Kurzweil-Stieltjes integral, measure equations and time scales

Milan Tvrdý

(based on joint research with **Giselle Antunes Monteiro** and **Antonín Slavík**)

Puebla, August 2020

Cordial greetings from Baroque Pearl of Central Europe

to Baroque Pearl of Central America


SEPTIEMBRE
PUEBLA

- $-\infty < a < b < \infty$, X is a Banach space,
- $f\colon [a,b] \to X$ is *regulated* on $[a,b]$, if
  $f(s+) := \lim\limits_{\tau \to s+} f(\tau) \in X$ for $s \in [a,b)$, $f(t-) := \lim\limits_{\tau \to t-} f(\tau) \in X$ for $t \in (a,b]$,
- $\Delta^+ f(s) = f(s+) - f(s)$, $\Delta^- f(t) = f(t) - f(t-)$, $\Delta f(t) = f(t+) - f(t-)$.
- $G = G([a,b],X)$ is the space of functions $f\colon [a,b] \to X$ regulated on $[a,b]$.
  (G is a Banach space with respect to the norm $\|f\|_\infty = \sup\limits_{t \in [a,b]} \|f(t)\|$).
  - regulated functions are uniform limits of finite step functions,
  - regulated functions have at most countably many points of discontinuity.
- $BV = BV([a,b],X) = \left\{ f\colon [a,b] \to X \colon \operatorname{var}_a^b f < \infty \right\}$ is the space of functions with *bounded variation* on $[a,b]$.

- $\mathcal{G} = \{\delta\colon [a,b] \to (0,1)\}$ are gauges on $[a,b]$.
- $\mathcal{P}=\{P=(D,\xi),\ D=\{a=\alpha_0<\alpha_1<\ldots<\alpha_m=b\},\ \xi=\{\xi_1,\ldots,\xi_m\}\in[a,b]^m,\ \xi_j\in[\alpha_{j-1},\alpha_j]\}$ are tagged divisions of $[a,b]$.
- $P=(D,\xi)\in\mathcal{P}$ is δ-fine if $[\alpha_{j-1},\alpha_j]\subset(\xi_j-\delta(\xi_j),\xi_j+\delta(\xi_j))$ for all j.
- For $F:[a,b]\to L(X)$, $g:[a,b]\to X$, $P=(D,\xi)\in\mathcal{P}$ define

$$S(F,dg,P)=\sum_{j=1}^{m}F(\xi_j)\,[g(\alpha_j)-g(\alpha_{j-1})]\,.$$

DEFINITION

$$I=\int_a^b F\,d[g] \quad\Longleftrightarrow\quad \begin{cases}\text{for each } \varepsilon>0 \text{ there is a gauge } \delta\in\mathcal{G} \text{ such that} \\ \qquad\qquad \Big|S(F,dg,P)-I\Big|<\varepsilon \\ \text{for every } \delta-\text{fine tagged division } P.\end{cases}$$

$$\int_c^c F\,d[g]=0.$$

- $\mathcal{G} = \{\delta\colon [a,b] \to (0,1)\}$ are gauges on $[a,b]$.
- $\mathcal{P}=\{P=(D,\xi),\ D=\{a=\alpha_0<\alpha_1<\ldots<\alpha_m=b\},\ \xi=\{\xi_1,\ldots,\xi_m\}\in[a,b]^m,\ \xi_j\in[\alpha_{j-1},\alpha_j]\}$ are tagged divisions of $[a,b]$.
- $P=(D,\xi)\in\mathcal{P}$ is δ-fine if $[\alpha_{j-1},\alpha_j]\subset(\xi_j-\delta(\xi_j),\xi_j+\delta(\xi_j))$ for all j.
- For $F:[a,b]\to L(X)$, $g:[a,b]\to X$, $P=(D,\xi)\in\mathcal{P}$ define

$$S(dF,g,P)=\sum_{j=1}^{m}[F(\alpha_j)-F(\alpha_{j-1})]\,g(\xi_j)\,.$$

DEFINITION

$$I=\int_a^b d[F]\,g \quad\Longleftrightarrow\quad \begin{cases}\text{for each } \varepsilon>0 \text{ there is a gauge } \delta\in\mathcal{G} \text{ such that}\\ \qquad\qquad \left|S(dF,g,P)-I\right|<\varepsilon\\ \text{for every } \delta-\text{fine tagged division } P.\end{cases}$$

$$\int_c^c d[F]\,g=0.$$

- RS $\cup$ improper integrals $\subset$ KS, LS $\subset$ KS,
  $X = \mathbb{R} \implies$ KS = PS.
- $F : [a,b] \to L(X)$ and $g : [a,b] \to X$ are regulated $\implies$

$$\int_a^b F \, d[g] \quad \text{and} \quad \int_a^b d[F]\, g \quad \text{exist whenever}$$

one of the functions F, g has a bounded variation.

ASSUME:

- $F,\, F_k \in G$ for $k \in \mathbb{N}$, $g \in BV$,
- $F_k \rightrightarrows F$.

THEN: $\displaystyle\int_a^t F_k\, d[g] \rightrightarrows \int_a^t F\;\; d[g]$ on $[a,b]$.

ASSUME:

- $F \in BV$, $g, g_k \in G$ for $n \in \mathbb{N}$,
- $g_k \rightrightarrows g$.

THEN: $\displaystyle\int_a^t F\, d[g_k] \rightrightarrows \int_a^t F\, d[g]$ on $[a,b]$.

ASSUME:

- $F,\, F_k \in G$, $g, g_k \in BV$ for $k \in \mathbb{N}$,
- $F_k \rightrightarrows F$, $g_k \rightrightarrows g$,
- $\alpha^* := \sup\{\mathrm{var}_a^b\, g_k : k \in \mathbb{N}\} < \infty$.

THEN: $\displaystyle\int_a^t F_k\, d[g_k] \rightrightarrows \int_a^t F\, d[g]$ on $[a,b]$.

BOUNDED CONVERGENCE THEOREM

(i) ASSUME:

- $F \in BV, \quad g, g_k \in G$ for $k \in \mathbb{N}$,
- $g_k(t) \to g(t)$ on $[a, b]$,
- $\|g_k\|_\infty \leq \gamma^* < \infty$ for $k \in \mathbb{N}$.

