. If φ is in addition finite-to-one then $\varphi^{\to}(\mathcal{K}_2)$ is the ideal. For $\varphi : M_2 \to M_1$ we write $\mathcal{K}_1 \leq_\varphi \mathcal{K}_2$ if $\mathcal{K}_1 \subseteq \varphi^{\to}(\mathcal{K}_2)$, i.e., $\varphi^{-1}(I) \in \mathcal{K}_2$ for any $I \in \mathcal{K}_1$. Then $\mathcal{K}_1 \leq_{\mathrm{K}} \mathcal{K}_2$ if there is a function $\varphi : M_2 \to M_1$ such that $\mathcal{K}_1 \leq_\varphi \mathcal{K}_2$, $\mathcal{K}_1 \leq_{\mathrm{KB}} \mathcal{K}_2$ if there is a finite-to-one function $\varphi : M_2 \to M_1$ such that $\mathcal{K}_1 \leq_\varphi \mathcal{K}_2$.

[7] In terms of ideals, $\mathcal{F}^d$-γ-cover.

[8] In terms of ideals, $f_n \xrightarrow{\mathcal{F}^{\mathrm{d}}} \mathbf{0}$.

[9] J. Gerlits and Zs. Nagy [25] considered arbitrary covers and families of functions, so they use notions of a γ-set and notions of Fréchet and strictly Fréchet space.

[10] The equivalence (a) ≡ (b) holds for arbitrary topological space.

Lemma 6.1. *Let $\mathcal{I}_1$, $\mathcal{I}_2$ be ideals on ω such that $\mathcal{I}_1 \leq_\varphi \mathcal{I}_2$, $\langle f_n :\ n \in \omega\rangle$ being a sequence of functions.*

(1) *If $\langle f_n :\ n \in \omega\rangle$ is $\mathcal{I}_1$-convergent then $\langle f_{\varphi(n)} :\ n \in \omega\rangle$ is $\mathcal{I}_2$-convergent.*
(2) *If $\langle f_n :\ n \in \omega\rangle$ is $\mathcal{I}_1$-quasi-normally convergent then $\langle f_{\varphi(n)} :\ n \in \omega\rangle$ is $\mathcal{I}_2$-quasi-normally convergent.*
(3) *If $\langle f_n :\ n \in \omega\rangle$ is $\mathcal{I}_1$-monotone sequence then $\langle f_{\varphi(n)} :\ n \in \omega\rangle$ is $\mathcal{I}_2$-monotone sequence.*
(4) *If $\langle U_n :\ n \in \omega\rangle$ is $\mathcal{I}_1$-γ-cover then $\langle U_{\varphi(n)} :\ n \in \omega\rangle$ is $\mathcal{I}_2$-γ-cover.*

Proof. (1) Let $\langle f_n :\ n \in \omega\rangle$ be an $\mathcal{I}_1$-convergent sequence of functions on X. We can assume that $f_n \xrightarrow{\mathcal{I}_1} \mathbf{0}$ and all functions are non-negative. Now we consider a sequence $\langle f_{\varphi(n)} :\ n \in \omega\rangle$. For $\varepsilon > 0$ and $x \in X$ the set $\{n :\ f_n(x) \geq \varepsilon\} \in \mathcal{I}_1$. Therefore by definition of φ we have $\varphi^{-1}(\{n :\ f_n(x) \geq \varepsilon\}) \in \mathcal{I}_2$. On the other hand, $\varphi^{-1}(\{n :\ f_n(x) \geq \varepsilon\}) = \left\{l :\ f_{\varphi(l)}(x) \geq \varepsilon\right\}$ and therefore $\left\{l :\ f_{\varphi(l)}(x) \geq \varepsilon\right\} \in \mathcal{I}_2$. Hence we have $f_{\varphi(n)} \xrightarrow{\mathcal{I}_2} \mathbf{0}$.

(2) We follow previous part of proof. We consider an $\mathcal{I}_1$-quasi-normally convergent sequence $\langle f_n :\ n \in \omega\rangle$ of functions on X with control sequence $\langle \varepsilon_n :\ n \in \omega\rangle$ $\mathcal{I}_1$-converging to 0, i.e., $\{n :\ f_n(x) \geq \varepsilon_n\} \in \mathcal{I}_1$ and $\{n :\ \varepsilon_n \geq \varepsilon\} \in \mathcal{I}_1$. Then by definition of φ we have $\left\{n :\ f_{\varphi(n)}(x) \geq \varepsilon_{\varphi(n)}\right\} = \varphi^{-1}(\{n :\ f_n(x) \geq \varepsilon_n\}) \in \mathcal{I}_2$ and similarly for $\langle \varepsilon_n :\ n \in \omega\rangle$. Hence, $f_{\varphi(n)} \xrightarrow{\mathcal{I}_2\mathrm{QN}} \mathbf{0}$ with control sequence $\langle \varepsilon_{\varphi(n)} :\ n \in \omega\rangle$.

(3) Let us assume that $\langle f_n :\ n \in \omega\rangle$ is $\mathcal{I}_1$-monotone sequence. Thus we have $\{n :\ f_n \nleq f_k\} \in \mathcal{I}_1$ for each $k \in \omega$. Consider a sequence $\langle f_{\varphi(n)} :\ n \in \omega\rangle$ and let $m \in \omega$. Then $\varphi^{-1}(\left\{n :\ f_n \nleq f_{\varphi(m)}\right\}) \in \mathcal{I}_2$. However, for any $i \in \left\{l :\ f_{\varphi(l)} \nleq f_{\varphi(m)}\right\}$ we have $\varphi(i) \in \left\{n :\ f_n \nleq f_{\varphi(m)}\right\}$.

(4) The proof of this part was shown in [42]. □

By [42], the property $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$ is preserved in the same way as properties $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$ and $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$ in Proposition 6.2.

Proposition 6.2. *Let $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{J}_1$, $\mathcal{J}_2$ be ideals on ω such that $\mathcal{I}_1 \leq_K \mathcal{I}_2$ and $\mathcal{J}_1 \leq_{KB} \mathcal{J}_2$, X being a topological space.*

(1) *If $\mathrm{C}_p(X)$ has the property $\mathrm{S}_1(\mathcal{I}_2\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}_1\text{-}\Gamma_{\mathbf{0}})$ then $\mathrm{C}_p(X)$ has the property $\mathrm{S}_1(\mathcal{I}_1\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}_2\text{-}\Gamma_{\mathbf{0}})$.*
(2) *If $\mathrm{C}_p(X)$ has the property $\mathrm{S}_1(\mathcal{I}_2\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}_1\text{-}\Gamma_{\mathbf{0}})$ then $\mathrm{C}_p(X)$ has the property $\mathrm{S}_1(\mathcal{I}_1\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}_2\text{-}\Gamma_{\mathbf{0}})$.*
(3) *If $\mathrm{C}_p(X)$ has $\left[\begin{smallmatrix}\mathcal{I}_2\text{-}\Gamma_{\mathbf{0}}\\ s\mathcal{J}_1\mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right]$ then $\mathrm{C}_p(X)$ has $\left[\begin{smallmatrix}\mathcal{I}_1\text{-}\Gamma_{\mathbf{0}}\\ s\mathcal{J}_2\mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right]$.*

Proof. We shall prove (2). (1) is shown similarly. (3) follows by (1) and Theorem 1.1.

Let X be a topological space such that $\mathrm{C}_p(X)$ has the property $\mathrm{S}_1(\mathcal{I}_2\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}_1\text{-}\Gamma_{\mathbf{0}})$. Since $\mathcal{I}_1 \leq_K \mathcal{I}_2$ and $\mathcal{J}_1 \leq_{KB} \mathcal{J}_2$ there are $\varphi \in {}^\omega\omega$ and finite-to-one $\psi \in {}^\omega\omega$ such that $\mathcal{I}_1 \leq_\varphi \mathcal{I}_2$ and $\mathcal{J}_1 \leq_\psi \mathcal{J}_2$.

Consider an $\mathcal{I}_1$-monotone sequence of functions $\langle\langle f_{n,m} :\ m \in \omega\rangle :\ n \in \omega\rangle$ such that $f_{n,m} \xrightarrow{\mathcal{I}_1} \mathbf{0}$ for each $n \in \omega$. By Lemma 6.1 sequences $\langle f_{n,\varphi(m)} :\ m \in \omega\rangle$ are $\mathcal{I}_2$-convergent and $\mathcal{I}_2$-monotone. Now we define non-negative functions

$$h_{n,k} = \begin{cases} \max\left\{f_{j,\varphi(k)} :\ \psi(j) = n\right\} & \text{if } n \in \psi(\omega), \\ 0 & \text{otherwise.}\end{cases}$$

Let $\varepsilon > 0$, $x \in X$, $n \in \omega$ and $i \in \{k :\ h_{n,k}(x) \geq \varepsilon\}$. Then there is j such that $\psi(j) = n$ and $f_{j,\varphi(i)}(x) \geq \varepsilon$, thus $i \in \left\{m :\ f_{j,\varphi(m)}(x) \geq \varepsilon\right\}$. Therefore

$$\{k :\ h_{n,k}(x) \geq \varepsilon\} \subseteq \bigcup_{\psi(j)=n} \left\{k :\ f_{j,\varphi(k)}(x) \geq \varepsilon\right\} \in \mathcal{I}_2$$

and we have $h_{n,k} \xrightarrow{\mathcal{I}_2} \mathbf{0}$ for all $n \in \omega$. Similarly, we also obtain that sequences $\langle h_{n,k} :\ k \in \omega\rangle$ are all $\mathcal{I}_2$-monotone. In addition, by the assumption there is a sequence $\langle k_n :\ n \in \omega\rangle$ such that $h_{n,k_n} \xrightarrow{\mathcal{J}_1} \mathbf{0}$, i.e., $\{n :\ h_{n,k_n}(x) \geq \varepsilon\} \in \mathcal{J}_1$.

