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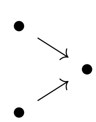
Abstract. We will show that separable unital AF-algebras whose Bratteli diagrams do not allow converging two nodes into one node, can be classified up to the tensor product with the universal UHF-algebra $\mathcal{Q}$ only by their trace spaces. That is, if $\mathcal{A}$ and $\mathcal{B}$ are such AF-algebras, then $T(A) = T(B)$ if and only if $\mathcal{A} \otimes \mathcal{Q} \cong \mathcal{B} \otimes \mathcal{Q}$.

1. Introduction

UHF-algebras were first studied and classified by Glimm [4]. To any UHF-algebra one assigns a unique a "supernatural number" (and vise versa) which is a complete isomorphism invariant. More generally, later separable AF-algebras were classified by Elliott [2] using their K_0-groups. In the unital case, Elliott's theorem assigns to a unital separable AF-algebra $\mathcal{A}$ a unique "dimension group with order-unit" $\langle K_0(\mathcal{A}), K_0(\mathcal{A})^+, [1_{\mathcal{A}}]_0\rangle$ (and vice versa), as a complete isomorphism invariant. Since then, the aim of Elliott's classification program has been to classify more classes of separable nuclear C*-algebras. The so called "strongly self-absorbing" C*-algebras play a particularly important role in the classification program. In fact, the classification results are usually obtained up to tensoring with one of the C*-algebras in the short list of known strongly self-absorbing C*-algebras (refer to [9] or [8]). In the classification of separable nuclear C*-algebras in addition to the K-theoretic data, the "trace space" of a C*-algebra, which is tightly related to the state space of the K_0-group, is also usually used as an invariant. In these notes we show that a fairly rich class of AF-algebras can be classified up to tensoring with the "universal UHF-algebra" $\mathcal{Q}$ (up to $\mathcal{Q}$-stability) using only the trace space. The universal UHF-algebra is strongly self-absorbing and is the UHF-algebras with K_0-group isomorphic to the additional group of all rational numbers $\mathbb{Q}$ with the usual ordering and 1 as order-unit.

The AF-algebras that we consider are unital and their Bratteli diagrams do not allow edges of the form



Equivalently, these separable unital AF-algebras are precisely the ones that arise as the limit of a direct sequence of finite-dimensional C*-algebras where the connecting maps have exactly one non-zero entry in each row in their representing matrices. We will denote this class of unital AF-algebras by $\overrightarrow{\mathfrak{D}}$. It contains all the UHF-algebras (for which Theorem 1.1 is a tautology), all the commutative C*-algebras $C(X)$ for compact, second countable and totally disconnected spaces X

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(because every such X is the inverse limit of an inverse sequence of finite sets and surjective maps). Also $\overrightarrow{\mathfrak{D}}$ is closed under direct sums and tensor products.

For a unital C*-algebra $\mathcal{A}$, the space of all tracial states of $\mathcal{A}$ is denoted by $T(\mathcal{A})$. Since $\mathcal{Q}$ has a unique tracial state, it is known and easy to check that $T(\mathcal{A}) \cong T(\mathcal{A} \otimes \mathcal{Q})$ for any unital C*-algebra $\mathcal{A}$. In particular, for unital AF-algebras $\mathcal{A}, \mathcal{B}$, if $\mathcal{A} \otimes \mathcal{Q} \cong \mathcal{B} \otimes \mathcal{Q}$ then we always have $T(\mathcal{A}) \cong T(\mathcal{B})$ (i.e. they are affinely homeomorphic simplices).

Theorem 1.1. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are unital AF-algebras in $\overrightarrow{\mathfrak{D}}$. Then $T(A) \cong T(B)$ if and only if $\mathcal{A} \otimes \mathcal{Q} \cong \mathcal{B} \otimes \mathcal{Q}$.*

For any unital AF-algebra $\mathcal{A}$, its trace space $T(\mathcal{A})$ is affinely homeomorphic to "state space" of the dimension group with order-unit $\langle K_0(\mathcal{A}), K_0(\mathcal{A})^+, [1_{\mathcal{A}}]_0\rangle$ via the map which sends τ to $K_0(\tau)$ (see [8, Proposition 1.5.5]). Therefore Theorem 1.1 (as well as the whole theory of unital AF-algebras) can be restated in the language of dimension groups with order-units and their state spaces (Theorem 2.7).