THEN: $\displaystyle\int_a^b d[F]\, g_k \to \int_a^b d[F]\, g.$

(ii) ASSUME:

- $g \in BV, \quad F, F_k \in G$ for $k \in \mathbb{N}$,
- $F_k(t) \to F(t)$ on $[a, b]$,
- $\|F_k\|_\infty \leq \varkappa^* < \infty$ for $k \in \mathbb{N}$.

THEN: $\displaystyle\int_a^b F_k\, d[g] \to \int_a^b F\, d[g].$

A sequence $\{f_k, g_k\}$ is **equi-integrable** if:

- $\displaystyle\int_a^b f_k \, d[g_k]$ exists for each $k \in \mathbb{N}$,
- for every $\varepsilon > 0$, there is a gauge δ on $[a, b]$ such that

$$\left| \int_a^b f_k \, d[g_k] - S(f_k, dg_k, P) \right| < \varepsilon$$

holds for each δ-fine partition P of $[a, b]$ and for every $k \in \mathbb{N}$.

A sequence $\{f_k, g_k\}$ is **equi-integrable** if:

- $\int_a^b f_k \; d[g_k]$ exists for each $k \in \mathbb{N}$,
- for every $\varepsilon > 0$, there is a gauge δ on $[a, b]$ such that

$$\left| \int_a^b f_k \; d[g_k] - S(f_k, dg_k, P) \right| < \varepsilon$$

holds for each δ-fine partition P of $[a, b]$ and for every $k \in \mathbb{N}$.

Equi-integrability Convergence Theorem

ASSUME: $\{f_k, g_k\}$ is equi-integrable,

$$f_k(t) \to f(t) \quad \text{and} \quad g_k(t) \to g(t) \quad \text{for} \;\; t \in [a, b].$$

THEN:
$$\int_a^b f \; d[g] = \lim_{k \to \infty} \int_a^b f_k \; d[g_k].$$

A sequence $\{f_k, g_k\}$ is **equi-integrable** if:

- $\int_a^b f_k \; d[g_k]$ exists for each $k \in \mathbb{N}$,
- for every $\varepsilon > 0$, there is a gauge δ on $[a,b]$ such that

$$\left| \int_a^b f_k \; d[g_k] - S(f_k, dg_k, P) \right| < \varepsilon$$

holds for each δ-fine partition P of $[a,b]$ and for every $k \in \mathbb{N}$.

Equi-integrability Convergence Theorem

ASSUME: $\{f_k, g_k\}$ is equi-integrable,

$$f_k(t) \to f(t) \quad \text{and} \quad g_k(t) \to g(t) \quad \text{for} \;\; t \in [a,b].$$

THEN: $$\int_a^b f \; d[g] = \lim_{k\to\infty} \int_a^b f_k \; d[g_k].$$

MOREOVER: $\int_a^t f_k \; d[g_k] \rightrightarrows \int_a^t f \; d[g]$ on $[a,b]$ whenever $\{g_k\}$ is uniformly bounded on $[a,b]$.

Preiss-Schwabik-Kurzweil (PSK) Convergence Theorem

<u>ASSUME</u>: $\dim X < \infty$, $g \in BV$ and f, $\{f_k\}$ are such that

(i) $\lim_{k\to\infty} f_k(t) = f(t)$ for every $t \in [a,b]$,

(ii) $\int_a^b f_k \, d[g]$ exists for every $k \in \mathbb{N}$,

(iii) $\Big| \sum_{j=1}^{\ell} \int_{\sigma_{j-1}}^{\sigma_j} f_{m_j} \, d[g] \Big| \le K < \infty$

for any $\ell \in \mathbb{N}$, $\{\sigma_0, \dots, \sigma_\ell\} \in \mathcal{D}[a,b]$ and $\{m_j\}_{j=1}^{\ell} \subset \mathbb{N}^\ell$.

<u>THEN</u>: $\{f_k, g\}$ is **equi-integrable** and $\int_a^t f_k \, d[g] \rightrightarrows \int_a^t f \, d[g]$.

Preiss-Schwabik-Kurzweil (PSK) Convergence Theorem

ASSUME: $\dim X < \infty$, $g \in BV$ and f, $\{f_k\}$ are such that

(i) $\lim\limits_{k\to\infty} f_k(t) = f(t)$ for every $t \in [a,b]$,

(ii) $\displaystyle\int_a^b f_k \, d[g]$ exists for every $k \in \mathbb{N}$,

(iii) $\displaystyle\Big|\sum_{j=1}^{\ell} \int_{\sigma_{j-1}}^{\sigma_j} f_{m_j} \, d[g]\Big| \le K < \infty$

for any $\ell \in \mathbb{N}$, $\{\sigma_0, \dots, \sigma_\ell\} \in \mathcal{D}[a,b]$ and $\{m_j\}_{j=1}^{\ell} \subset \mathbb{N}^\ell$.

THEN: $\{f_k, g\}$ is **equi-integrable** and $\displaystyle\int_a^t f_k \, d[g] \rightrightarrows \int_a^t f \, d[g]$.

If $\dim X < \infty$, then Bounded Convergence Theorem follows from PSK Theorem.

Preiss-Schwabik-Kurzweil (PSK) Convergence Theorem

<u>ASSUME</u>: $\dim X < \infty$, $g \in BV$ and f, $\{f_k\}$ are such that

(i) $\lim\limits_{k\to\infty} f_k(t) = f(t)$ for every $t \in [a,b]$,

(ii) $\displaystyle\int_a^b f_k \, d[g]$ exists for every $k \in \mathbb{N}$,

(iii) $\displaystyle\Big|\sum_{j=1}^{\ell} \int_{\sigma_{j-1}}^{\sigma_j} f_{m_j} \, d[g]\Big| \le K < \infty$

for any $\ell \in \mathbb{N}$, $\{\sigma_0, \dots, \sigma_\ell\} \in \mathcal{D}[a,b]$ and $\{m_j\}_{j=1}^{\ell} \subset \mathbb{N}^\ell$.

<u>THEN</u>: $\{f_k, g\}$ is **equi-integrable** and $\displaystyle\int_a^t f_k \, d[g] \rightrightarrows \int_a^t f \, d[g]$.

If $\dim X < \infty$, then Bounded Convergence Theorem follows from PSK Theorem.