Finally, let $\langle m_n :\ n \in \omega\rangle$ be a sequence such that $m_n = \varphi(k_{\psi(n)})$. For $\varepsilon > 0$, $x \in X$ and $f_{j,m_j}(x) \geq \varepsilon$ we can say that $h_{\psi(j),k_{\psi(j)}}(x) \geq \varepsilon$, thus we have

$$\{n :\ f_{n,m_n}(x) \geq \varepsilon\} \subseteq \left\{i :\ h_{\psi(i),k_{\psi(i)}}(x) \geq \varepsilon\right\}.$$

Additionally, by Lemma 6.1 we obtain $h_{\psi(i),k_{\psi(i)}} \xrightarrow{\mathcal{J}_2} \mathbf{0}$ and hence $f_{n,m_n} \xrightarrow{\mathcal{J}_2} \mathbf{0}$. Therefore $\mathrm{C}_p(X)$ has the property $\mathrm{S}_1(\mathcal{I}_1\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}_2\text{-}\Gamma_{\mathbf{0}})$ as well. □

Since an ideal $\mathcal{J}$ has a pseudounion if and only if $\mathcal{J} \leq_{KB}$ Fin and if and only if $\mathcal{J} \leq_K$ Fin we obtain

Corollary 6.3. *Let X be a topological space, $\mathcal{I}, \mathcal{J}$ being ideals with pseudounions.*

(1) $\mathrm{C}_p(X)$ *is an* $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$*-space if and only if* $\mathrm{C}_p(X)$ *is an* $\mathrm{S}_1(\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$*-space.*
(2) $\mathrm{C}_p(X)$ *is an* $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$*-space if and only if* $\mathrm{C}_p(X)$ *is an* $\mathrm{S}_1(\Gamma^{\mathrm{m}}_{\mathbf{0}}, \Gamma_{\mathbf{0}})$*-space.*

7. The connection between coverings and functions

M. Scheepers [37, 7] has shown that the Hurewicz property is a covering characterization of the monotonic sequence selection property and later its connection to quasi-normal convergence appeared as well [10, 7]. Similarly, L. Bukovský [4] and M. Sakai [34] described an $\mathrm{S}_1(\Gamma, \Gamma)$-space in terms of semicontinuous functions. We provide an ideal version of these results.

We say that a topological space X has **$\mathcal{J}$-Hurewicz property**[11] if for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there are finite $\mathcal{V}_n \subset \mathcal{U}_n$, $n \in \omega$ such that for each $x \in X$, $\{n \in \omega : x \notin \bigcup \mathcal{V}_n\} \in \mathcal{J}$. $\mathcal{J}$-Hurewicz property was introduced by P. Das [15]. P. Szewczak and B. Tsaban [43] investigated $\mathcal{J}$-Hurewicz property and they showed

$$\text{Hurewicz} \longrightarrow \mathcal{J}\text{-Hurewicz} \longrightarrow \text{Menger}$$

$\mathcal{J}$-Hurewicz property is a covering characterization of $\mathrm{S}_1(\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$.

Proposition 7.1. *If X is a perfectly normal topological space then the following are equivalent. Moreover, if X is arbitrary topological space then* (a) $\equiv$ (b)*.*

(a) $\mathrm{C}_p(X)$ *has* $\left[{\Gamma^{\mathrm{m}}_{\mathbf{0}} \atop \mathrm{s}\mathcal{J}\mathrm{QN}_{\mathbf{0}}}\right]$.
(b) $\mathrm{C}_p(X)$ *has the property* $\mathrm{S}_1(\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$.
(c) X *possesses a $\mathcal{J}$-Hurewicz property.*

Proof. The implication (c) → (b) has been shown by D. Chandra [13].

(b) → (a) Consider a monotone sequence $\langle f_n : n \in \omega \rangle$ such that $f_n \to \mathbf{0}$. Let $\varepsilon_n \to 0$. The similar way as proof of Theorem 7 in [7] we set $f_{n,m} = \frac{1}{\varepsilon_n} f_m$ for each $n \in \omega$. By the property $\mathrm{S}_1(\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$ there is a sequence $\langle m_n : n \in \omega \rangle$ such that $f_{n,m_n} \xrightarrow{\mathcal{J}} \mathbf{0}$. Thus we have $f_{m_n} \xrightarrow{\mathcal{J}\mathrm{QN}} \mathbf{0}$ with control sequence $\langle \varepsilon_n : n \in \omega \rangle$.

(a) → (c) The proof of this implication is based on proof by M. Scheepers [37], Lemma 3. Let $\mathcal{U}_n = \langle U_{n,m} : m \in \omega \rangle$ be a countable open cover of X for each $n \in \omega$. Since X is perfectly normal, for each open set $U_{n,m}$ we have an increasing sequence $\langle F_{n,m,k} : k \in \omega \rangle$ of closed sets such that $U_{n,m} = \bigcup_{k \in \omega} F_{n,m,k}$. By Urysohn's Lemma there are monotone sequences of continuous functions $\langle f_{n,m,k} : k \in \omega \rangle$ with values in $[0, 1]$ such that

$$f_{n,m,k}(x) = \begin{cases} 1 & x \notin U_{n,m}, \\ 0 & x \in F_{n,m,k}, \end{cases}$$

for each $n, m, k \in \omega$. Next we define functions $h_{n,m}$ such that $h_{n,m}(x) = \left| \prod_{j \leq m} f_{n,j,m}(x) \right|$ which are continuous. Moreover, we know that $\langle h_{n,m} : m \in \omega \rangle$ are monotone sequences such that $h_{n,m} \to \mathbf{0}$.

Let $\delta_n = 1$ and $\varepsilon_n = 2^{-n}$ for $n \in \omega$. By Lemma 4.1 there is sequence of continuous monotone functions $\langle g_m : m \in \omega \rangle$ converging to zero such that 4.2 holds. Note that $k_n = n$. Since $\mathrm{C}_p(X)$ is a $[\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathrm{s}\mathcal{J}\mathrm{QN}_{\mathbf{0}}]$-space there exists $\langle m_n : n \in \omega \rangle$ such that $g_{m_n} \xrightarrow{\mathrm{s}\mathcal{J}\mathrm{QN}} \mathbf{0}$.

[11] P. Szewczak and B. Tsaban [43, 44] say $\mathcal{J}$-Menger property instead of $\mathcal{J}$-Hurewicz property.

Finally, let $\mathcal{V}_n = \{U_{n,j} : j \le m_n\}$ for all $n \in \omega$. If we consider any $x \in X$ and $n \in \omega$ such that $g_{m_n}(x) < 2^{-n}$ then $|\prod_{j \le m_n} f_{n,j,m_n}(x)| < \frac{1}{2^n} \le 1$. Therefore there exists $j \le m_n$ such that $x \in U_{n,j}$ and we obtain $\{n : x \notin \bigcup \mathcal{V}_n\} \subseteq \{n : g_{m_n}(x) \ge 2^{-n}\} \in \mathcal{J}$. Thus X possesses a $\mathcal{J}$-Hurewicz property. $\square$

Hence, let us summarize basic relations of $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$-space for $\mathrm{C}_p(X)$.

$$\begin{array}{ccccc}
\text{Hurewicz} \equiv \mathrm{S}_1(\Gamma^{\mathrm{m}}_{\mathbf{0}}, \Gamma_{\mathbf{0}}) & \longrightarrow & \mathrm{S}_1(\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}}) & \longrightarrow & \text{Menger} \\
\uparrow & & \uparrow & & \\
\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \Gamma_{\mathbf{0}}) & \longrightarrow & \mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}}) & & \\
\uparrow & & & & \\
\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}}) & & & &
\end{array}$$

Diagram 3: Monotonic selection principles for functions.

An $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-space can be characterized in terms of upper semicontinuous functions, similarly to an $\mathrm{S}_1(\Gamma, \Gamma)$-space.

Proposition 7.2. *Let $\mathcal{I}, \mathcal{J}$ be ideals on ω. Then the following statements are equivalent.*

(a) *X is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-space.*
(b) *$\mathrm{USC}_p(X)$ has the property $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$.*
(c) *$\mathrm{USC}_p(X)$ has $\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}} \\ \mathrm{s}\mathcal{J}\mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right]$.*

Proof. The proof of implication (b) → (c) is similar to proof of implication (b) → (a) of Bukovský–Das–Šupina's Theorem 1.1, see [6].

(a) → (b) We will follow the proof of Theorem 13 in [4]. Let $\langle f_{n.m} : m \in \omega\rangle$ be upper semicontinuous functions such that $f_{n,m} \xrightarrow{\mathcal{I}} \mathbf{0}$ for all $n \in \omega$, $\langle x_m : m \in \omega\rangle$ being an arbitrary sequence of distinct points of X. Define open sets $U_{n,m}$ as

$$U_{n,m} = \{x \in X : 2^n f_{n,m}(x) < 1\} \setminus \{x_m\}.$$

Sequence $\langle U_{n,m} : m \in \omega\rangle$ is an $\mathcal{I}$-γ-cover. Since X is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-space there exists sequence $\langle m_n : n \in \omega\rangle$ such that $\langle U_{n,m_n} : n \in \omega\rangle$ is a $\mathcal{J}$-γ-cover on X. Let $\varepsilon > 0$. Then there is n_0 such that $\varepsilon \ge 2^{-n_0}$ and

$$\{n : f_{n,m_n}(x) \ge \varepsilon\} \subseteq n_0 \cup \{n : x \notin U_{n,m_n}\} \in \mathcal{J}.$$

Hence $f_{n,m_n} \xrightarrow{\mathcal{J}} \mathbf{0}$.

(c) → (a) We will pursue the method of proof of Theorem 2.2 in [34]. Let $\langle U_{n,m} : m \in \omega\rangle$ be open $\mathcal{I}$-γ-cover of X for each $n \in \omega$. We put $V_{n,m} = U_{0,m} \cap \cdots \cap U_{n,m}$ for all $n, m \in \omega$. Note that $\langle V_{n,m} : m \in \omega\rangle$ is $\mathcal{I}$-γ-cover of X. Now we define functions f_m as

$$f_m(x) = \begin{cases} 1 & \text{if } x \in X \setminus V_{0,m}, \\ \frac{1}{k+2} & \text{if } x \in V_{k,m} \setminus V_{k+1,m}, \\ 0 & \text{otherwise.} \end{cases}$$

Every f_m is upper semicontinuous. Let $\varepsilon > 0$ and $x \in X$. By definition of functions f_m one can easily see that $\{m : f_m(x) \ge \varepsilon\} \subseteq \{m : x \notin V_{n,m}\} \in \mathcal{I}$. Therefore $f_m \xrightarrow{\mathcal{I}} \mathbf{0}$.