Preliminaries

An (partially) ordered abelian group G with a fixed order-unit u is denoted by $\langle G, u\rangle$. Suppose $G_1, G_2, \dots G_k$ are ordered abelian groups. The tensor product $G = G_1 \otimes G_2 \otimes \cdots \otimes G_k$, as defined in [6], is an ordered abelian group with the positive cone G^+ defined as the collection of all finite sums of the elements of the set

$$\{x_1 \otimes \cdots \otimes x_k : x_i \in G_i^+\}.$$

If u_i is an order-unit of G_i then $u_1 \otimes \cdots \otimes u_k$ is an order-unit of G ([6, Lemma 2.4]). Tensor products of positive homomorphisms are positive homomorphisms and if G and H are respective direct limits of sequences (G_i, α_i^j) and (H_i, β_i^j) of ordered abelian groups, then $G \otimes H$ is the direct limit of the sequence $(G_i \otimes H_i, \alpha_i^j \otimes \beta_i^j)$ ([6, Lemma 2.2]).

We always consider the abelian group of rational numbers $\mathbb{Q}$ with its usual ordering and the order-unit 1. In the following write simply $\mathbb{Q}$ instead of $\langle \mathbb{Q}, 1\rangle$. There is a one-to-one correspondence between the subgroups of $\mathbb{Q}$ which contain 1 and the supernatural numbers (see [7, Proposition 7.4.3]). For a supernatural number n we denote the corresponding subgroup of $\mathbb{Q}$ by $\mathbb{Q}_n$. Multiplication of supernatural numbers is defined naturally as an extension of the multiplication of the natural numbers. If m is also a supernatural number then $\mathbb{Q}_n \otimes \mathbb{Q}_m \cong \mathbb{Q}_{nm}$, i.e., they are isomorphic as ordered groups with order-units. In particular, $\mathbb{Q}_n \otimes \mathbb{Q} \cong \mathbb{Q}$ for every supernatural number n.

Recall that an ordered abelian group is archimedean if $nx \geq y$ for every $n \in \mathbb{N}$ implies that $x \leq 0$.

Proposition 1.2. *Suppose G is an ordered abelian group and n is a supernatural number. Then G is archimedean if and only if $G \otimes \mathbb{Q}_n$ is archimedean.*

Proof. The map $x \to x \otimes 1$ is a positive embedding from G into $G \otimes \mathbb{Q}_n$. Therefore if G is not archimedean it is clear that $G \otimes \mathbb{Q}_n$ is not archimedean. Assume G is archimedean. Suppose $x, y \in G \otimes \mathbb{Q}_n$ and $nx \leq y$ for every natural number n. Suppose $x = \sum_{i=1}^r x_i \otimes \frac{a_i}{b_i}$ and $y = \sum_{j=1}^s y_j \otimes \frac{c_j}{d_j}$. By changing the sign of the integers a_i and c_j, if necessary, we may assume b_i and d_j are positive natural numbers for every $i \leq r$ and $j \leq s$. Let $b = b_1 \dots b_r > 0$, $d = d_1 \dots d_s > 0$, $k_i = a_i b / b_i \in \mathbb{N}$

and $l_j = c_j d/d_j \in \mathbb{N}$. Then we have $x = (\sum_i k_i x_i) \otimes \frac{1}{b}$ and $y = (\sum_j l_j y_j) \otimes \frac{1}{d}$. For every n

$$y - nx = (\sum_j l_j y_j) \otimes \frac{1}{d} - n(\sum_i k_i x_i) \otimes \frac{1}{b} = (b \sum_j l_j y_j - nd \sum_i k_i x_i) \otimes \frac{1}{bd} \geq 0.$$

Since $\frac{1}{bd} > 0$ we must have $b \sum_j l_j y_j - nd \sum_i k_i x_i \geq 0$ for every n. Since G is archimedean this implies that $d \sum_i k_i x_i \leq 0$. As d is a positive integer, we have $\sum_i k_i x_i \leq 0$ which implies that $x = (\sum_i k_i x_i) \otimes \frac{1}{b} \leq 0$. □

For any non-negative integer r, the ordered abelian group $\mathbb{Z}^r$ equipped with the positive cone

$$(\mathbb{Z}^r)^+ = \{(x_1, x_2, \dots, x_r) \in \mathbb{Z}^r : x_i \geq 0\}.$$

is called a *simplicial group*. A (not necessarily countable) partially ordered abelian group $\langle G, G^+ \rangle$ is called a *dimension group* if it is directed, unperforated interpolation group. We refer the reader to [5] for definitions and more on dimension groups. By a well-known result of Effros, Handelman and Shen [1] any dimension group (with order-unit) is isomorphic to a direct limit of a direct system of simplicial groups (with order-units) and positive (order-unit preserving) homomorphisms, in the category of ordered abelian groups (with order-units). In particular, a countable dimension group $\langle G, G^+, u \rangle$ with order-unit u (we would simply write $\langle G, u \rangle$ in the following) is isomorphic to the direct limit of a sequence simplicial groups and normalized (order-unit preserving) positive homomorphisms

$$\langle \mathbb{Z}^{r_1}, u_1 \rangle \xrightarrow{\alpha_1^2} \langle \mathbb{Z}^{r_2}, u_2 \rangle \xrightarrow{\alpha_2^3} \langle \mathbb{Z}^{r_3}, u_3 \rangle \xrightarrow{\alpha_3^4} \dots \langle G, u \rangle.$$