Open questions:

- $\dim X$ is not finite,
- $\{g_k\}$ instead of g.

$$\text{(L)} \qquad x(t) = \widetilde{x} + \int_{t_0}^{t} d[A]\,x + f(t) - f(t_0)\,, \quad t \in [a,b]\,.$$

$$\text{(L)} \qquad x(t) = \widetilde{x} + \int_{t_0}^{t} d[A]\,x + f(t) - f(t_0)\,, \quad t \in [a,b]\,.$$

THEOREM (Schwabik)

ASSUME:

- $A \in BV$ and $t_0 \in [a,b]$.
- $$\big[I - \Delta^- A(t)\big]^{-1} \in \mathcal{L}(X) \text{ for } t \in (t_0, b\,],$$
  $$\big[I + \Delta^+ A(s)\big]^{-1} \in \mathcal{L}(X) \text{ for } s \in [a, t_0).$$

THEN: For each $f \in G$ and $\widetilde{x} \in X$, (L) has 1! solution $x \in G$.

$$x_k(t) = \widetilde{x}_k + \int_a^t d[A_k]\,x + f_k(t) - f_k(a), \quad t \in [a,b].$$

$$x(t) = \widetilde{x} + \int_a^t d[A]\,x + f(t) - f(a), \quad t \in [a,b].$$

$$x_k(t) = \widetilde{x}_k + \int_a^t d[A_k]\,x + f_k(t) - f_k(a), \quad t \in [a,b].$$

$$x(t) = \widetilde{x} + \int_a^t d[A]\,x + f(t) - f(a), \quad t \in [a,b].$$

$$A_k, A \in BV, \quad f_k, f \in G, \quad \widetilde{x}_k, \widetilde{x} \in X \quad \text{for } k \in \mathbb{N}.$$

$$x_k(t) = \widetilde{x}_k + \int_a^t d[A_k]\,x + f_k(t) - f_k(a), \quad t \in [a,b].$$

$$x(t) = \widetilde{x} + \int_a^t d[A]\,x + f(t) - f(a), \quad t \in [a,b].$$

$$A_k, A \in BV, \quad f_k, f \in G, \quad \widetilde{x}_k, \widetilde{x} \in X \quad \text{for } k \in \mathbb{N}.$$

THEOREM

ASSUME:

- $\left[I - \Delta^- A(t)\right]^{-1} \in \mathcal{L}(X)$ for $t \in (a,b]$,
- $A_k \rightrightarrows A$ on $[a,b]$, $\alpha^* := \sup\{\mathrm{var}_a^b A_k : k \in \mathbb{N}\} < \infty$,
- $\widetilde{x}_k \to \widetilde{x}$, $f_k \rightrightarrows f$ on $[a,b]$.

THEN: $x_k \rightrightarrows x$ on $[a,b]$.

$$x_k' = P_k(t)\,x_k, \qquad x_k(a) = \widetilde{x}\,,$$
$$x' = \ P(t)\,x, \qquad x(a) = \widetilde{x}\,.$$

Kurzweil & Vorel, 1957

ASSUME:

- $\|P_k\|_1 \leq p^* < \infty$ for $k \in \mathbb{N}$,
- $\displaystyle\int_a^t P_k\, ds \rightrightarrows \int_a^t P\, ds.$

THEN: $x_k \rightrightarrows x$ on $[a,b]$.

Opial, 1967

ASSUME:

- $\displaystyle\lim_{k\to\infty}\left[\left\|\int_a^t P_k\, ds - \int_a^t P\, ds\right\|_\infty \big(1 + \|P_k\|_1\big)\right] = 0.$

THEN: $x_k \rightrightarrows x$ on $[a,b]$.

$$x_k(t) = \widetilde{x} + \int_a^t d[A_k]\, x_k(s) + f_k(t) - f_k(a), \quad t \in [a,b], \qquad (\mathrm{GL}_k)$$

$$x(t) = \widetilde{x} + \int_a^t d[A]\, x(s) + f(t) - f(a), \qquad t \in [a,b]. \qquad (\mathrm{GL})$$

THEOREM

ASSUME: $A_k \in BV$, $f_k \in G$ for $k \in \mathbb{N}$,

- $A \in BV$, $f \in BV$, $\widetilde{x} \in X$,
- $[I - \Delta^- A(t)]^{-1} \in L(X)$ for $t \in (a,b]$,
- $\lim_{k\to\infty} (1 + \mathrm{var}_a^b\, A_k)\|A_k - A\|_\infty = 0$,
- $\lim_{k\to\infty} \Big(1 + \mathrm{var}_a^b\, A_k\Big)\|f_k - f\|_\infty = 0$.

THEN: (GL) has a unique solution $x \in BV$ on $[a,b]$.

MOREOVER: (GL_k) has a unique solution x_k for k sufficiently large and $x_k \rightrightarrows x$.

Let

$$A(t) = \begin{pmatrix} 0 & P(t) \\ Q(t) & 0 \end{pmatrix}, \quad f(t) = \begin{pmatrix} g(t) \\ h(t) \end{pmatrix}, \quad X = Y \times Y \quad \text{and} \quad \widetilde{x} = \begin{pmatrix} \widetilde{y} \\ \widetilde{z} \end{pmatrix},$$

where $P, Q \in BV([a, b], L(Y))$ and $g, h \in BV([a, b], Y)$, Y is a Banach space, $\widetilde{y}, \widetilde{z} \in Y$.

Let

$$A(t) = \begin{pmatrix} 0 & P(t) \\ Q(t) & 0 \end{pmatrix}, \quad f(t) = \begin{pmatrix} g(t) \\ h(t) \end{pmatrix}, \quad X = Y \times Y \quad \text{and} \quad \widetilde{x} = \begin{pmatrix} \widetilde{y} \\ \widetilde{z} \end{pmatrix},$$

where $P, Q \in BV([a,b], L(Y))$ and $g, h \in BV([a,b], Y)$, Y is a Banach space, $\widetilde{y}, \widetilde{z} \in Y$.

Then

$$x(t) = \widetilde{x} + \int_a^t d[A]\, x + f(t) - f(a)$$

reduces to

Let

$$A(t) = \begin{pmatrix} 0 & P(t) \\ Q(t) & 0 \end{pmatrix}, \quad f(t) = \begin{pmatrix} g(t) \\ h(t) \end{pmatrix}, \quad X = Y \times Y \quad \text{and} \quad \widetilde{x} = \begin{pmatrix} \widetilde{y} \\ \widetilde{z} \end{pmatrix},$$

where $P, Q \in BV([a,b], L(Y))$ and $g, h \in BV([a,b], Y)$, Y is a Banach space, $\widetilde{y}, \widetilde{z} \in Y$.