Since $\mathrm{USC}_p(X)$ is an $[\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathrm{s}\mathcal{J}\mathrm{QN}_{\mathbf{0}}]$-space there exists sequence $\langle m_n : n \in \omega\rangle$ such that $f_{m_n} \xrightarrow{\mathrm{s}\mathcal{J}\mathrm{QN}} \mathbf{0}$. By Theorem 4.2 we can consider control sequence $\langle \delta_n : n \in \omega\rangle$ such that $\delta_n = \frac{1}{n+1}$ for each $n \in \omega$. Then

$$\{n : x \notin V_{n,m_n}\} \subseteq \left\{n : f_{m_n}(x) \ge \frac{1}{n+1}\right\} \in \mathcal{J},$$

hence $\langle V_{n,m_n}:\ n\in\omega\rangle$ is a $\mathcal{J}$-γ-cover of X. $\square$

8. Ideals and their cardinal invariants

We shall investigate combinatorial properties of cardinal invariant $\lambda(\mathcal{I},\mathcal{J})$ introduced in [42]. Let $\mathcal{A}\subseteq\mathcal{P}(\omega)$. Similarly to slaloms in [2], a sequence $s\in{}^{\omega}\mathcal{A}$ will be called an **$\mathcal{A}$-slalom**. However, slalom in the sense of A. Blass [2] does not coincide with our Fin-slalom since we do not control the size of n-th coordinate of Fin-slalom. Hence, a Fin-slalom s is called a slalom if $|s(n)|=n$ for each n. We say that a function $\varphi\in{}^{\omega}\omega$ **$\mathcal{J}$-goes** through $\mathcal{A}$-slalom s if $\{n:\ \varphi(n)\in s(n)\}\in\mathcal{J}^d$, i.e., $\{n:\ \varphi(n)\in\omega\setminus s(n)\}\in\mathcal{J}$. We say that φ goes through $\mathcal{I}$-slalom instead of φ Fin-goes through $\mathcal{I}$-slalom.

Let us recall a standard result by T. Bartoszynski [3] regarding slaloms. He has shown that

$$\mathrm{add}(\mathcal{N})=\min\{|\mathcal{R}|:\ \mathcal{R}\subseteq{}^{\omega}\omega,\ (\forall\text{ slalom } s)(\exists\varphi\in\mathcal{R})\ \neg(\varphi\text{ goes through } s)\}.$$

If one substitutes slalom by Fin-slalom then A. Blass [2] mentions that

$$\mathfrak{b}=\min\{|\mathcal{R}|:\ \mathcal{R}\subseteq{}^{\omega}\omega,\ (\forall\text{Fin-slalom } s)(\exists\varphi\in\mathcal{R})\ \neg(\varphi\text{ goes through } s)\}.$$

Thus the bounding number $\mathfrak{b}$ is the minimal cardinality of a family $\mathcal{R}$ of functions from ${}^{\omega}\omega$ such that there is no single Fin-slalom with all functions from $\mathcal{R}$ going through this Fin-slalom. On the contrary, we are interested in the minimal cardinality of a family of $\mathcal{I}^d$-slaloms such that there is no single function which $\mathcal{J}$-goes through all of them.

The above mentioned invariant is called $\lambda(\mathcal{I},\mathcal{J})$ in [42] and we restate in more formal way its definition. We say that a function $\varphi\in{}^{\omega}\omega$ **$\mathcal{J}$-omits** $\mathcal{A}$-slalom s if $\{n:\ \varphi(n)\in s(n)\}\in\mathcal{J}$.

$$\begin{aligned}\lambda(\mathcal{I},\mathcal{J})&=\min\left\{|\mathcal{R}|:\ \mathcal{R}\text{ contains }\mathcal{I}^d\text{-slaloms, }(\forall\varphi\in{}^{\omega}\omega)(\exists s\in\mathcal{R})\ \neg(\varphi\ \mathcal{J}\text{-goes through } s)\right\}\\&=\min\left\{\kappa:\ (\exists\mathcal{R}\subseteq{}^{\omega}\mathcal{I}^d)(|\mathcal{R}|=\kappa\ \wedge\ (\forall\varphi\in{}^{\omega}\omega)(\exists s\in\mathcal{R})\ \{n:\varphi(n)\notin s(n)\}\in\mathcal{J}^+)\right\}\\&=\min\{|\mathcal{R}|:\ \mathcal{R}\text{ contains }\mathcal{I}\text{-slaloms, }\ (\forall\varphi\in{}^{\omega}\omega)(\exists s\in\mathcal{R})\ \neg(\varphi\ \mathcal{J}\text{-omits } s)\}\\&=\min\left\{\kappa:\ (\exists\mathcal{R}\subseteq{}^{\omega}\mathcal{I})(|\mathcal{R}|=\kappa\ \wedge\ (\forall\varphi\in{}^{\omega}\omega)(\exists s\in\mathcal{R})\ \{n:\varphi(n)\in s(n)\}\in\mathcal{J}^+)\right\}.\end{aligned}$$

Hence, if $\mathcal{R}$ contains $\mathcal{I}^d$-slaloms and $|\mathcal{R}|<\lambda(\mathcal{I},\mathcal{J})$ then there is a function $\varphi\in{}^{\omega}\omega$ which $\mathcal{J}$-goes through every $s\in\mathcal{R}$. If $\mathcal{R}$ contains $\mathcal{I}$-slaloms and $|\mathcal{R}|<\lambda(\mathcal{I},\mathcal{J})$ then there is a function $\varphi\in{}^{\omega}\omega$ which $\mathcal{J}$-omits every $s\in\mathcal{R}$. By [42], $\lambda(\mathrm{Fin},\mathcal{J})=\mathfrak{b}_{\mathcal{J}}$, if $\mathcal{I}_1\leq_K\mathcal{I}_2$ and $\mathcal{J}_1\leq_{KB}\mathcal{J}_2$ then $\lambda(\mathcal{I}_2,\mathcal{J}_1)\leq\lambda(\mathcal{I}_1,\mathcal{J}_2)$ and if $\mathcal{I},\mathcal{J}$ are not tall ideals then $\lambda(\mathcal{I},\mathcal{J})=\mathfrak{b}$, see [6, 42].[12]

We recall invariant $\mathtt{cov}^*(\mathcal{I})$ introduced in [26] for tall ideal $\mathcal{I}$. Let $\mathcal{I}$ be a tall ideal then

$$\begin{aligned}\mathtt{cov}^*(\mathcal{I})&=\min\{|\mathcal{A}|:\ \mathcal{A}\subseteq\mathcal{I}\wedge(\forall S\in[\omega]^{\omega})(\exists A\in\mathcal{A})\ |S\cap A|=\omega\}\\&=\min\{|\mathcal{A}|:\ \mathcal{A}\subseteq\mathcal{I}\wedge\mathcal{A}\text{ does not have a pseudounion}\}\\&=\min\left\{|\mathcal{A}|:\ \mathcal{A}\subseteq\mathcal{I}^d\wedge\mathcal{A}\text{ does not have a pseudointersection}\right\}.\end{aligned}$$

Actually, $\mathfrak{p}=\min\{\kappa:\ (\exists\text{ an ideal }\mathcal{I})\ \mathtt{cov}^*(\mathcal{I})=\kappa\}$. If $\mathcal{I}_1\leq_K\mathcal{I}_2$ then $\mathtt{cov}^*(\mathcal{I}_2)\leq\mathtt{cov}^*(\mathcal{I}_1)$, see [27].

M. Repický [32] introduced cardinal invariant $\mathfrak{k}_{\mathcal{I},\mathcal{J}}$. If $\mathcal{I}\leq_K\mathcal{J}$ then $\mathfrak{k}_{\mathcal{I},\mathcal{J}}=\infty$ and if $\mathcal{I}\not\leq_K\mathcal{J}$ then

$$\begin{aligned}\mathfrak{k}_{\mathcal{I},\mathcal{J}}&=\min\{|\mathcal{A}|:\ \mathcal{A}\subseteq\mathcal{I}\wedge\mathcal{A}\not\leq_K\mathcal{J}\}\\&=\min\left\{|\mathcal{A}|:\ \mathcal{A}\subseteq\mathcal{I}\wedge(\forall\varphi\in{}^{\omega}\omega)(\exists A\in\mathcal{A})\ \{n:\varphi(n)\in A\}\in\mathcal{J}^+\right\}\end{aligned}$$

Hence, if $\mathcal{I}$ is tall then $\mathfrak{k}_{\mathcal{I},\mathrm{Fin}}=\mathtt{cov}^*(\mathcal{I})$. If $\mathcal{I}\not\leq_K\mathcal{J}$ then $\mathfrak{p}\leq\mathfrak{k}_{\mathcal{I},\mathcal{J}}\leq\mathfrak{c}$.

We are now ready to prove the main result of this section.

[12] It follows by relation of $\lambda(\mathcal{I},\mathcal{J})$ to covering property $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma,\mathcal{J}\text{-}\Gamma)$ which is equivalent to $\mathrm{S}_1(\Gamma,\Gamma)$ for not tall $\mathcal{I},\mathcal{J}$, see our Sections 7 and 9.

Theorem 8.1.

(1) *If* $\mathcal{I} \not\leq_K \mathcal{J}$ *then* $\lambda(\mathcal{I},\mathcal{J}) \leq \min\{\mathfrak{k}_{\mathcal{I},\mathcal{J}}, \mathfrak{b}_{\mathcal{J}}\}$.
(2) *If* $\mathcal{I} \not\leq_K \mathcal{J}$ *and* $\mathcal{J} \leq_K \mathcal{I}$ *then* $\lambda(\mathcal{I},\mathcal{J}) = \min\{\mathfrak{k}_{\mathcal{I},\mathcal{J}}, \lambda(\mathcal{J},\mathcal{J})\}$.
(3) *If* $\mathcal{I}$ *is tall then* $\lambda(\mathcal{I},\mathrm{Fin}) = \min\{\mathtt{cov}^*(\mathcal{I}), \mathfrak{b}\}$.

Proof. (3) is a particular case of (2).