Tensor product of two dimension groups with order-units is again a dimension group with order-unit. In fact, their tensor products correspond to the tensor products of the corresponding AF-algebras. That is, if $\mathcal{A}$ and $\mathcal{B}$ are AF-algebras, then $\langle K_0(\mathcal{A} \otimes \mathcal{B}), [1]_{\mathcal{A} \otimes \mathcal{B}} \rangle \cong \langle K_0(\mathcal{A}), [1]_{\mathcal{A}} \rangle \otimes \langle K_0(\mathcal{B}), [1]_{\mathcal{B}} \rangle$ ([6, Proposition 3.4]).

We are only concerned here with countable dimension groups and therefore by a dimension group we mean a countable one. Clearly $\mathbb{Q}$ and its subgroups (with the inherited ordering) are dimension groups. Every ordered subgroup of $\mathbb{Q}$ containing 1 is isomorphic as an ordered abelian group with order-unit to the limit of a sequence of simplicial groups and normalized positive homomorphisms $\langle \mathbb{Z}, n_1 \rangle \to \langle \mathbb{Z}, n_2 \rangle \to \dots$ for some sequence $\{n_i\}$ of natural numbers, such that $n_i | n_{i+1}$ for each i. We say such a sequence $\{n_i\}$ of natural numbers is *associated* to the supernatural number n if the limit of the directed sequence $\langle \mathbb{Z}, n_1 \rangle \to \langle \mathbb{Z}, n_2 \rangle \to \dots$ is isomorphic to $\mathbb{Q}_n$. Note that this is different from the usual unique representation of n as extended powers of prime numbers, since it is not even uniquely associated to n. However, it is convenient to use for our purpose. For a natural number m and $x = (x_1, x_2, \dots, x_r) \in \mathbb{Z}^r$ we write mx for $(mx_1, mx_2, \dots, mx_r)$. If $\langle G, u \rangle = \varinjlim(\mathbb{Z}^{r_i}, u_i, \alpha_i^j)$ is a dimension group with order-unit and $\{n_i\}$ is any sequence associated to the supernatural number n, then $\langle G, u \rangle \otimes \mathbb{Q}_n = \varinjlim(\mathbb{Z}^{r_i}, n_i u_i, \frac{n_j}{n_i} \alpha_i^j)$ (cf. [6, Lemma 2.2]).

Definition 1.3. Define the equivalence relation $\sim_{\mathbb{Q}}$ on the set of all dimension groups with order-units by

$$\langle G, u \rangle \sim_{\mathbb{Q}} \langle H, v \rangle \quad \Leftrightarrow \quad \langle G, u \rangle \otimes \mathbb{Q} \cong \langle H, v \rangle \otimes \mathbb{Q}.$$

Equivalently, $\langle G, u \rangle \sim_{\mathbb{Q}} \langle H, v \rangle$ if and only if there are supernatural numbers n, m such that $\langle G, u \rangle \otimes \mathbb{Q}_m \cong \langle H, v \rangle \otimes \mathbb{Q}_n$ as ordered abelian groups with order-units.

2. Dimension groups with positive non-mixing connecting maps

We say a homomorphism $\alpha : \mathbb{Z}^r \to \mathbb{Z}^s$ is *non-mixing* if $\alpha(x_1, \dots, x_r) = (k_1 x_{i_1}, \dots, k_s x_{i_s})$ where $i_j \in \{1, \dots r\}$ for each $j \leq s$. A non-mixing homomorphism is positive if and only if each k_j is a positive integer. Let $\overrightarrow{\mathfrak{D}}$ denote the class of all dimension groups G that arise as the limit of a direct sequence of simplicial groups and positive non-mixing homomorphisms (G_i, α_i^j). Since positive non-mixing homomorphisms send order-units to order-units, the image of each order-unit of each G_i in G is an order-unit of G. Note that positive non-mixing homomorphisms are not necessarily embeddings. A dimension groups that is inductive limit of a sequence of simplicial groups with injective connecting maps is called *ultrasimplicial* [3]. Not all dimension groups are ultrasimplicial [3, Example 2.7].

Proposition 2.1. *Suppose G is a dimension group in $\overrightarrow{\mathfrak{D}}$*

(1) G is ultrasimplicial.
(2) G is archimedean.