Then

$$x(t) = \widetilde{x} + \int_a^t d[A]\, x + f(t) - f(a)$$

reduces to

$$y(t) = \widetilde{y} + \int_a^t d[P]\, z + g(t) - g(a), \quad z(t) = \widetilde{z} + \int_a^t d[Q]\, y + h(t) - h(a)$$

Let

$$A(t) = \begin{pmatrix} 0 & P(t) \\ Q(t) & 0 \end{pmatrix}, \quad f(t) = \begin{pmatrix} g(t) \\ h(t) \end{pmatrix}, \quad X = Y \times Y \quad \text{and} \quad \widetilde{x} = \begin{pmatrix} \widetilde{y} \\ \widetilde{z} \end{pmatrix},$$

where $P, Q \in BV([a,b], L(Y))$ and $g, h \in BV([a,b], Y)$, Y is a Banach space, $\widetilde{y}, \widetilde{z} \in Y$.

Then

$$x(t) = \widetilde{x} + \int_a^t d[A]\, x + f(t) - f(a)$$

reduces to

$$y(t) = \widetilde{y} + \int_a^t d[P]\, z + g(t) - g(a), \quad z(t) = \widetilde{z} + \int_a^t d[Q]\, y + h(t) - h(a)$$

and $\quad [I_X - \Delta^- A(t)]^{-1} \in L(X) \quad$ iff

either $[I_Y - \Delta^- Q(t)\, \Delta^- P(t)]^{-1} \in L(Y)$ or $[I_Y - \Delta^- P(t)\, \Delta^- Q(t)]^{-1} \in L(Y)$.

Consider systems

$$\left.\begin{aligned} y_k(t) &= \widetilde{y}_k + \int_a^t d[P_k]\, z_k + g_k(t) - g_k(a), \\ z_k(t) &= \widetilde{z}_k + \int_a^t d[Q_k]\, y_k + h_k(t) - h_k(a), \end{aligned}\right\} \qquad (S_k)$$

$$\left.\begin{aligned} y(t) &= \widetilde{y} + \int_a^t d[P]\, z + g(t) - g(a), \\ z(t) &= \widetilde{z} + \int_a^t d[Q]\, y + h(t) - h(a). \end{aligned}\right\} \qquad (S)$$

Consider systems

$$\left.\begin{aligned} y_k(t) &= \widetilde{y}_k + \int_a^t d[P_k]\, z_k + g_k(t) - g_k(a), \\ z_k(t) &= \widetilde{z}_k + \int_a^t d[Q_k]\, y_k + h_k(t) - h_k(a), \end{aligned}\right\} \qquad (S_k)$$

$$\left.\begin{aligned} y(t) &= \widetilde{y} + \int_a^t d[P]\, z + g(t) - g(a), \\ z(t) &= \widetilde{z} + \int_a^t d[Q]\, y + h(t) - h(a). \end{aligned}\right\} \qquad (S)$$

COROLLARY

<u>ASSUME</u>: $P, P_k, Q, Q_k \in BV([a,b], L(Y))$, $g, h \in BV([a,b], Y)$, $g_k, h_k \in G([a,b], Y)$ $\widetilde{y}, \widetilde{z} \in Y$,

- $[I_Y - \Delta^- Q(t)\,\Delta^- P(t)]^{-1} \in L(Y)$ or $[I_Y - \Delta^- P(t)\,\Delta^- Q(t)]^{-1} \in L(Y)$ for $t \in (a,b]$,
- $\lim_{k\to\infty} \|\widetilde{y}_k - \widetilde{y}\|_Y = 0$, $\lim_{k\to\infty} \|\widetilde{z}_k - \widetilde{z}\|_Y = 0$,
- $\lim_{k\to\infty} \big(1 + \operatorname{var}_a^b P_k + \operatorname{var}_a^b Q_k\big)\big(\|P_k - P\|_\infty + \|Q_k - Q\|_\infty\big) = 0$,
- $\lim_{k\to\infty} \big(1 + \operatorname{var}_a^b P_k + \operatorname{var}_a^b Q_k\big)\big(\|g_k - g\|_\infty + \|h_k - h\|_\infty\big) = 0$.

Consider systems

$$\left.\begin{aligned} y_k(t) &= \widetilde{y}_k + \int_a^t d[P_k]\, z_k + g_k(t) - g_k(a), \\ z_k(t) &= \widetilde{z}_k + \int_a^t d[Q_k]\, y_k + h_k(t) - h_k(a), \end{aligned}\right\} \qquad (S_k)$$

$$\left.\begin{aligned} y(t) &= \widetilde{y} + \int_a^t d[P]\, z + g(t) - g(a), \\ z(t) &= \widetilde{z} + \int_a^t d[Q]\, y + h(t) - h(a). \end{aligned}\right\} \qquad (S)$$

COROLLARY

ASSUME: $P, P_k, Q, Q_k \in BV([a,b], L(Y))$, $g, h \in BV([a,b], Y)$, $g_k, h_k \in G([a,b], Y)$ $\widetilde{y}, \widetilde{z} \in Y$,

- $[I_Y - \Delta^- Q(t)\, \Delta^- P(t)]^{-1} \in L(Y)$ or $[I_Y - \Delta^- P(t)\, \Delta^- Q(t)]^{-1} \in L(Y)$ for $t \in (a,b]$,
- $\lim_{k\to\infty} \|\widetilde{y}_k - \widetilde{y}\|_Y = 0$, $\lim_{k\to\infty} \|\widetilde{z}_k - \widetilde{z}\|_Y = 0$,
- $\lim_{k\to\infty} \left(1 + \mathrm{var}_a^b P_k + \mathrm{var}_a^b Q_k\right) \left(\|P_k - P\|_\infty + \|Q_k - Q\|_\infty\right) = 0$,
- $\lim_{k\to\infty} \left(1 + \mathrm{var}_a^b P_k + \mathrm{var}_a^b Q_k\right) \left(\|g_k - g\|_\infty + \|h_k - h\|_\infty\right) = 0$.

THEN:

- (S) has a unique solution $(y, z) \in BV([a,b], Y \times Y)$ on $[a,b]$,
- (S_k) has a unique solution $(y_k, z_k) \in G([a,b], Y \times Y)$ on $[a,b]$ for k sufficiently large,
- $\lim_{k\to\infty} \left(\|y_k - y\|_\infty + \|z_k - z\|_\infty\right) = 0$.