(1) We begin with the inequality $\lambda(\mathcal{I},\mathcal{J}) \leq \mathfrak{k}_{\mathcal{I},\mathcal{J}}$. The inequality $\lambda(\mathcal{I},\mathcal{J}) \leq \mathfrak{b}_{\mathcal{J}}$ is due to [42]. Suppose that $\mathcal{A} \subseteq \mathcal{I}$, $|\mathcal{A}| = \mathfrak{k}_{\mathcal{I},\mathcal{J}}$ and $\mathcal{A} \not\leq_K \mathcal{J}$, i.e.,

$$(\forall \varphi \in {}^{\omega}\omega)(\exists A \in \mathcal{A})\ \{n :\ \varphi(n) \in A\} \in \mathcal{J}^+.$$

Let $\mathcal{R}$ be a family of constant sequences of elements from $\mathcal{A}$. Then $|\mathcal{R}| = \mathfrak{k}_{\mathcal{I},\mathcal{J}}$ and

$$(\forall \varphi \in {}^{\omega}\omega)(\exists s \in \mathcal{R})\ \{n :\ \varphi(n) \in s(n)\} \in \mathcal{J}^+.$$

(2) If $\mathcal{J} \leq_K \mathcal{I}$ then $\lambda(\mathcal{I},\mathcal{J}) \leq \lambda(\mathcal{J},\mathcal{J})$ by [42]. To prove the inequality $\min\{\mathfrak{k}_{\mathcal{I},\mathcal{J}}, \lambda(\mathcal{J},\mathcal{J})\} \leq \lambda(\mathcal{I},\mathcal{J})$ let us take $\omega \leq \kappa < \min\{\mathfrak{k}_{\mathcal{I},\mathcal{J}}, \lambda(\mathcal{J},\mathcal{J})\}$, $\mathcal{R}$ being a family of cardinality κ containing $\mathcal{I}$-slaloms. Consider a family

$$\mathcal{A} = \{A \subseteq \omega :\ (\exists s \in \mathcal{R})(\exists n \in \omega)\ A = s(n)\} \subseteq \mathcal{I}.$$

Since $|\mathcal{A}| \leq \omega \cdot \kappa = \kappa$ there is $\psi \in {}^{\omega}\omega$ such that $\psi^{-1}(A) \in \mathcal{J}$ for all $A \in \mathcal{A}$. We define a family $\mathcal{P}$ of $\mathcal{J}$-slaloms as

$$\mathcal{P} = \big\{p :\ (\exists s \in \mathcal{R})\ p(n) = \psi^{-1}(s(n))\big\} \subseteq {}^{\omega}\mathcal{J}.$$

In addition, $|\mathcal{P}| \leq |\mathcal{R}|$. Therefore there exists function $\varphi \in {}^{\omega}\omega$ such that for each $p \in \mathcal{P}$ we have $\{n :\ \varphi(n) \in p(n)\} \in \mathcal{J}$. Let us take arbitrary $s \in \mathcal{R}$. Then

$$\{n :\ \psi(\varphi(n)) \in s(n)\} = \big\{n :\ \varphi(n) \in \psi^{-1}(s(n))\big\} \in \mathcal{J}.$$

Therefore $\kappa < \lambda(\mathcal{I},\mathcal{J})$. □

Adding an upper bound for $\lambda(\mathcal{I},\mathcal{J})$ from [42], we have

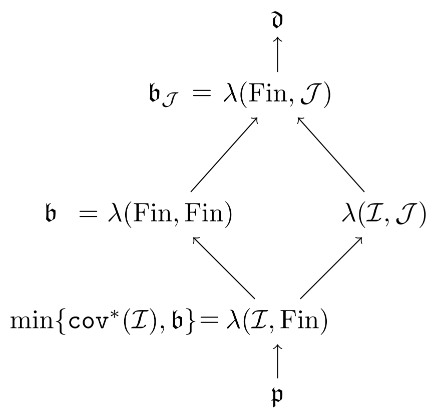

Diagram 4: Cardinal $\lambda(\mathcal{I},\mathcal{J})$.

$\mathcal{J}$ is a **weak** P($\mathcal{I}$)**-ideal** [42] if for any family $\{A_n :\ n \in \omega\} \subseteq \mathcal{J}$ there is $A \in \mathcal{J}^+$ such that $A \cap A_n \in \mathcal{I}$ for any $n \in \omega$. Note that we will write a weak P-ideal instead of a weak P(Fin)-ideal. We can also describe weak P-ideals by Katetov ordering. An ideal $\mathcal{I}$ is not a weak P-ideal if and only if $\mathrm{Fin}\times\mathrm{Fin} \leq_K \mathcal{I}$ [22, 29, 30].

Corollary 8.2.

(1) *If* $\mathtt{cov}^*(\mathcal{I}) \geq \mathfrak{b}$ *then* $\mathfrak{b} \leq \lambda(\mathcal{I},\mathcal{J}) \leq \mathfrak{b}_{\mathcal{J}}$.
(2) *If* $\mathtt{cov}^*(\mathcal{I}) \geq \mathfrak{b}$ *and* $\mathcal{J}$ *has Baire property then* $\lambda(\mathcal{I},\mathcal{J}) = \mathfrak{b}$.
(3) *If* $\mathtt{cov}^*(\mathcal{I}) \leq \mathfrak{b}$ *then* $\lambda(\mathcal{I},\mathrm{Fin}) = \mathtt{cov}^*(\mathcal{I})$.
(4) *If* $\mathtt{cov}^*(\mathcal{I}) \geq \mathfrak{b}$ *then* $\lambda(\mathcal{I},\mathrm{Fin}) = \mathfrak{b}$.
(5) *There is an ultrafilter* $\mathcal{U}$ *such that* $\mathfrak{p} = \lambda(\mathcal{U},\mathrm{Fin})$.
(6) $\mathfrak{p} = \min\{\kappa :\ (\exists \text{ an ideal } \mathcal{I})\ \lambda(\mathcal{I},\mathrm{Fin}) = \kappa\}$.
(7) *If* $\mathcal{I}$ *is not a weak* P*-ideal then* $\lambda(\mathcal{I},\mathrm{Fin}) = \mathtt{cov}^*(\mathcal{I})$.

Proof. (1) follows directly by Theorem 8.1. (2) By B. Farkas and L. Soukup [19] and Mathias, Jalali–Naini and Talagrand Theorem[13] we have $\mathfrak{b}_{\mathcal{J}} = \mathfrak{b}$ for ideal $\mathcal{J}$ with Baire property. (3) - (4) Direct consequence of Theorem 8.1. (5) - (6) Since $\mathfrak{p} = \min\{\kappa :\ (\exists \text{ a filter } \mathcal{F})\ \mathtt{cov}^*(\mathcal{F}) = \kappa\}$ there is an ultrafilter $\mathcal{U}$ such that $\mathfrak{p} = \mathtt{cov}^*(\mathcal{U})$. (7) If $\mathcal{I}$ is not a weak P-ideal then $\mathrm{Fin}\times\mathrm{Fin} \leq_K \mathcal{I}$. Thus, see e.g. [27], we know that $\mathtt{cov}^*(\mathcal{I}) \leq \mathtt{cov}^*(\mathrm{Fin}\times\mathrm{Fin}) = \mathfrak{b}$. Therefore by (3) we have $\lambda(\mathcal{I},\mathrm{Fin}) = \mathtt{cov}^*(\mathcal{I})$. □

R.M. Canjar, P. Nyikos and J. Ketonen [11] showed that if an ultrafilter $\mathcal{U}$ is not a P-point then $\mathtt{cov}^*(\mathcal{U}) \leq \mathfrak{b}$. Although it follows by the fact that $\mathtt{cov}^*(\mathcal{I}) \leq \mathtt{cov}^*(\mathrm{Fin}\times\mathrm{Fin}) = \mathfrak{b}$ for $\mathrm{Fin}\times\mathrm{Fin} \leq_K \mathcal{I}$, it is evident after one has seen part (2) of our Corollary 8.2.

Using notation from [27], $\mathcal{Z}$, nwd, $\mathcal{ED}$, $\mathcal{R}$, $\mathcal{S}$ and conv denote an asymptotic density zero ideal, nowhere dense ideal, eventually different ideal, random graph ideal, Solecki's ideal and ideal on $\mathbb{Q} \cap [0,1]$ generated by sequences convergent in $[0,1]$, respectively. All these ideals are Borel and one can find more about their $\mathtt{cov}^*(\mathcal{I})$ characteristics in [27]. Thus by part (7) of Corollary 8.2 we have

$$\lambda(\mathcal{I},\mathcal{J}) = \mathfrak{b}$$

for $\mathcal{I} \in \{\mathrm{Fin}, \mathrm{Fin}\times\mathrm{Fin}, \mathcal{R}, \mathrm{conv}, \mathcal{ED}\}$ and $\mathcal{J} \in \{\mathrm{Fin}, \mathrm{Fin}\times\mathrm{Fin}, \emptyset\times\mathrm{Fin}, \mathrm{Fin}\times\emptyset, \mathcal{S}, \mathcal{R}, \mathrm{conv}, \mathcal{ED}, \mathcal{Z}, \mathrm{nwd}\}$.

Moreover, if $\mathcal{I}$ is a Borel ideal on ω and $\mathtt{cov}^*(\mathcal{I}) \geq \mathrm{cov}(\mathcal{M})$ then $\lambda(\mathcal{I},\mathcal{J}) \geq \lambda(\mathrm{nwd},\mathcal{J})$, see [27]. Possible values of $\lambda(\mathcal{I},\mathcal{J})$ are discussed in Section 10 and values of $\lambda(\mathcal{I},\mathrm{Fin})$ for well-known ideals are summarized in Table 1.

Finally, we may also consider version of $\lambda(\mathcal{I},\mathrm{Fin})$ for instance for families $\mathcal{A} \subseteq [\omega]^\omega$ such that $\mathrm{Fin} \subseteq \mathcal{A}$. Namely, for the value

$$\lambda(\mathcal{A},\mathrm{Fin}) = \min\{|\mathcal{R}| :\ \mathcal{R} \subset {}^\omega\mathcal{A},\ (\forall \varphi \in {}^\omega\omega)(\exists s \in \mathcal{R})\ |\{n : \varphi(n) \in s(n)\}| = \omega)\}$$

we have $\lambda(\mathcal{A},\mathrm{Fin}) = \min\left\{k :\ \{A_0,\dots A_k\} \subseteq \mathcal{A} \text{ and } \bigcup_{i=0}^k A_i = \omega\right\}$ for $\mathcal{A}$ not having finite union property, e.g., $\lambda(\mathcal{P}(\omega),\mathrm{Fin}) = 1$, or there is an ideal $\mathcal{I}$ on ω such that $\mathcal{A} \subseteq \mathcal{I}$ and then $\mathfrak{p} \leq \lambda(\mathcal{I},\mathrm{Fin}) \leq \lambda(\mathcal{A},\mathrm{Fin}) \leq \mathfrak{b}$. Indeed, to prove the last inequality let $\{A_i :\ i \in \omega\}$ be a family of sets from $\mathcal{A}^d$ with empty intersection, $\{f_\alpha :\ \alpha \in \kappa\}$ being a family of functions from ${}^\omega\omega$, $\kappa < \lambda(\mathcal{A},\mathrm{Fin})$. We set $s_\alpha(n)$ to be A_i such that $f_\alpha(n) < \min A_i$ which exists by Pigeonhole principle. There is a function φ which goes through every $\mathcal{A}^d$-slalom s_α. One can see that φ is a bound for $\{f_\alpha :\ \alpha \in \kappa\}$. Hence, $\lambda(\mathcal{A},\mathrm{Fin}) \leq \mathfrak{b}$.

9. Critical cardinality

Cardinal invariant $\lambda(\mathcal{I},\mathcal{J})$ was in [42] introduced such that $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma,\mathcal{J}\text{-}\Gamma)) = \lambda(\mathcal{I},\mathcal{J})$.[14] Hence, by (5.1) we have $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}},\mathcal{J}\text{-}\Gamma_{\mathbf{0}})) \geq \lambda(\mathcal{I},\mathcal{J})$ and we will show that the reversed inequality holds as well, see Corollary 9.4. The following can be considered as an ideal analogue of a result by M. Sakai [35].