Proof. (1) Suppose G is the limit of a sequence (G_i, α_i^j) of simplicial groups and positive non-mixing homomorphisms. Restricting each α_i^{i+1} to $G_i/\ker(\alpha_i^{i+1})$ and projecting its image to $G_{i+1}/\ker(\alpha_{i+1}^{i+2})$ yields a sequence with injective maps with the same limit G.

(2) By (1) we can assume α_i^j are injective positive non-mixing homomorphisms. If $z \in G_i$ and $z \not\leq 0$ then $\alpha_i^j(z) \not\leq 0$ for any $j \geq i$ and therefore $\alpha_i^\infty(z) \not\leq 0$. Suppose $x, y \in G$ are such that $nx \leq y$ for every $n \in \mathbb{N}$, and x_i, y_i are such that $\alpha_i^\infty(x_i) = x$ and $\alpha_i^\infty(y_i) = y$. Then $nx - y \leq 0$ implies that $nx_i - y_i \leq 0$ for every n. Since G_i is archimedean we have $x_i \leq 0$ and therefore $x \leq 0$. □

Next we will show that for dimension groups in $\overrightarrow{\mathfrak{D}}$ changing the order-unit results in $\sim_{\mathbb{Q}}$-equivalent (Definition 1.3) dimension groups. First we need the following fairly trivial lemma.

Lemma 2.2. *Suppose $\alpha : \mathbb{Z}^r \to \mathbb{Z}^s$ is a positive non-mixing homomorphism of simplicial groups and $\gamma : \mathbb{Z}^r \to \mathbb{Z}^r$ is a homomorphism defined by $\gamma(x_1, \dots, x_r) = (l_1 x_1, \dots, l_r x_r)$ for positive integers $l_1, \dots l_r$. Then there are natural numbers n, positive integers $l'_1, \dots, l'_s$ such that if $\eta : \mathbb{Z}^s \to \mathbb{Z}^s$ is defined by $\eta(x_1, \dots, x_s) = (l'_1 x_1, \dots, l'_s x_s)$, then $\eta \circ \alpha \circ \gamma = n\alpha$, i.e., the following diagram commutes.*

$$\begin{array}{ccc} \mathbb{Z}^r & \xrightarrow{n\alpha} & \mathbb{Z}^s \\ \downarrow{\scriptstyle\gamma} & & \uparrow{\scriptstyle\eta} \\ \mathbb{Z}^r & \xrightarrow{\alpha} & \mathbb{Z}^s \end{array}$$

Proof. Suppose $\alpha(x_1, \dots, x_r) = (k_1 x_{i_1}, \dots, k_s x_{i_s})$ for $i_j \in \{1, \dots r\}$. Let $n = k_1 \dots k_s l_{i_1} \dots l_{i_s}$ and define $\eta(x_1, \dots, x_s) = (\frac{n}{k_1 l_{i_1}} x_1, \dots, \frac{n}{k_s l_{i_s}} x_s)$. Then $\eta \circ \alpha \circ \gamma = n\alpha$. □

Proposition 2.3. *If $G \in \overrightarrow{\mathfrak{D}}$ and u, w are order-units of G then $\langle G, u \rangle \sim_{\mathbb{Q}} \langle G, w \rangle$.*

Proof. Suppose $G = \varinjlim (G_i, \alpha_i^j)$ where (G_i, α_i^j) is a direct sequence of simplicial groups and positive non-mixing homomorphisms. Without loss of generality assume that there are $u_1, w_1 \in G_1$ such that $\alpha_1^\infty(u_1) = u$ and $\alpha_1^\infty(w_1) = w$. Set $u_i = \alpha_1^i(u_1)$ and $w_i = \alpha_1^i(w_1)$. Since each u_i and w_i are order-units of G_i (because α_1^i is order-unit preserving) the convex subgroups (ideals) of G_i generated by u_i and w_i are all of G_i. Therefore we have $\langle G, u \rangle = \varinjlim (G_i, u_i, \alpha_i^j)$ and $\langle G, w \rangle = \varinjlim (G_i, w_i, \alpha_i^j)$; cf. [5, Corollary 3.18].

Suppose $G_1 = \mathbb{Z}^{r_1}$ and let $m_1 = u_{11}\dots u_{1r_1}$ where $u_1 = (u_{11},\dots,u_{1r_1})$. We can find $\gamma_1 : \langle G_1, u_1\rangle \to \langle G_1, m_1 w_1\rangle$ given by $(x_1 \dots, x_{r_1}) \to (k_1 x_1, \dots, k_{r_1} x_{r_1})$ where $k_i = \frac{u_{11}\dots u_{1r_1}}{u_i} w_i$. Starting wit γ_1 use Lemma 2.2 recursively to find sequences of natural numbers $\{n_i\}$ and $\{m_i\}$ and normalized positive homomorphism γ_i and η_i such that the following diagram commutes