Meng and Zhang:

$$dy^{\bullet} + d[\mu_k(t)]\,y = 0, \quad y(0) = \widetilde{y},\ y^{\bullet}(0) = \widetilde{z},\ k \in \mathbb{N}\,, \tag{MZ$_k$}$$

where $\mu_k \in BV$ are right-continuous, $\widetilde{y}, \widetilde{z} \in \mathbb{R}$, $y \in AC$ and $y^{\bullet}$ is generalized right-derivative of y.

$y^{\bullet}(t) = y'(t)$ for a.e. $t \in [a,b]$ and $y^{\bullet}(t) = \lim_{s \to t+} \frac{y(s) - y(t)}{s - t}$ for all $t \in [a,b)$.

Meng and Zhang:

$$dy^{\bullet} + d[\mu_k(t)]\,y = 0, \quad y(0) = \widetilde{y},\ y^{\bullet}(0) = \widetilde{z},\ k \in \mathbb{N}\,, \tag{MZ$_k$}$$

where $\mu_k \in BV$ are right-continuous, $\widetilde{y}, \widetilde{z} \in \mathbb{R}$, $y \in AC$ and $y^{\bullet}$ is generalized right-derivative of y.

They proved that the weak* convergence $\mu_k \to \mu$

Meng and Zhang:

$$dy^{\bullet} + d[\mu_k(t)]\,y = 0, \quad y(0) = \widetilde{y},\ y^{\bullet}(0) = \widetilde{z},\ k \in \mathbb{N}, \qquad (\mathrm{MZ}_k)$$

where $\mu_k \in BV$ are right-continuous, $\widetilde{y}, \widetilde{z} \in \mathbb{R}$, $y \in AC$ and $y^{\bullet}$ is generalized right-derivative of y.

They proved that the weak* convergence $\mu_k \to \mu$

$\mu_k \rightharpoonup^* \mu$ in $BV_R[0,1]$ iff :

- $\mu_k(1) \to \mu(1)$,
- $\mathrm{var}_0^1\, \mu_k \le \alpha^* < \infty$ for $k \in \mathbb{N}$,
- $\int_\alpha^\beta \mu_k\, dt \to \int_\alpha^\beta \mu_0\, dt$ for all $[\alpha, \beta] \subset [0,1]$.

Meng and Zhang:

$$d\,y^{\bullet} + d[\mu_k(t)]\,y = 0, \quad y(0) = \widetilde{y},\ y^{\bullet}(0) = \widetilde{z},\ k \in \mathbb{N}\,, \tag{MZ$_k$}$$

where $\mu_k \in BV$ are right-continuous, $\widetilde{y}, \widetilde{z} \in \mathbb{R}$, $y \in AC$ and $y^{\bullet}$ is generalized right-derivative of y.

They proved that the weak* convergence $\mu_k \to \mu$ yields

$$y_k \rightrightarrows y,\ y_k^{\bullet} \to y^{\bullet} \text{ in weak* topology and } y_k^{\bullet}(1) \to y^{\bullet}(1).$$

Meng and Zhang:

$$d\,y^{\bullet} + d[\mu_k(t)]\,y = 0, \quad y(0) = \widetilde{y},\ y^{\bullet}(0) = \widetilde{z},\ k \in \mathbb{N}\,, \tag{MZ$_k$}$$

where $\mu_k \in BV$ are right-continuous, $\widetilde{y}, \widetilde{z} \in \mathbb{R}$, $y \in AC$ and $y^{\bullet}$ is generalized right-derivative of y.

They proved that the weak* convergence $\mu_k \to \mu$ yields

$$y_k \rightrightarrows y,\ y_k^{\bullet} \to y^{\bullet} \text{ in weak* topology and } y_k^{\bullet}(1) \to y^{\bullet}(1).$$

(S$_k$) reduce to (MZ$_k$) when

$$[a,b] = [0,1],\ X = \mathbb{R},\ P_k(t) = t,\ Q_k(t) = \mu_k(t),\ g_k, h_k \text{ are constant and } x_k = \big(y_k, y_k^{\bullet}\big)^T.$$

Similarly, (S) reduces to

$$d\,y^{\bullet} + d[\mu(t)]\,y = 0, \quad y(0) = \widetilde{y},\ y^{\bullet}(0) = \widetilde{z} \tag{MZ}$$

if

$$[a,b] = [0,1],\ X = \mathbb{R},\ P(t) = t,\ Q(t) = \mu(t),\ g, h \text{ are constant and } x = \big(y, y^{\bullet}\big)^T.$$

Meng and Zhang:

$$d\,y^{\bullet} + d[\mu_k(t)]\,y = 0, \quad y(0) = \widetilde{y},\, y^{\bullet}(0) = \widetilde{z},\; k \in \mathbb{N}\,, \tag{MZ$_k$}$$

where $\mu_k \in BV$ are right-continuous, $\widetilde{y}, \widetilde{z} \in \mathbb{R}$, $y \in AC$ and $y^{\bullet}$ is generalized right-derivative of y.

They proved that the weak* convergence $\mu_k \to \mu$ yields

$$y_k \rightrightarrows y,\; y_k^{\bullet} \to y^{\bullet} \text{ in weak* topology and } y_k^{\bullet}(1) \to y^{\bullet}(1).$$

(S_k) reduce to (MZ_k) when

$$[a,b] = [0,1],\; X = \mathbb{R},\; P_k(t) = t,\; Q_k(t) = \mu_k(t),\; g_k, h_k \text{ are constant and } x_k = \big(y_k, y_k^{\bullet}\big)^T.$$

Similarly, (S) reduces to

$$d\,y^{\bullet} + d[\mu(t)]\,y = 0, \quad y(0) = \widetilde{y},\, y^{\bullet}(0) = \widetilde{z} \tag{MZ}$$

if

$$[a,b] = [0,1],\; X = \mathbb{R},\; P(t) = t,\; Q(t) = \mu(t),\; g, h \text{ are constant and } x = \big(y, y^{\bullet}\big)^T.$$

Our **COROLLARY** yields

$$\lim_{k\to\infty} \big(\|y_k - y\|_\infty + \|y_k^{\bullet} - y^{\bullet}\|_\infty\big) = 0.$$

whenever

$$\lim_{k\to\infty} (1 + \operatorname{var}_0^1 \mu_k)\|\mu_k - \mu\|_\infty = 0.$$

Time scales: nonempty and closed subset $\mathbb{T}$ of $\mathbb{R}$.

Time scales: nonempty and closed subset $\mathbb{T}$ of $\mathbb{R}$.