[13] $\mathcal{J}$ has Baire property if and only if $\mathcal{J}$ is meager if and only if $\mathrm{Fin} \leq_{RB} \mathcal{J}$ (see, e.g., [18]).
[14] $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma,\mathcal{J}\text{-}\Gamma)$-space) denotes the minimal cardinality of a perfectly normal space which is not an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma,\mathcal{J}\text{-}\Gamma)$-space, $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}},\mathcal{J}\text{-}\Gamma_{\mathbf{0}}))$ denotes the minimal cardinality of a perfectly normal space X such that $\mathrm{C}_p(X)$ is not an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}},\mathcal{J}\text{-}\Gamma_{\mathbf{0}})$-space etc.

Theorem 9.1. *Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω, D being a discrete topological space. Then the following statements are equivalent.*

(a) *D is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-space.*
(b) *$\mathrm{C}_p(D)$ has $\left[{\mathcal{I}\text{-}\Gamma_{\mathbf{0}} \atop s\mathcal{J}\mathrm{QN}_{\mathbf{0}}}\right]$.*
(c) *$\mathrm{C}_p(D)$ has the property $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$*
(d) *$\mathrm{C}_p(D)$ has the property $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$.*
(e) *$|D| < \lambda(\mathcal{I}, \mathcal{J})$.*

Proof. (b) $\equiv$ (c) by Theorem 1.1 and (a) $\equiv$ (e) by Theorem 7.3 in [42]. The implication (a) $\to$ (b) is by (5.1) and (c) $\to$ (d) is trivial.

$\neg$(e) $\to$ $\neg$(d). Let us take a family $\mathcal{R} = \{s_x : x \in D\} \subseteq {}^{\omega}\mathcal{I}$ of $\mathcal{I}$-slaloms. So $|\mathcal{R}| = |D|$. Now we define functions $f_{n,m}$ such that

$$f_{n,m}(x) = \begin{cases} 1 & m \in s_x(n), \\ 0 & \text{otherwise.} \end{cases}$$

Note that for any $k \in \omega$ we have $\{m : f_{n,m}(x) \nleq f_{n,k}(x)\} \subseteq s_x(n) \in \mathcal{I}$. Therefore sequences $\langle f_{n,m} : m \in \omega\rangle$ are $\mathcal{I}$-monotone. Take $\varepsilon < 1$. Then for each $n \in \omega$ we have $\{m : f_{n,m}(x) \geq \varepsilon\} \subseteq s_x(n) \in \mathcal{I}$, so $f_{n,m} \xrightarrow{\mathcal{I}} \mathbf{0}$. For any $\varphi \in {}^{\omega}\omega$ there is $x \in D$ such that $\left\{n : f_{n,\varphi(n)}(x) \geq \varepsilon\right\} \supseteq \{n : \varphi(n) \in s_x(n)\} \in \mathcal{J}^+$, thus $\mathrm{C}_p(D)$ does not have $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$. ☐

Lemma 9.2 and Theorem 9.3 are partially covered in Theorem 3.2 by [28] and in Lemma 4.5 by [32].

Lemma 9.2 (A. Kwela–M. Repický). *Let $\mathcal{I}$ be a tall ideal on ω. If $|X| < \mathtt{cov}^*(\mathcal{I})$ then X has $\left[{\mathcal{I}\text{-}\Gamma \atop \Gamma}\right]$ and $\mathrm{C}_p(X)$ has both, $\left[{\mathcal{I}\mathrm{QN}_{\mathbf{0}} \atop \mathrm{QN}_{\mathbf{0}}}\right]$ and $\left[{\mathcal{I}\text{-}\Gamma_{\mathbf{0}} \atop \Gamma_{\mathbf{0}}}\right]$.*

Proof. A. Kwela [28] proved that if $|X| < \mathtt{cov}^*(\mathcal{I})$ then $\mathrm{C}_p(X)$ is $[\mathcal{I}\mathrm{QN}_{\mathbf{0}}, \mathrm{QN}_{\mathbf{0}}]$-space. To prove that X is $[\mathcal{I}\text{-}\Gamma, \Gamma]$-space and $\mathrm{C}_p(X)$ is $[\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}}]$-space, let us consider continuous $f_n \xrightarrow{\mathcal{I}} \mathbf{0}$ and an open $\mathcal{I}$-γ-cover $\langle U_n : n \in \omega\rangle$. One can easily see that $f_{n_m} \to \mathbf{0}$ and $\langle U_{n_m} : m \in \omega\rangle$ is a γ-cover where $\{n_m : m \in \omega\}$ is a pseudointersection of a family of sets

$$A^m_x = \left\{n : f_n(x) < 2^{-m}\right\}, \ A_x = \{n : x \in U_n\}, \ x \in X, \ m \in \omega.$$ ☐

Theorem 9.3 (A. Kwela–M. Repický). *Let D be a discrete topological space. Then the following statements are equivalent.*

(a) *$|D| < \mathtt{cov}^*(\mathcal{I})$.*
(b) *$\mathrm{C}_p(D)$ has $\left[{\mathcal{I}\mathrm{QN}_{\mathbf{0}} \atop \mathrm{QN}_{\mathbf{0}}}\right]$.*
(c) *$\mathrm{C}_p(D)$ has $\left[{\mathcal{I}\text{-}\Gamma_{\mathbf{0}} \atop \Gamma_{\mathbf{0}}}\right]$.*
(d) *D has the property $\left[{\mathcal{I}\text{-}\Gamma \atop \Gamma}\right]$.*

Proof. The equivalence between (a) and (b) was proven by A.Kwela in [28] and the implication (a) $\to$ (c) follows by Lemma 9.2.

To prove the implication (c) $\to$ (d), let $\langle U_n : n \in \omega\rangle$ be an $\mathcal{I}$-γ-cover of D and we define

$$f_n(x) = \begin{cases} 0 & x \in U_n, \\ 1 & \text{otherwise.} \end{cases}$$

We have $f_n \xrightarrow{\mathcal{I}} \mathbf{0}$ and there is $\langle n_m : m \in \omega\rangle$ such that $f_{n_m} \to \mathbf{0}$. One can see that $\langle U_{n_m} : n \in \omega\rangle$ is a γ-cover of D.

$\neg$(a) $\to$ $\neg$(d) Let $\mathcal{A} \subseteq \mathcal{I}$ be a family such that $|\mathcal{A}| = |D| \geq \mathtt{cov}^*(\mathcal{I})$ and $\mathcal{A}$ does not have a pseudounion. Define $U_n = \{A \in \mathcal{A} : n \notin A\}$. $\langle U_n : n \in \omega\rangle$ is an $\mathcal{I}$-γ-cover of a family $\mathcal{A}$. Let us consider $S \in [\omega]^\omega$. Since $\mathcal{A}$ does not have a pseudounion there exists $A \in \mathcal{A}$ such that $\{n \in S : A \notin U_n\} = \{n \in S : n \in A\} = S \cap A \in [\omega]^\omega$. □

It has been shown in [42] that any topological space of cardinality less than $\lambda(\mathcal{I}, \mathcal{J})$ is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-space. Consequently, we obtain

Corollary 9.4. *Let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(\omega)$ be ideals.*

(1) $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})) = \mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})) = \mathrm{non}(\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}}\\ s\mathcal{J}\mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right]) = \lambda(\mathcal{I}, \mathcal{J})$.

(2) $\mathrm{non}(\mathrm{S}_1(\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})) = \mathrm{non}(\mathrm{S}_1(\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})) = \mathrm{non}(\left[\begin{smallmatrix}\Gamma_{\mathbf{0}}\\ s\mathcal{J}\mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right]) = \mathfrak{b}_{\mathcal{J}}$.

If $\mathcal{I}$ is tall then

(3) $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma)) = \mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})) = \mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{m}_{\mathbf{0}}, \Gamma_{\mathbf{0}})) = \mathrm{non}(\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}}\\ \mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right]) = \min\{\mathtt{cov}^*(\mathcal{I}), \mathfrak{b}\}$.

(4) **(A. Kwela–M. Repický)** $\mathrm{non}(\left[\begin{smallmatrix}\mathcal{I}\mathrm{QN}_{\mathbf{0}}\\ \mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right]) = \mathrm{non}(\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}}\\ \Gamma_{\mathbf{0}}\end{smallmatrix}\right]) = \mathrm{non}(\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma\\ \Gamma\end{smallmatrix}\right]) = \mathtt{cov}^*(\mathcal{I})$.

Moreover, according to Proposition 7.1 about relation between $\mathcal{J}$-Hurewicz property and the property $\mathrm{S}_1(\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$ and according to part (2) of Corollary 9.4 it is obvious that $\mathrm{non}(\mathcal{J}\text{-Hurewicz}) = \mathfrak{b}_{\mathcal{J}}$ which was proved by S.G. da Silva [40]. In addition to Corollary 9.4, (3), stating $\mathrm{non}([\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathrm{QN}_{\mathbf{0}}]\text{-space}) = \min\{\mathtt{cov}^*(\mathcal{I}), \mathfrak{b}\}$, A. Kwela [28] introduced cardinal invariant $\kappa(\mathcal{J})$ for any weak P-ideal $\mathcal{J}$ such that $\mathrm{non}([\Gamma_{\mathbf{0}}, \mathcal{J}\mathrm{QN}_{\mathbf{0}}]\text{-space}) = \kappa(\mathcal{J})$. He has shown $\mathfrak{p} \leq \kappa(\mathcal{J}) \leq \mathfrak{d}$ for every P-ideal $\mathcal{J}$ and $\kappa(\mathcal{J}) = \mathfrak{b}$ for any F_σ ideal $\mathcal{J}$. $\mathrm{non}([\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\mathrm{QN}_{\mathbf{0}}]\text{-space})$ in general is investigated by M. Repický [32].

M. Staniszewski [41] introduced and investigated cardinal invariant $\mathbf{b}(\mathcal{I}, \mathcal{J}, \mathcal{K})$ which is naturally related to an $[\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, (\mathcal{J}, \mathrm{Fin})\text{-e}]$-space. For instance, it follows that $\mathrm{non}([\mathrm{Fin}, (\mathcal{J}, \mathrm{Fin})\text{-e}]\text{-space}) \geq \mathbf{b}(\mathcal{I}, \mathrm{Fin}, \mathrm{Fin})$.

We shall prove that subsets of the Baire space which are $\left[\begin{smallmatrix}\Gamma_{\mathbf{0}}\\ s\mathcal{J}\mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right]$ or which have either $\mathrm{S}_1(\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$ or $\mathrm{S}_1(\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$ are all bounded in $({}^\omega\omega, \leq^{\mathcal{J}})$.

Proposition 9.5.