$$\begin{array}{cccccccc} \langle G_1,u_1\rangle & \xrightarrow{n_1\alpha_1^2} & \langle G_2,n_1u_2\rangle & \xrightarrow{\alpha_2^3} & \langle G_3,n_1u_3\rangle & \xrightarrow{n_2\alpha_3^4} & \langle G_4,n_2n_1u_4\rangle & \dots \\ \downarrow{\scriptstyle\gamma_1} & & {\scriptstyle\eta_1}\uparrow & & \downarrow{\scriptstyle\gamma_2} & & {\scriptstyle\eta_2}\uparrow & \\ \langle G_1,m_1w_1\rangle & \xrightarrow{\alpha_1^2} & \langle G_2,m_1w_2\rangle & \xrightarrow{m_2\alpha_2^3} & \langle G_3,m_2m_1w_3\rangle & \xrightarrow{\alpha_3^4} & \langle G_4,m_2m_1w_4\rangle & \dots \end{array}$$

This intertwining provides an isomorphism between $\langle G,u\rangle \otimes \mathbb{Q}_n$ and $\langle G,w\rangle \otimes \mathbb{Q}_m$, where n and m are the supernatural numbers associated to the sequences $\{n_i \dots n_1\}_{i\in\mathbb{N}}$ and $\{m_i \dots m_1\}_{i\in\mathbb{N}}$, respectively. □

Corollary 2.4. *Suppose $G, H \in \overrightarrow{\mathfrak{D}}$ and $\langle G,w\rangle$ and $\langle H,z\rangle$ are isomorphic for some order-units w, z. Then for every u order-unit of G and v order-unit of H we have $\langle G,u\rangle \sim_{\mathbb{Q}} \langle H,v\rangle$.*

Proof. By Proposition 2.3 and our assumption we have $\langle G,u\rangle \sim_{\mathbb{Q}} \langle G,w\rangle \cong \langle H,z\rangle \sim_{\mathbb{Q}} \langle H,v\rangle$. □

2.1. **State spaces.** Let $\langle G,u\rangle$ be an ordered abelian group with order-unit u. The set $S(G,u)$ of all states on $\langle G,u\rangle$, called the *state space* of $\langle G,u\rangle$, when equipped with the relative topology endowed from $\mathbb{R}^G$, is a compact convex subset of the locally convex topological vector space $\mathbb{R}^G$. Given a normalized positive homomorphism $\alpha : \langle G,u\rangle \to \langle H,v\rangle$, one associates an affine and continuous map $S(\alpha) : S(H,v) \to S(G,u)$ defined by $S(\alpha)(s) = s \circ \alpha$, for every $s \in S(H,v)$. In fact, S is a contravariant and continuous functor from the category of ordered abelian groups with order-unit into the category of compact convex sets. If $\langle G,u\rangle$ is a dimension group with an order-unit, then $S(G,u)$ is a Choquet simplex ([5, Theorem 10.17]). Conversely, a Choquet simplex is affinely homeomorphic to the state space of a (countable) dimension group with an order-unit ([5, Corollary 14.9]).

If G is a ordered abelian group and $u_1, u_2 \in G^+$ are order-units of G, then the map $\phi : S(G,u_1) \to S(G,u_2)$ defined by $\phi(s)(x) = \frac{1}{s(u_2)} s(x)$ is a (not necessarily affine) homeomorphism. In fact, there are dimension groups G with order-units u and v such that $S(G,u)$ is not affinely homeomorphic to $S(G,v)$ (see [5, Example 6.18]).

Proposition 2.5. *For every dimension group with order-unit $\langle G,u\rangle$ and supernatural number n we have $S(G,u) \cong S(\langle G,u\rangle \otimes \mathbb{Q}_n)$. If $\langle G,u\rangle \sim_{\mathbb{Q}} \langle H,v\rangle$, then $S(G,u) \cong S(H,v)$.*

Proof. If $\langle G,u\rangle = \varinjlim(G_i, u_i, \alpha_i^j)$ where each $\langle G_i,u_i\rangle$ is a simplicial group with order-unit, then for any supernatural number n and an associated sequence $\{n_i\}$, we have $\langle G,u\rangle \otimes \mathbb{Q}_n = \varinjlim(G_i, n_i u_i, \frac{n_j}{n_i}\alpha_i^j)$. The following diagram commutes.