Forward jump operator:

$$\sigma(t) := \begin{cases} \inf \big((t,b] \cap \mathbb{T}\big) & \text{if } t \in \mathbb{T} \text{ and } t < \sup \mathbb{T}, \\ t & \text{if } t = \sup \mathbb{T}. \end{cases}$$

Time scales: nonempty and closed subset $\mathbb{T}$ of $\mathbb{R}$.

Forward jump operator:

$$\sigma(t) := \begin{cases} \inf \big((t,b] \cap \mathbb{T}\big) & \text{if } t \in \mathbb{T} \text{ and } t < \sup \mathbb{T}, \\ t & \text{if } t = \sup \mathbb{T}. \end{cases}$$

If $\sigma(t) = t$ and $t < \sup \mathbb{T}$, then t is said to be **right-dense**.

Time scales: nonempty and closed subset $\mathbb{T}$ of $\mathbb{R}$.

Forward jump operator:

$$\sigma(t) := \begin{cases} \inf\big((t,b]\cap\mathbb{T}\big) & \text{if } t\in\mathbb{T} \text{ and } t<\sup\mathbb{T}, \\ t & \text{if } t=\sup\mathbb{T}. \end{cases}$$

If $\sigma(t)=t$ and $t<\sup\mathbb{T}$, then t is said to be **right-dense**.

$f:\mathbb{T}\to\mathbb{R}$ is **rd-continuous**, if it is regulated on $\mathbb{T}$ and continuous at all right-dense points in $\mathbb{T}$.

Time scales: nonempty and closed subset $\mathbb{T}$ of $\mathbb{R}$.

Forward jump operator:

$$\sigma(t) := \begin{cases} \inf\big((t,b]\cap\mathbb{T}\big) & \text{if } t\in\mathbb{T} \text{ and } t<\sup\mathbb{T}, \\ t & \text{if } t=\sup\mathbb{T}. \end{cases}$$

If $\sigma(t)=t$ and $t<\sup\mathbb{T}$, then t is said to be **right-dense**.

$f:\mathbb{T}\to\mathbb{R}$ is **rd-continuous**, if it is regulated on $\mathbb{T}$ and continuous at all right-dense points in $\mathbb{T}$.

A couple of functions $\delta=(\delta_L,\delta_R)$ positive on $[a,b]\cap\mathbb{T}$ is a $\Delta-$**gauge** on $[a,b]\cap\mathbb{T}$ if

$$\sigma(t)\le t+\delta_R(t) \text{ for } t\in[a,b]\cap\mathbb{T}.$$

(D,ξ) is a **tagged division** of $[a,b]\cap\mathbb{T}$ if

$$\xi=\{\xi_1,\ldots\xi_{\nu(D)}\} \quad \text{and} \quad \xi_i\in[\alpha_{i-1},\alpha_i)\cap\mathbb{T} \quad \text{for } i\in\{1,\ldots,\nu(D)\}.$$

If $\delta=(\delta_L,\delta_R)$ is a gauge on $[a,b]\cap\mathbb{T}$, then (D,ξ) is δ-**fine** if

$$[\alpha_{i-1},\alpha_i]\subset[\xi_i-\delta_L(\xi_i),\xi_i+\delta_R(\xi_i)] \quad \text{for all } i.$$

Time scales: nonempty and closed subset $\mathbb{T}$ of $\mathbb{R}$.

Forward jump operator:
$$\sigma(t) := \begin{cases} \inf\big((t,b]\cap\mathbb{T}\big) & \text{if } t\in\mathbb{T} \text{ and } t<\sup\mathbb{T}, \\ t & \text{if } t=\sup\mathbb{T}. \end{cases}$$

If $\sigma(t)=t$ and $t<\sup\mathbb{T}$, then t is said to be **right-dense**.

$f:\mathbb{T}\to\mathbb{R}$ is **rd-continuous**, if it is regulated on $\mathbb{T}$ and continuous at all right-dense points in $\mathbb{T}$.

A couple of functions $\delta=(\delta_L,\delta_R)$ positive on $[a,b]\cap\mathbb{T}$ is a $\Delta-$**gauge** on $[a,b]\cap\mathbb{T}$ if
$$\sigma(t)\le t+\delta_R(t) \text{ for } t\in[a,b]\cap\mathbb{T}.$$

(D,ξ) is a **tagged division** of $[a,b]\cap\mathbb{T}$ if
$$\xi=\{\xi_1,\dots\xi_{\nu(D)}\} \quad\text{and}\quad \xi_i\in[\alpha_{i-1},\alpha_i)\cap\mathbb{T} \quad\text{for } i\in\{1,\dots,\nu(D)\}.$$

If $\delta=(\delta_L,\delta_R)$ is a gauge on $[a,b]\cap\mathbb{T}$, then (D,ξ) is δ-**fine** if
$$[\alpha_{i-1},\alpha_i]\subset[\xi_i-\delta_L(\xi_i),\xi_i+\delta_R(\xi_i)] \quad\text{for all } i.$$

DEFINITION (Kurzweil-Stieltjes Δ-integral)

Let $f,g:[a,b]\cap\mathbb{T}\to\mathbb{R}$, then
$$I=\int_a^b f\,\Delta g \qquad (\textbf{KS } \Delta\textbf{-integral})$$
iff for every $\varepsilon>0$ there is a gauge δ such that
$$\Big|\sum_{i=1}^{\nu(D)} f(\xi_i)\,[g(\alpha_i)-g(\alpha_{i-1})]-I\Big|<\varepsilon \quad \text{for all } \delta-\text{fine tagged divisions } (D,\xi) \text{ of } [a,b]\cap\mathbb{T}.$$

Put $\widetilde{\sigma}(t) := \inf\left([t,b] \cap \mathbb{T}\right)$ **for** $t \in [a,b]$.

Put $\widetilde{\sigma}(t) := \inf\big([t,b]\cap\mathbb{T}\big)$ **for** $t \in [a,b]$.

$\widetilde{\sigma}$ is nondecreasing, left-continuous and

$$\Delta^+\widetilde{\sigma}(t) = \begin{cases} \sigma(t) - t & \text{if } t \in \mathbb{T}, \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION (Slavík)

ASSUME: $f, g : [a,b]\cap\mathbb{T} \to \mathbb{R}^n$ and $\widetilde{f} = f$ on $[a,b]\cap\mathbb{T}$.