(1) *Let $X \subseteq {}^\omega\omega$. If $\mathrm{C}_p(X)$ has the property $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$, then X is bounded in $({}^\omega\omega, \leq^{\mathcal{J}})$.*

(2) *Let $\mathcal{I}$ be a tall ideal. If $\mathcal{A} \subseteq \mathcal{I}$ has $\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma\\ \Gamma\end{smallmatrix}\right]$ or $\mathrm{C}_p(\mathcal{A})$ has $\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}}\\ \Gamma_{\mathbf{0}}\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}\mathcal{I}\mathrm{QN}_{\mathbf{0}}\\ \mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right]$ then $\mathcal{A}$ has a pseudounion.*

(3) *Let $\mathcal{I}$ be a tall ideal. If $\mathcal{A} \subseteq \mathcal{I} \cap [\omega]^\omega$ and $\mathrm{C}_p(\mathcal{A})$ is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$ then $\mathcal{A}$ has a pseudounion and the family of increasing enumerations of its elements is bounded in $({}^\omega\omega, \leq^*)$.*

Proof.

(1) Let X be unbounded in $({}^\omega\omega, \leq^{\mathcal{J}})$, i.e., $(\forall \varphi \in {}^\omega\omega)(\exists x \in X)\ \{n : \varphi(n) \in x(n)\} \in \mathcal{J}^+$.

Now we define functions $f_{n,m}$ for all $x \in X$ and $n, m \in \omega$ in the same way as in the proof of above mentioned Theorem 9.3, i.e.,

$$f_{n,m}(x) = \begin{cases} 1 & m \in x(n), \\ 0 & \text{otherwise.} \end{cases}$$

One can easily see that for every $n \in \omega$ a sequence $\langle f_{n,m} : m \in \omega\rangle$ is almost monotone and convergent to $\mathbf{0}$. Moreover, $\langle f_{n,m} : m \in \omega\rangle$ are sequences of continuous functions since $\{x \in X : x(n) = k\}$ is open for any $n, k \in \omega$ and $\{x \in X : f_{n,m}(x) = 1\} = \{x \in X : m < x(n)\}$, $\{x \in X : f_{n,m}(x) = 0\} = \{x \in X : x(n) \leq m\}$. Similarly to the proof of Theorem 9.3 we can conclude that $\mathrm{C}_p(X)$ does not have $\mathrm{S}_1(\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$. Hence, $\mathrm{C}_p(X)$ has neither $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$ as one can see in Diagram 3.

(2) Recall the set U_n in the proof of $\neg$(a) $\to$ $\neg$(d) in Theorem 9.3, it is open and the characteristic function f_n of its complement is continuous in the Cantor topology of $\mathcal{P}(\omega)$. Moreover, for S and A as in the end of the proof, $\{n \in S : f_n(x) \geq \varepsilon_n\} = S \cap A$ for any control $\langle \varepsilon_n : n \in \omega\rangle$.

(3) Let $\mathcal{A} \subseteq \mathcal{I} \cap [\omega]^\omega$ such that $\mathrm{C}_p(\mathcal{A})$ is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$. By Proposition 5.2, $\mathrm{C}_p(\mathcal{A})$ is also $[\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}}]$-space and $\mathrm{S}_1(\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$-space. Therefore by part (2) $\mathcal{A}$ has a pseudounion. Moreover, $\mathcal{A}$ is bounded in $({}^\omega\omega, \leq^*)$ from

part (1). $\square$

By part (1) of Proposition 9.5 and by our Proposition 7.1, we have proved Proposition 2.2 in [40]. Moreover, part (1) of our Proposition 9.5 is a consequence of our Proposition 7.1 and Proposition 4.4 by P. Szewczak and B. Tsaban [43]. Part (2) of Proposition 9.5 appears in more general form in [32], Lemma 4.3.

By Proposition 9.5, $\mathrm{C}_p({}^{\omega}\omega)$ has neither the property $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$ nor the property $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$. In fact, it says more.

Corollary 9.6.

(1) *If $\mathfrak{b}_{\mathcal{J}} \le \mu \le \mathfrak{c}$ then there is a set X of reals of cardinality μ such that $\mathrm{C}_p(X)$ has neither the property $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$ nor the property $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$.*

(2) *If $\mathtt{cov}^*(\mathcal{I}) \le \mu \le \mathfrak{c}$ then there is a set X of reals of cardinality μ such that $\mathrm{C}_p(X)$ has neither the property $\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}}\\ \Gamma_{\mathbf{0}}\end{smallmatrix}\right]$ nor the property $\left[\begin{smallmatrix}\mathcal{I}\mathrm{QN}_{\mathbf{0}}\\ \mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right]$ and X does not have $\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma\\ \Gamma\end{smallmatrix}\right]$.*

(3) *If $\min\{\mathtt{cov}^*(\mathcal{I}), \mathfrak{b}\} \le \mu \le \mathfrak{c}$ then there is a set X of reals of cardinality μ such that $\mathrm{C}_p(X)$ does not have the property $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$ and X does not have $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma)$.*

10. Consistency and relations

Problem 4.2 in [6] asks whether every $[\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\mathrm{QN}_{\mathbf{0}}]$-space $\mathrm{C}_p(X)$ is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$-space. The problem is by Theorem 1.1 equivalent to the question whether every $[\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\mathrm{QN}_{\mathbf{0}}]$-space $\mathrm{C}_p(X)$ is an $[\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathrm{s}\mathcal{J}\mathrm{QN}_{\mathbf{0}}]$-space. First of all, let us recall that

$$\mathrm{C}_p(X) \text{ has } \left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}}\\ \mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right] \Leftrightarrow \mathrm{C}_p(X) \text{ has } \left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}}\\ \mathrm{sQN}_{\mathbf{0}}\end{smallmatrix}\right] \Leftrightarrow \mathrm{C}_p(X) \text{ is an } \mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})\text{-space.}$$

On the other hand, by Observation 5.1, if $\mathrm{C}_p(X)$ is an $[\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathrm{s}\mathcal{J}\mathrm{QN}_{\mathbf{0}}]$-space then $\mathrm{Ind}_{\mathbb{Z}}(X) = 0$. Moreover, by Corollary 9.4 we have $\mathrm{non}([\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathrm{s}\mathcal{J}\mathrm{QN}_{\mathbf{0}}]\text{-space}) = \lambda(\mathcal{I}, \mathcal{J})$. However, by [42] if $\mathcal{J}$ is not a weak $\mathrm{P}(\mathcal{I})$-ideal then any topological space is $[\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\mathrm{QN}_{\mathbf{0}}]$-space. Thus by [42] we get negative answer to Problem 4.2 in [6]:

Proposition 10.1. *Let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(\omega)$ be ideals such that $\mathcal{J}$ is not a weak $\mathrm{P}(\mathcal{I})$-ideal. Then there exists an $[\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\mathrm{QN}_{\mathbf{0}}]$-space which is not an $[\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, s\mathcal{J}\mathrm{QN}_{\mathbf{0}}]$-space.*

However, beside two aforementioned cases we do not know a lot about relations between these notions. In fact, another notion naturally fits in between.

$$\mathrm{C}_p(X) \text{ has } \left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}}\\ \mathrm{s}\mathcal{J}\mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right] \to \mathrm{C}_p(X) \text{ has } \left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}}\\ (\mathcal{J},\mathrm{Fin})\text{-e}\end{smallmatrix}\right] \to \mathrm{C}_p(X) \text{ has } \left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}}\\ \mathcal{J}\mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right].$$

As we have already mentioned, there are known relations of $\mathtt{cov}^*(\mathcal{I})$ to cardinal invariants in Cichoń's diagram, see [27]. Thus by part (2) of Corollary 8.2 for ideals $\mathcal{Z}$, nwd and $\mathcal{S}$ we have

Proposition 10.2. *Let $\mathcal{J}$ be arbitrary Borel ideal.*

(1) *If $\mathrm{cov}(\mathcal{N}) \ge \mathfrak{b}$ then $\lambda(\mathcal{Z}, \mathcal{J}) = \mathfrak{b}$.*
(2) *If $\mathrm{cov}(\mathcal{M}) \ge \mathfrak{b}$ then $\lambda(\mathrm{nwd}, \mathcal{J}) = \mathfrak{b}$.*
(3) *If $\mathrm{non}(\mathcal{N}) \ge \mathfrak{b}$ then $\lambda(\mathcal{S}, \mathcal{J}) = \mathfrak{b}$.*

Since every ideal considered in Proposition 10.2 is Borel, in particular,

$$\text{if } \min\{\mathrm{cov}(\mathcal{N}), \mathrm{cov}(\mathcal{M})\} \ge \mathfrak{b} \text{ then } \lambda(\mathcal{Z}, \mathcal{Z}) = \lambda(\mathrm{nwd}, \mathrm{nwd}) = \lambda(\mathcal{S}, \mathcal{S}) = \mathfrak{b}.$$

We summarize several set theoretic assumptions for $\lambda(\mathcal{I}, \mathcal{J})$ or $\lambda(\mathcal{I}, \mathrm{Fin})$ to be or not to be the bounding number or $\mathtt{cov}^*(\mathcal{I})$.

Proposition 10.3. *Let $\mathcal{I}$ be arbitrary ideal.*

(1) *If $\mathfrak{p} = \mathfrak{b}$ then $\lambda(\mathcal{I}, \mathrm{Fin}) = \mathfrak{b}$.*

(2) *If* $\mathfrak{b} = \mathfrak{c}$ *then* $\lambda(\mathcal{I}, \mathrm{Fin}) = \mathtt{cov}^*(\mathcal{I})$.
(3) *If* $\mathfrak{b} < \mathtt{cov}^*(\mathcal{I})$ *then* $\lambda(\mathcal{I}, \mathrm{Fin}) < \mathtt{cov}^*(\mathcal{I})$.
(4) *If* $\mathfrak{p} = \mathfrak{d}$ *then* $\lambda(\mathcal{I}, \mathcal{J}) = \mathfrak{b}$.
(5) *If* $\mathrm{cf}(\mathfrak{d}) < \mathfrak{d}$ *then* $\lambda(\mathcal{I}, \mathcal{J}) < \mathfrak{d}$.