$$\begin{array}{cccccccc} \langle G_1,u_1\rangle & \xrightarrow{\alpha_1^2} & \langle G_2,u_2\rangle & \xrightarrow{\alpha_2^3} & \langle G_3,u_3\rangle & \xrightarrow{\alpha_3^4} & \dots & \langle G,u\rangle \\ \downarrow{\scriptstyle n_1} & & \downarrow{\scriptstyle n_2} & & \downarrow{\scriptstyle n_3} & & & \\ \langle G_1,n_1u_1\rangle & \xrightarrow{\frac{n_2}{n_1}\alpha_1^2} & \langle G_2,n_2u_2\rangle & \xrightarrow{\frac{n_3}{n_2}\alpha_2^3} & \langle G_3,n_3u_3\rangle & \xrightarrow{\frac{n_4}{n_3}\alpha_3^4} & \dots & \langle G,u\rangle \otimes \mathbb{Q}_n \end{array}$$

After applying the state functor S to the above diagram we get a commuting diagram where the inverse limit of the first row is $S(G,u)$ and the inverse limit of the second row is $S(\langle G,u\rangle \otimes \mathbb{Q}_n)$ (cf. [5, Proposition 6.14]). Clearly $S(n_i) : S(G_i, n_i u_i) \to S(G_i, u_i)$ is an affine homeomorphism and therefore $S(G,u) \cong S(\langle G,u\rangle \otimes \mathbb{Q}_n)$. The second statement follows immediately. □

Corollary 2.6. *Suppose u and v are order-units of a dimension group $G \in \overrightarrow{\mathfrak{D}}$. Then $S(G,u)$ is affinely homeomorphic to $S(G,v)$.*

Proof. Follows from Proposition 2.3 and Proposition 2.5. □

In general $S(G,u) \cong S(H,v)$ does not imply $\langle G,u\rangle \sim_{\mathbb{Q}} \langle H,v\rangle$, not even in the finite-dimensional case. For example, simply let $\langle G,u\rangle$ denote the simplicial group $\langle \mathbb{Z}^2, 1\rangle$ and $\langle H,v\rangle$ denote the group $\mathbb{Q}^2$ with the strict ordering $\ll$ and order-unit 1. Note that $\langle H,v\rangle$ is a dimension group since it is a directed and unperforated interpolation group. Clearly $S(G,u) \cong S(H,v) \cong \Delta_1$ but $\langle G,u\rangle \not\sim_{\mathbb{Q}} \langle H,v\rangle$, since the left side is archimedean while the right side is not, and tensoring with $\mathbb{Q}$ does not change the state of being archimedean (Proposition 1.2). However, we will show that for dimension groups G, H in $\overrightarrow{\mathfrak{D}}$ we have $S(G,u) \cong S(H,v)$ implies that $\langle G,u\rangle \sim_{\mathbb{Q}} \langle H,v\rangle$.

Given a compact convex set K in a linear topological space $\mathrm{Aff}(K)$ denotes the collection of all affine continuous real-valued functions on K. Let 1_K denote the constant function with value 1 on K. Equipped with the pointwise ordering, $\langle \mathrm{Aff}(K), 1_k\rangle$ is an ordered real vector space with order-unit. With the supremum norm $\mathrm{Aff}(K)$, as a norm-closed subspace of $C(K,\mathbb{R})$, is an "ordered real Banach space". A compact convex K is a Choquet simplex if and only if $\mathrm{Aff}(K)$ is a (uncountable) dimension group (cf. [5, Theorem 11.4]).

For any affine continuous map $\nu : K \to L$ between compact convex sets, the map $A(\nu) : \langle \mathrm{Aff}(L), 1_L\rangle \to \langle \mathrm{Aff}(K), 1_K\rangle$ defined by $A(\nu)(f) = f \circ \nu$, for $f \in \mathrm{Aff}(L)$ is an order-unit preserving positive homomorphism.

2.2. **The main result.** Suppose $\langle G,u\rangle$ is an ordered abelian group with order-unit. Then there is a natural normalized positive homomorphism $\varphi : \langle G,u\rangle \to \langle \mathrm{Aff}(S(G,u)), 1\rangle$ defined by $x \to \widehat{x}$ where $\widehat{x}(s) = s(x)$ for any state s. The map φ is called the *natural affine representation* of $\langle G,u\rangle$ and it is an embedding if and only if G is archimedean (cf. [5, Theorem 7.7]).

Note that $\mathrm{Aff}(S(\mathbb{Z}^r, u))$ is isomorphic to $\langle \mathbb{R}^r, 1\rangle$ as ordered real Banach space. Any normalized positive homomorphism $\beta : \langle \mathbb{R}^r, 1\rangle \to \langle \mathbb{R}^s, 1\rangle$ is of the form

$$\beta(x_1, \dots, x_r) = (\sum_i^r a_{i1} x_i, \dots, \sum_i^r a_{is} x_i)$$

where $a_{ij} \in \mathbb{R}$ and $\sum_i^r a_{ij} = 1$ for every $1 \le j \le s$. We say β is *rational* if each a_{ij} is a rational number.