THEN:

$$\int_a^b f(t)\,\Delta g(t) \quad \text{exists} \quad \Longleftrightarrow \quad \int_a^b \widetilde{f}(t)\,d[g(\widetilde{\sigma}(t))] \quad \text{exists}$$

and, in such a case,

$$\int_a^b f(t)\,\Delta g(t) = \int_a^b \widetilde{f}(t)\,d[g(\widetilde{\sigma}(t)).$$

Consider equation

$$y(t) = y(t_0) + \int_a^t f(y(s), s)\,\Delta s \quad \text{for } t \in \mathbb{T}\,, \tag{D}$$

where $t_0 \in \mathbb{T}$, $f : \mathbb{R}^n \times \mathbb{T} \to \mathbb{R}^n$ and $y : \mathbb{T} \to \mathbb{R}^n$.

Consider equation

$$y(t) = y(t_0) + \int_a^t f(y(s), s)\,\Delta s \quad \text{for } t \in \mathbb{T}\,, \tag{D}$$

where $t_0 \in \mathbb{T}$, $f : \mathbb{R}^n \times \mathbb{T} \to \mathbb{R}^n$ and $y : \mathbb{T} \to \mathbb{R}^n$.

THEOREM (Slavík)

ASSUME: $a, b \in \mathbb{T}$, $t_0 \in [a, b] \cap \mathbb{T}$, $B \subset \mathbb{R}^n$, $f : B \times ([a, b] \cap \mathbb{T}) \to \mathbb{R}^n$, $\widetilde{f} : B \times [a, b] \to \mathbb{R}^n$ and

$$\widetilde{f}(y, t) = f(y, t) \quad \text{for } (y, t) \in B \times ([a, b] \cap \mathbb{T}).$$

THEN:

- if $y : [a, b] \cap \mathbb{T} \to \mathbb{R}^n$ is a solution of (D) on $[a, b] \cap \mathbb{T}$, then $x = y \circ \widetilde{\sigma}$ is a solution of

$$x(t) = x(t_0) + \int_{t_0}^t \widetilde{f}(x(s), s)\, d[\widetilde{\sigma}(s)] \quad \text{for } t \in [a, b]\,. \tag{G}$$

- If $x : [a, b] \to \mathbb{R}^n$ satisfies (G), then there is a solution y of (D) such that $x = y \circ \widetilde{\sigma}$.

Let $P : [a,b] \cap \mathbb{T} \longrightarrow \mathcal{L}(\mathbb{R}^n)$ and $q : [a,b] \cap \mathbb{T} \longrightarrow \mathbb{R}^n$. Then

$$y(t) = y(t_0) + \int_{t_0}^{t} \big[P(s)\,y(s) + q(s)\big]\,\Delta s\,, \quad t \in [a,b] \cap \mathbb{T}\,, \tag{LD}$$

is equivalent (in the above sense) to

$$x(t) = x(t_0) + \int_{t_0}^{t} d[A(s)]\,x(s) + b(t) - b(t_0) \quad \text{for } t \in [a,b]\,, \tag{LG}$$

where

$$A(t) = \int_{t_0}^{t} P\; d[\widetilde{\sigma}] \quad \text{and} \quad b(t) = \int_{t_0}^{t} q\; d[\widetilde{\sigma}]\,.$$

Let $P : [a,b] \cap \mathbb{T} \to \mathcal{L}(\mathbb{R}^n)$ and $q : [a,b] \cap \mathbb{T} \to \mathbb{R}^n$. Then

$$y(t) = y(t_0) + \int_{t_0}^{t} \big[P(s)\, y(s) + q(s)\big]\, \Delta s\,, \quad t \in [a,b] \cap \mathbb{T}\,, \tag{LD}$$

is equivalent (in the above sense) to

$$x(t) = x(t_0) + \int_{t_0}^{t} d[A(s)]\, x(s) + b(t) - b(t_0) \quad \text{for } t \in [a,b]\,, \tag{LG}$$

where

$$A(t) = \int_{t_0}^{t} P\; d[\widetilde{\sigma}] \quad \text{and} \quad b(t) = \int_{t_0}^{t} q\; d[\widetilde{\sigma}]\,.$$

A, b are left-continuous, $\Delta^+ A(t) = \begin{cases} 0 & \text{if } t \notin \mathbb{T}, \\ P(t)\,(\sigma(t) - t) & \text{if } t \in \mathbb{T}, \end{cases}$

Let $P : [a,b]\cap\mathbb{T} \to \mathcal{L}(\mathbb{R}^n)$ and $q : [a,b]\cap\mathbb{T}\to\mathbb{R}^n$. Then

$$y(t) = y(t_0) + \int_{t_0}^{t} \big[P(s)\,y(s) + q(s)\big]\,\Delta s\,, \quad t\in[a,b]\cap\mathbb{T}\,, \tag{LD}$$

is equivalent (in the above sense) to

$$x(t) = x(t_0) + \int_{t_0}^{t} d[A(s)]\,x(s) + b(t) - b(t_0) \quad \text{for } t\in[a,b]\,, \tag{LG}$$

where

$$A(t) = \int_{t_0}^{t} P\; d[\widetilde{\sigma}] \quad \text{and} \quad b(t) = \int_{t_0}^{t} q\; d[\widetilde{\sigma}]\,.$$

A, b are left-continuous, $\Delta^+ A(t) = \begin{cases} 0 & \text{if } t\notin\mathbb{T}, \\ P(t)\,(\sigma(t)-t) & \text{if } t\in\mathbb{T}, \end{cases}$

THEOREM

ASSUME:

- P and q are Kurzweil Δ-integrable on $[a,b]\cap\mathbb{T}$,
- $(I + P(t)\,(\sigma(t)-t))$ is invertible for $t\in[a,t_0)\cap\mathbb{T}$,
- there is a Kurzweil Δ-integrable function $m : [a,b]\cap\mathbb{T}\to\mathbb{R}$ such that

$$\left|\int_{t_1}^{t_2} P(s)\,\Delta s\right| \le \int_{t_1}^{t_2} m(s)\,\Delta s \quad \text{for } t_1,t_2\in[a,b]\cap\mathbb{T} \text{ with } t_1\le t_2.$$

THEN: (LD) has a unique solution $y : [a,b]\cap\mathbb{T}\to\mathbb{R}^n$.

$$y(t) = \widetilde{y} + \int_a^t \big[P(s)\,y(s) + q(s)\big]\,\Delta s\,, \qquad t \in [a,b] \cap \mathbb{T}\,, \tag{LD}$$

$$y(t) = \widetilde{y}_k + \int_a^t \big[P_k(s)\,y(s) + q_k(s)\big]\,\Delta s\,, \quad t \in [a,b] \cap \mathbb{T}\,. \tag{LD_k}$$