Table 1 contains possible values of $\lambda(\mathcal{I}, \mathrm{Fin})$ for $\mathcal{I}$ being one of standard ideals considered in [27]. If the statement in the heading of the Table 1 is provable, then we put checkmark in the corresponding line. If such inequality is not true then crossmark is used. If we know hat the statement holds under some additional set theoretical assumption, we specify which one. A remark in the parenthesis is a complementary information when the exact value of $\lambda(\mathcal{I}, \mathcal{J})$ or $\mathtt{cov}^*(\mathcal{I})$ is known. Finally, question mark is an open question.[15]

$\mathcal{I}$	$\lambda(\mathcal{I}, \mathrm{Fin}) = \mathfrak{b}$	$\lambda(\mathcal{I}, \mathrm{Fin}) < \mathfrak{b}$	$\lambda(\mathcal{I}, \mathrm{Fin}) = \mathtt{cov}^*(\mathcal{I})$	$\lambda(\mathcal{I}, \mathrm{Fin}) < \mathtt{cov}^*(\mathcal{I})$
Fin	✓	×	×	×
Fin × Fin	✓	×	✓	×
Fin × ∅	✓	×	×	×
∅ × Fin	✓	×	×	×
$\mathcal{S}$	$\mathtt{non}(\mathcal{N}) \geq \mathfrak{b}$	$\mathtt{non}(\mathcal{N}) < \mathfrak{b}$	$\mathtt{non}(\mathcal{N}) \leq \mathfrak{b}$	$\mathtt{non}(\mathcal{N}) > \mathfrak{b}$
		$(\mathtt{non}(\mathcal{N}))$	$(\mathtt{non}(\mathcal{N}))$	$(\mathtt{cov}^*(\mathcal{I}) = \mathtt{non}(\mathcal{N}))$
$\mathcal{Z}$	$\mathtt{cov}(\mathcal{N}) \geq \mathfrak{b}$	Hechler model	$\mathrm{non}(\mathcal{N}) \leq \mathfrak{b}$	?
			$\mathrm{non}(\mathcal{M}) = \mathfrak{b}$	
$\mathcal{I}_f$	$\mathtt{cov}(\mathcal{N}) \geq \mathfrak{b}$	Hechler model	$\mathtt{non}(\mathcal{M}) = \mathfrak{b}$	$\mathtt{cov}(\mathcal{N}) > \mathfrak{b}$
			$(\leq \mathtt{non}(\mathcal{M}))$	
$\mathcal{ED}$	✓	×	$\mathtt{non}(\mathcal{M}) = \mathfrak{b}$	$\mathtt{non}(\mathcal{M}) > \mathfrak{b}$
			$(\mathtt{non}(\mathcal{M}))$	$(\mathtt{cov}^*(\mathcal{I}) = \mathtt{non}(\mathcal{M}))$
$\mathcal{R}$	✓	×	$\mathfrak{b} = \mathfrak{c}$	$\mathfrak{b} < \mathfrak{c}$
			$(\mathtt{cof}(\mathcal{I}))$	$(\mathtt{cov}^*(\mathcal{I}) = \mathtt{cof}(\mathcal{I}))$
$\mathtt{conv}$	✓	×	$\mathfrak{b} = \mathfrak{c}$	$\mathfrak{b} < \mathfrak{c}$
			$(\mathtt{cof}(\mathcal{I}))$	$(\mathtt{cov}^*(\mathcal{I}) = \mathtt{cof}(\mathcal{I}))$
$\mathtt{nwd}$	$\mathtt{cov}(\mathcal{M}) \geq \mathfrak{b}$	$\mathtt{cov}(\mathcal{M}) < \mathfrak{b}$	$\mathtt{cov}(\mathcal{M}) \leq \mathfrak{b}$	$\mathtt{cov}(\mathcal{M}) > \mathfrak{b}$
		$(\mathtt{cov}(\mathcal{M}))$	$(\mathtt{cov}(\mathcal{M}))$	$(\mathtt{cov}^*(\mathcal{I}) = \mathtt{cov}(\mathcal{M}))$
$\mathcal{ED}_{\mathrm{Fin}}$	$\mathtt{cov}(\mathcal{N}) \geq \mathfrak{b}$	Hechler model	$\mathtt{non}(\mathcal{M}) = \mathfrak{b}$	Random model

Table 1: Well-known ideals and $\lambda(\mathcal{I}, \mathrm{Fin})$.

By Theorem 8.1 and Corollaries 8.2, 9.4 we obtain Proposition 10.4. Statements (1) - (2) and (4) can be reformulated for $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$-space and $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$-space as well.

Proposition 10.4.

(1) *If* $\mathfrak{b} = \mathfrak{c}$ *then* $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma)) = \mathtt{cov}^*(\mathcal{I})$ *for every tall ideal* $\mathcal{I}$.

[15] A model obtained from a model of **CH** by adding at least $\aleph_2$-many random reals.

(2) *If* $\mathfrak{b} < \mathsf{cov}^*(\mathcal{I})$ *then* $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma)) < \mathsf{cov}^*(\mathcal{I})$ *for every tall ideal* $\mathcal{I}$.
(3) *If* $\mathfrak{p} = \mathfrak{b}$ *then* $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma)) = \mathfrak{b}$.
(4) *If* $\mathsf{cov}^*(\mathcal{I}) < \mathfrak{b}$ *then* $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma)) < \mathfrak{b}$.
(5) *If* $\mathfrak{b}_{\mathcal{J}} < \mathfrak{d}$ *then* $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)) < \mathfrak{d}$.

G. Di Maio and Lj.D.R. Kočinac [16] investigate $\mathrm{S}_1(\text{s-}\Gamma, \text{s-}\Gamma)$-space, i.e., our $\mathrm{S}_1(\mathcal{Z}\text{-}\Gamma, \mathcal{Z}\text{-}\Gamma)$-space. Thus we have shown that if $\mathrm{cov}(\mathcal{N}) \geq \mathfrak{b}$ then $\mathrm{non}(\mathrm{S}_1(\text{s-}\Gamma, \text{s-}\Gamma)) = \mathfrak{b}$.

In Diagram 5 we summarize relations of investigated notions of the present paper. We hope that properties of topological space X in one diagram with properties of topological space $\mathrm{C}_p(X)$ does not lead to confusion.

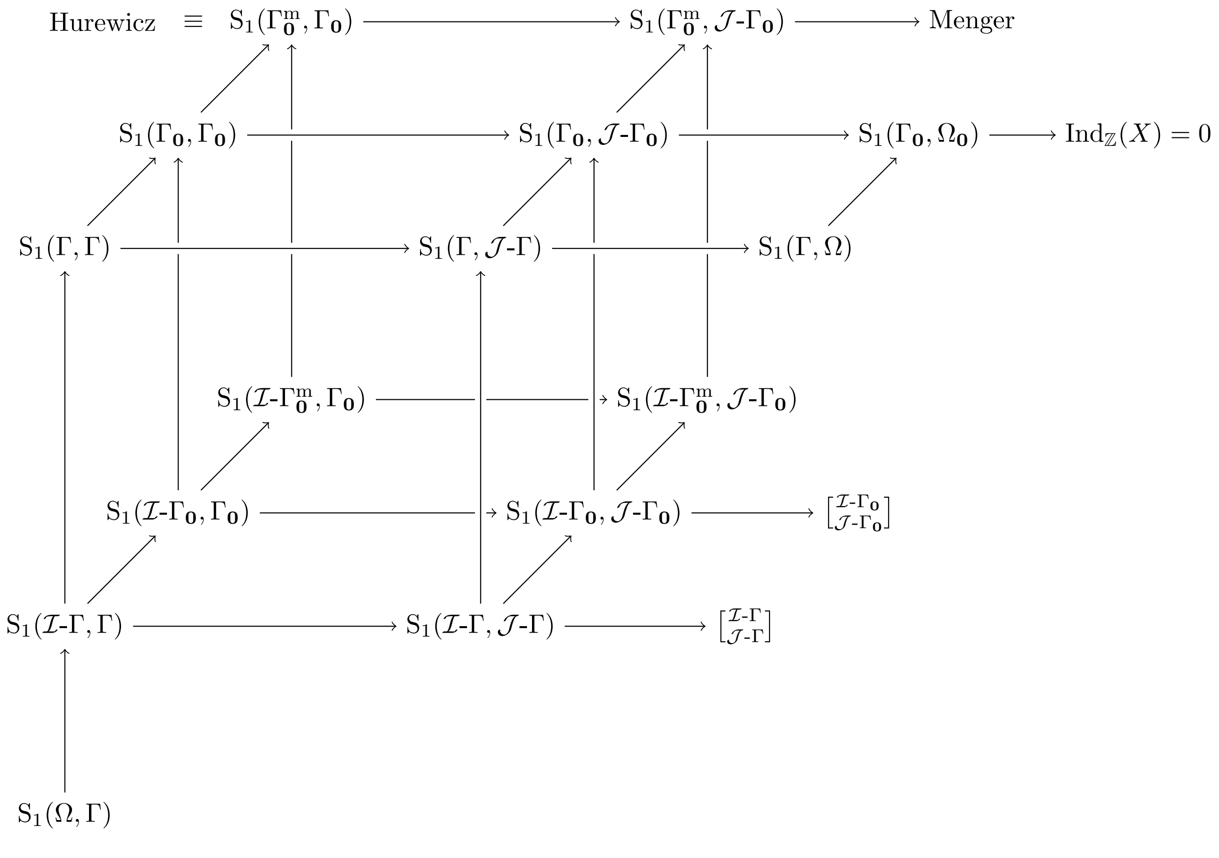

Diagram 5: The overall relations of investigated properties.

Proposition 10.5.

(1) *If* $\mathfrak{p} < \mathfrak{b}$ *there is an* $\mathrm{S}_1(\Gamma, \Gamma)$*-space* X *such that* $\mathrm{C}_p(X)$ *is not an* $\mathrm{S}_1(\mathcal{U}\text{-}\Gamma_{\mathbf{0}}^m, \Gamma_{\mathbf{0}})$*-space.*
(2) *If* $\mathsf{cov}^*(\mathcal{I}) < \mathfrak{b}$ *there is an* $\mathrm{S}_1(\Gamma, \Gamma)$*-space* X *such that* $\mathrm{C}_p(X)$ *is not an* $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}^m, \Gamma_{\mathbf{0}})$*-space.*
(3) *For any* $\mathfrak{b}$*-Sierpiński set* S *there is an ultrafilter* $\mathcal{U}$ *such that* $\mathrm{C}_p(S)$ *is not an* $\mathrm{S}_1(\mathcal{U}\text{-}\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$*-space (but* S *is an* $\mathrm{S}_1(\Gamma, \Gamma)$*-space).*
(4) *If* $\mathfrak{b} < \mathfrak{b}_{\mathcal{U}}$ *then there is an* $\mathrm{S}_1(\Gamma, \mathcal{U}\text{-}\Gamma)$*-space* X *such that* $\mathrm{C}_p(X)$ *is not an* $\mathrm{S}_1(\Gamma_{\mathbf{0}}^m, \Gamma_{\mathbf{0}})$*-space.*
(5) *If* $\mathfrak{b}_{\mathcal{J}} < \mathfrak{d}$ *then there is an* $\mathrm{S}_1(\Gamma, \Omega)$*-space* X *such that* $\mathrm{C}_p(X)$ *is not an* $\mathrm{S}_1(\Gamma_{\mathbf{0}}^m, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$*-space.*
(6) *If* $\mathfrak{b} < \mathsf{cov}^*(\mathcal{I})$ *then there is an* $[\mathcal{I}\text{-}\Gamma, \Gamma]$ *-space* X *such that* $\mathrm{C}_p(X)$ *is not an* $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}^m, \Gamma_{\mathbf{0}})$*-space.*

Proof. (1) By Corollaries 9.4 and 8.2 there is an ultrafilter $\mathcal{U}$ such that $\mathrm{non}(\mathrm{S}_1(\mathcal{U}\text{-}\Gamma_{\mathbf{0}}^m, \Gamma_{\mathbf{0}})) = \mathfrak{p}$. However, $\mathrm{non}(\mathrm{S}_1(\Gamma, \Gamma)) = \mathfrak{b}$. (2) If $\mathsf{cov}^*(\mathcal{I}) < \mathfrak{b}$ then by Corollaries 9.4 and 8.2 we have $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}^m, \Gamma_{\mathbf{0}})) = \mathsf{cov}^*(\mathcal{I})$.