Theorem 2.7. *Suppose G, H are in $\overrightarrow{\mathfrak{D}}$ and $u \in G$ and $v \in H$ are order-units. Then $S(G,u) \cong S(H,v)$ if and only if $\langle G,u\rangle \sim_{\mathbb{Q}} \langle H,v\rangle$.*

Proof. We only need to prove the forward direction. First assume $S(G,u)$ and $S(H,v)$ are finite-dimensional simplices. If $S(G,u) \cong S(H,v)$ then $\mathrm{Aff}(S(G,u)) \cong \mathrm{Aff}(S(H,v)) \cong \mathbb{R}^k$ for some k. Since G and H are archimedean (Proposition 2), they are isomorphic to countable subgroups of $\mathbb{R}^k$ via positive homomorphisms which send u and v to 1. This together with the fact that every countable additive subgroup of $\mathbb{R}$ is isomorphic to $\mathbb{Z}$ or $\mathbb{Q}$ implies that $\langle G,u\rangle \otimes \mathbb{Q} \cong \langle H,v\rangle \otimes \mathbb{Q}$.

' Now assume $S(G,u)$ and $S(H,v)$ are infinite-dimensional. Suppose $\langle G,u\rangle = \varinjlim(G_i, u_i, \alpha_i^j)$ and $\langle H,v\rangle = \varinjlim(H_i, v_i, \beta_i^j)$, where $\langle G_i, u_i\rangle$ and $\langle H_i, v_i\rangle$ are simplicial groups. Without loss of generality we can assume that $G_i \cong H_i \cong \mathbb{Z}^i$. Let $A = \langle \mathrm{Aff}(S(G,u)), 1\rangle$, $B = \langle \mathrm{Aff}(S(H,v)), 1\rangle$, $A_i = \langle \mathrm{Aff}(S(G_i,u_i)), 1\rangle$ and $B_i = \langle \mathrm{Aff}(S(H_i,v_i)), 1\rangle$. Let $\widetilde{\alpha}_i^j : A_i \to A_j$ denote the map $A(s(\alpha_i^j))$ and $\widetilde{\beta}_i^j : B_i \to B_j$ denote the map $A(s(\beta_i^j))$. Also let $\varphi : \langle G,u\rangle \to A$, $\psi : \langle H,v\rangle \to B$, $\varphi_i : \langle G_i,u_i\rangle \to A_i$ and $\psi : \langle H_i,v_i\rangle \to B_i$ be the respective natural affine representations. Since $\langle G,u\rangle$ and $\langle H,v\rangle$

are archimedean, the natural maps φ and ψ are embeddings and therefore we may identify $\langle G, u\rangle$ with $\langle \widetilde{G}, 1_A\rangle$ and $\langle H, v\rangle$ with $\langle \widetilde{H}, 1_B\rangle$ as ordered subgroups of A and B, respectively. We may also identify each $\langle G_i, u_i\rangle$ with $\langle \widetilde{G}_i, 1_{A_i}\rangle$, its image under the map φ_i which sends $(x_1, \ldots, x_i) \in \mathbb{Z}^i$ to $(x_1/u_{i1}, \ldots, x_i/u_{ii}) \in \mathbb{Q}^i$, where $u_i = (u_{i1}, \ldots, u_{ii})$, as an ordered subgroup of A_i of rational coordinates. It is easy to check that the diagram

$$\begin{array}{ccccccc}
\langle G_1, u_1\rangle \xrightarrow{\alpha_1^2} \ldots & & \langle G_i, u_i\rangle & \xrightarrow{\alpha_i^{i+1}} & \langle G_{i+1}, u_{i+1}\rangle \longrightarrow \ldots & & \langle G, u\rangle \\
\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_i} & & \downarrow{\scriptstyle \varphi_{i+1}} & & \downarrow{\scriptstyle \varphi} \\
\langle \widetilde{G}_1, 1_{A_1}\rangle \xrightarrow{\widetilde{\alpha}_1^2} \ldots & & \langle \widetilde{G}_i, 1_{A_i}\rangle & \xrightarrow{\widetilde{\alpha}_i^{i+1}} & \langle \widetilde{G}_{i+1}, 1_{A_{i+1}}\rangle \longrightarrow \ldots & & \langle \widetilde{G}, 1_A\rangle
\end{array}$$

commutes, i.e. $\widetilde{\alpha}_i^{i+1}(\widehat{x}) = \widehat{\alpha_i^{i+1}(x)}$ and $\widetilde{\alpha}_i^{\infty}(\widehat{x}) = \widehat{\alpha_i^{\infty}(x)}$ for every i and $x \in G_i$ (this is true for any positive homomorphism and not just the non-mixing ones). Therefore

$$(*) \qquad \langle G, u\rangle \cong \langle \widetilde{G}, 1_A\rangle \cong \varinjlim(\widetilde{G}_i, 1_{A_i}, \widetilde{\alpha}_i^j)$$

and similarly,

$$(**) \qquad \langle H, v\rangle \cong \langle \widetilde{H}, 1_B\rangle \cong \varinjlim(\widetilde{H}_i, 1_{B_i}, \widetilde{\beta}_i^j),$$

where $\widetilde{H}_i$ is a subgroup of $\mathbb{Q}^i$ in B_i. Since $S(G, u) \cong S(H, v)$, the ordered real Banach spaces $\langle A, 1\rangle$ and $\langle B, 1\rangle$ are isomorphic, which induces an approximate intertwining between the systems $(A_i, 1_{A_i}, \widetilde{\alpha}_i^j)$ and $(B_i, 1_{B_i}, \widetilde{\beta}_i^j)$ as follows. Let X_i denote the natural set of generators of $\widetilde{G}_i$ and Y_i denote the natural set of generators of $\widetilde{H}_i$ as subgroups of $\mathbb{Q}^i$ (note that $|X_i| = |Y_i| = i$). Find increasing sequences of natural numbers $\{k_i\}$ and $\{l_i\}$ and normalized rational positive homomorphisms $\gamma_i : A_{k_i} \to B_{l_i}$ and $\eta_i : B_{l_i} \to A_{k_{i+1}}$ and finitely generated subgroups of rational coordinates $E_i \subseteq \mathbb{Q}^{k_i} \subseteq A_{k_i}$ and $F_i \subseteq \mathbb{Q}^{l_i} \subseteq B_{k_i}$ such that for every i we have

- $X_{k_i} \subseteq E_i$ and $Y_{l_i} \subseteq F_i$,
- $\widetilde{\alpha}_{k_i}^{k_{i+1}}[E_i] \subseteq E_{i+1}$, $\widetilde{\beta}_{l_i}^{l_{i+1}}[F_i] \subseteq F_{i+1}$, $\gamma_i[E_i] \subseteq F_i$ and $\eta_i[F_i] \subseteq E_{i+1}$,
- $\|\eta_i \circ \gamma_i(x) - \alpha_{k_i}^{k_{i+1}}(x)\| < 1/2^i$ and $\|\gamma_{i+1} \circ \eta_i(y) - \beta_{l_i}^{l_{i+1}}(y)\| < 1/2^i$ for every $x \in E_i$ and $y \in F_i$.

Finding such an approximate intertwining between $(E_i, 1_{A_{k_i}}, \widetilde{\alpha}_i^j)$ and $(F_i, 1_{B_{l_i}}, \widetilde{\beta}_i^j)$ is an standard argument and it implies that the two sequences have isomorphic limits. Since E_i and F_i are finitely generated and $X_{k_i} \subseteq E_i$ and $Y_{l_i} \subseteq F_i$, we have $\langle E_i, 1_{A_{k_i}}\rangle \cong \langle \mathbb{Z}^{k_i}, w_i\rangle$ and $\langle F_i, 1_{B_{l_i}}\rangle \cong \langle \mathbb{Z}^{l_i}, z_i\rangle$ for some order-units w_i, z_i. Note that in particular, $E_i \cong \widetilde{G}_{k_i}$ and $F_i \cong \widetilde{H}_{l_i}$. Hence by $(*)$ and $(**)$ for some order-units $w \in G$ and $z \in H$ we have

$$\langle G, w\rangle = \varinjlim(E_i, 1_{A_{k_i}}, \widetilde{\alpha}_{k_i}^{k_j}) = \varinjlim(\mathbb{Z}^{k_i}, w_i, \alpha_{k_i}^{k_j})$$

and

$$\langle H, z\rangle = \varinjlim(F_i, 1_{B_{l_i}}, \widetilde{\beta}_{l_i}^{l_j}) = \varinjlim(\mathbb{Z}^{l_i}, z_i, \beta_{l_i}^{l_j}).$$

Therefore $\langle G, w\rangle \cong \langle H, z\rangle$. By Corollary 2.4 we have $\langle G, u\rangle \sim_{\mathbb{Q}} \langle H, v\rangle$. □

Let $\overrightarrow{\mathfrak{D}}$ also denote the class of unital AF-algebras $\mathcal{A}$ such that the dimension group $K_0(\mathcal{A})$ belongs to $\overrightarrow{\mathfrak{D}}$.

Corollary 2.8. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are unital AF-algebras in $\overrightarrow{\mathfrak{D}}$. Then $T(A) \cong T(B)$ if and only if $\mathcal{A} \otimes \mathcal{Q} \cong \mathcal{B} \otimes \mathcal{Q}$.*

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