(3) Any $\mathfrak{b}$-Sierpiński set is an $\mathrm{S}_1(\Gamma,\Gamma)$-space which is not a γ-set. By Theorem 5.4, if S is a $\mathfrak{b}$-Sierpiński set then there is an ultrafilter $\mathcal{U}$ such that S is not an $\mathrm{S}_1(\mathcal{U}\text{-}\Gamma,\Gamma)$-space. (4) and (6) follow from the fact that $\mathrm{non}(\mathrm{S}_1(\Gamma,\mathcal{U}\text{-}\Gamma)) = \mathfrak{b}_{\mathcal{U}}$, $\mathrm{non}([\mathcal{I}\text{-}\Gamma,\Gamma]\text{-space}) = \mathtt{cov}^*(\mathcal{I})$ and $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^m_{\mathbf{0}},\Gamma_{\mathbf{0}})) = \mathfrak{b}$. □

If $\mathrm{cf}(\mathfrak{d}) < \mathfrak{d}$ then the assumption of part (5) in Proposition 10.5 is fulfilled. R.M. Canjar [12] has shown that there is an ultrafilter $\mathcal{U}$ such that $\mathfrak{b}_{\mathcal{U}} = \mathrm{cf}(\mathfrak{d})$. Therefore if $\mathrm{cf}(\mathfrak{d}) = \mathfrak{d}$ then $\mathrm{non}(\mathrm{S}_1(\Gamma,\mathcal{U}\text{-}\Gamma)) = \mathrm{non}(\mathrm{S}_1(\Gamma,\Omega)) = \mathfrak{d}$. Moreover, M. Sakai [33] has shown $\mathrm{non}(\mathrm{S}_1(\Gamma_{\mathbf{0}},\Omega_{\mathbf{0}})) = \mathfrak{d}$.

The implication $\mathrm{S}_1(\Omega,\Gamma) \to \mathrm{S}_1(\Gamma,\Gamma)$ is irreversible ($\mathrm{non}(\mathrm{S}_1(\Omega,\Gamma)) = \mathfrak{p}$ by [24]) and the implication $\mathrm{S}_1(\Gamma_{\mathbf{0}},\Gamma_{\mathbf{0}}) \to \mathrm{S}_1(\Gamma^m_{\mathbf{0}},\Gamma_{\mathbf{0}})$ is irreversible even in **ZFC** (the real line). The implication $\mathrm{S}_1(\Gamma,\Gamma) \to \mathrm{S}_1(\Gamma_{\mathbf{0}},\Gamma_{\mathbf{0}})$ can be consistently equivalence (Laver model, see [31]) and it is not known whether it can be reversed (Scheepers Conjecture [39]). However, we do not know how it is in case of ideal versions of preceding implications when $\mathcal{I}$, $\mathcal{J}$ are fixed ideals not equal to Fin. $\mathrm{C}_p(\mathbb{R})$ is an $\mathrm{S}_1(\Gamma^m_{\mathbf{0}},\mathcal{I}\text{-}\Gamma_{\mathbf{0}})$-space which is not an $\mathrm{S}_1(\Gamma_{\mathbf{0}},\mathcal{I}\text{-}\Gamma_{\mathbf{0}})$-space since $\mathbb{R}$ is not zero-dimensional.

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[1] Архангельский А.В. (Arkhangel'skiĭ A.V.), *Спектр частот топологического пространства и классификация пространств,* ДАН СССР, **206**:2 (1972), 265–268. English translation *The frequency spectrum of a topological space and the classification of spaces,* Soviet Math. Dokl. **13** (1972), 1185–1189.

[2] Blass A.: *Combinatorial cardinal characteristics of the Continuum* in: Foreman M., Kanamori A. (eds) Handbook of Set Theory. Springer, Dordrecht (2010), 24–27.

[3] Bartoszyński T.: *Additivity of measure implies additivity of category*, Trans. Amer. Math. Soc. **281** (1984), 209–213.

[4] Bukovský L.: *On* wQN$_*$ *and* wQN* *spaces*, Topology Appl. **156** (2008), 24–27.

[5] Bukovský L., The Structure of the Real Line, Monogr. Mat., Springer-Birkhauser, Basel, 2011.

[6] Bukovský L., Das P. and Šupina J.: *Ideal quasi-normal convergence and related notions*, Colloq. Math. **146** (2017), 265–281.

[7] Bukovský L. and Haleš J.: *On Hurewicz properties,* Topology Appl. **132** (2003), 71–79.

[8] Bukovský L. and Haleš J.: QN-*spaces,* wQN-*spaces and covering properties,* Topology Appl. **154** (2007), 848–858.

[9] Bukovský L., Recław I. and Repický M., *Spaces not distinguishing pointwise and quasinormal convergence of real functions,* Topology Appl. **41** (1991), 25–40.

[10] Bukovský L., Recław I. and Repický M.: *Spaces not distinguishing convergences of real-valued functions*, Topology Appl. **112** (2001), 13–40.

[11] Canjar R.M.: *Mathias forcing which does not add dominating reals,* Proc. Amer. Math. Soc. **104** (1988), 1269-1248.

[12] Canjar R.M.: *Cofinalities countable ultraproducts: The existence theorem,* Notre Dame J. Form. Log. **30** (1989), 539-542.

[13] Chandra D.: *Some remarks on sequence selection properties using ideals,* Mat. Vesnik **68** (2016), 39-44.

[14] Das P. and Chandra D., *Spaces not distinguishing pointwise and $\mathcal{I}$-quasinormal convergence of real functions*, Comment. Math. Univ. Carolin. **54** (2013), 83–96.

[15] Das P., *Certain types of open covers and selection principles using ideals,* Houston J. Math. **39** (2013), 637–650.

[16] Di Maio G. and Kočinac Lj.D.R., *Statistical convergence in topology,* Topology Appl., **156** (2008), 28–45.

[17] Engelking R., General Topology, Heldermann, Berlin, 1989.

[18] Farah I., *Analytic quotients: theory of liftings for quotients over analytic ideals on the integers*, Mem. Amer. Math. Soc. **148** (2000), no. 702

[19] Farkas B. and Soukup L., *More on cardinal invariants of analytic P-ideals*, Comment. Math. Univ. Carolin. **50** (2009), 281–295.

[20] Filipów R. and Staniszewski M., *On ideal equal convergence*, Cent. Eur. J. Math **12** (2014), 896–910.

[21] Filipów R. and Staniszewski M., *Pointwise versus equal (quasi-normal) convergence via ideals*, J. Math. Anal. Appl. **422** (2015), 995–1006.

[22] Filipów R. and Szuca P., *Three kinds of convergence and the associated $\mathcal{I}$-Baire classes*, J. Math. Anal. Appl. **391** (2012), 1–9.

[23] Fremlin D.H.: SSP *and* wQN, preprint, 2003.

[24] Galvin F. and Miller A.W., *γ-sets and other singular sets of real numbers,* Topology Appl. **17** (1984), 145–155.

[25] Gerlits J. and Nagy Zs., *Some properties of* $\mathrm{C}_p(X)$*, I,* Topology Appl. **14** (1982), 151–161.

[26] Hernández F. and Hrušák M., *Cardinal invariants of analytic P-ideals,* Canad.J.Math. **59** (2007), 575–595.

[27] Hrušák M., *Combinatorics of filters and ideals,* Contemp. Math. **533** (2011), 29–69.

[28] Kwela A., *Ideal weak QN-space*, Topology Appl. **240** (2018), 98–115.

[29] Laczkovich M. and Recław I., *Ideal limits of sequences of continuous functions*, Fund. Math. **203** (2009), 39–46.

[30] Laflamme C., *Filter games and combinatorial properties of winning strategies*, Contemp. Math. **192** (1996), 51–67.

[31] Miller A.W. and Tsaban B., *Point-cofinite covers in the Laver model,* Proc. Amer. Math. Soc. **138** (2010), 3313–3321.

[32] Repický M., *Spaces not distinguishing ideal convergences of real-valued functions*, preprint.

[33] Sakai M.: *The sequence selection properties of* $C_p(X)$, Topology Appl. **154** (2007), 552–560.
[34] Sakai M.: *Selection principles and upper semicontinuous functions,* Colloq. Math. **117** (2009), 251–256.
[35] Sakai M.: *The Ramsey property for* $C_p(X)$, Acta Math. Hungar. **128** (2010), 96–105.
[36] Scheepers M.: *Combinatorics of open covers I: Ramsey theory*, Topology Appl. **69** (1996), 31–62.
[37] Scheepers M.: *A sequential property of* $C_p(X)$ *and a covering property of Hurewicz,* Proc. Amer. Math. Soc. **125** (1997), 2789–2795.
[38] Scheepers M.: $C_p(X)$ *and Archangel'skiĭs* α_i*-spaces*, Topology Appl. **89** (1998), 256–275.
[39] Scheepers M.: *A sequential convergence in* $C_p(X)$ *and a covering property*, East-West J. Math. **1** (1999), 207–214.
[40] da Silva S.G.: *The* $\mathcal{I}$*-Hurewicz property and bounded families modulo an ideal*, Questions Answers Gen. Topology 36 (2018), 31-38.
[41] Staniszewski M.: *On ideal equal convergence II*, J. Math. Anal. Appl. **451** (2017), 1179–1197.
[42] Šupina J.: *Ideal* QN*-spaces*, J. Math. Anal. Appl. **434** (2016) 477–491.
[43] Szewczak P. and Tsaban B.: *Products of Menger spaces: A combinatorial approach*, Ann. Pure Appl. Logic **168** (2017) 1–18.
[44] Szewczak P. and Tsaban B.: *Products of Menger spaces II.: General spaces*, preprint.