COROLLARY

ASSUME:

P, $P_k\colon [a,b] \cap \mathbb{T} \to \mathcal{L}(\mathbb{R}^n)$, q, $q_k\colon [a,b] \cap \mathbb{T} \to \mathbb{R}^n$ for $k \in \mathbb{N}$ are rd-continuous in $[a,b] \cap \mathbb{T}$,

$$\alpha_k = \sup_{t \in [a,b] \cap \mathbb{T}} \|P_k(t)\|_{L(\mathbb{R}^n)} + \sup_{t \in [a,b] \cap \mathbb{T}} \|q_k(t)\|_{\mathbb{R}^n} \quad \text{for } k \in \mathbb{N}\,,$$

$$\lim_{k \to \infty} \|\widetilde{y}_k - \widetilde{y}\|_{\mathbb{R}^n} = 0\,,$$

$$\lim_{k \to \infty} \sup_{t \in [a,b] \cap \mathbb{T}} \left\| \int_a^t (P_k(s) - P(s))\,\Delta s \right\|_{L(\mathbb{R}^n)} \big[1 + \alpha_k\big] = 0\,,$$

$$\lim_{k \to \infty} \sup_{t \in [a,b] \cap \mathbb{T}} \left\| \int_a^t (q_k(s) - q(s))\,\Delta s \right\|_{L(\mathbb{R}^n)} \big[1 + \alpha_k\big] = 0\,.$$

THEN: (LD) has a solution y, (LD_k) has a solution y_k for $k \in \mathbb{N}$ sufficiently large and

$$\lim_{k \to \infty} \sup_{t \in [a,b] \cap \mathbb{T}} \|y_k(t) - y(t)\|_{\mathbb{R}^n} = 0\,.$$

G. Antunes Monteiro, A. Slavík and M. Tvrdý.
***Kurzweil-Stieltjes integral and its applications*.**

(World Scientific, 2019)

- Preface
- Introduction
- Functions of bounded variation
- Absolutely continuous functions
- Regulated functions
- Riemann-Stieltjes integral
- Kurzweil-Stieltjes integral
  (included: existence of the integral, integration by parts, absolute integrability, convergence theorems, integration over elementary sets, relation to the Perron-Stieltjes integral, relation to the Lebesgue-Stieltjes integral)
- Generalized linear differential equations
  (included: existence and uniqueness of solutions, a priori estimates of solutions, continuous dependence of solutions on parameters, fundamental matrices, variation of constants formula)
- Miscellaneous additional topics
  (included: functionals on spaces of regulated functions, adjoint classes of Kurzweil-Stieltjes integrable functions, distributions, generalized elementary functions, integration on time scales, dynamic equations on time scales)
- Bibliography, Subject index, Symbol index

References

- G.A. Monteiro and M. Tvrdý. On Kurzweil-Stieltjes integral in Banach space. *Math. Bohem.* **137** (2012), 365–381.
- G.A. Monteiro and M. Tvrdý. Generalized linear differential equations in a Banach space: Continuous dependence on a parameter. *Discrete Contin. Dyn. Syst.* **33** (1) (2013), 283–303,
- G.A. Monteiro and A. Slavík. Extremal solutions of measure differential equations. *J. Math. Anal. Appl.* **444** (2016), 568–597.
- G.A. Monteiro, U.M. Hanung and M. Tvrdý. Bounded convergence theorem for abstract Kurzweil–Stieltjes integral. *Monatshefte für Mathematik* **180** (2016) 409–434.
- A. Slavík. Dynamic equations on time scales and generalized ordinary differential equations. *J. Math. Anal. Appl.* **385** (2012), 534–550.
- M. Federson, J.G. Mesquita and A. Slavík. Basic results for functional differential and dynamic equations involving impulses. em Math. Nachr. **286** (2013), 181–204.

References

- G.A. Monteiro and M. Tvrdý. On Kurzweil-Stieltjes integral in Banach space. *Math. Bohem.* **137** (2012), 365–381.
- G.A. Monteiro and M. Tvrdý. Generalized linear differential equations in a Banach space: Continuous dependence on a parameter. *Discrete Contin. Dyn. Syst.* **33** (1) (2013), 283–303,
- G.A. Monteiro and A. Slavík. Extremal solutions of measure differential equations. *J. Math. Anal. Appl.* **444** (2016), 568–597.
- G.A. Monteiro, U.M. Hanung and M. Tvrdý. Bounded convergence theorem for abstract Kurzweil–Stieltjes integral. *Monatshefte für Mathematik* **180** (2016) 409–434.
- A. Slavík. Dynamic equations on time scales and generalized ordinary differential equations. *J. Math. Anal. Appl.* **385** (2012), 534–550.
- M. Federson, J.G. Mesquita and A. Slavík. Basic results for functional differential and dynamic equations involving impulses. em Math. Nachr. **286** (2013), 181–204.
- G. Meng and M. Zhang. Dependence of solutions and eigenvalues of measure differential equations on measures. *J. Differential Equations* **254** (2013), 2196–2232.
- Y.X. Liu, G.L. Shi and J. Yan. Dependence of solutions and eigenvalues of third order linear measure differential equations on measures. *Sci.China, Math.*, accepted 2019.

References

- Š. Schwabik. *Generalized Ordinary Differential Equations*. World Scientific, 1992.
- Š. Schwabik. Abstract Perron-Stieltjes integral. *Math. Bohem.* **121**(1996), 425–447.
- Š. Schwabik. Linear Stieltjes integral equations in Banach spaces. *Math. Bohem.* **124** (1999), 433–457.
- Š. Schwabik. Linear Stieltjes integral equations in Banach spaces II; Operator valued solutions. *Math. Bohem.* **125** (2000), 431–454.

References

- Š. Schwabik. *Generalized Ordinary Differential Equations*. World Scientific, 1992.
- Š. Schwabik. Abstract Perron-Stieltjes integral. *Math. Bohem.* **121**(1996), 425–447.
- Š. Schwabik. Linear Stieltjes integral equations in Banach spaces. *Math. Bohem.* **124** (1999), 433–457.
- Š. Schwabik. Linear Stieltjes integral equations in Banach spaces II; Operator valued solutions. *Math. Bohem.* **125** (2000), 431–454.
- R.M. Dudley and R. Norvaiša. *Concrete functional calculus*. Springer Monographs in Mathematics. Springer, New York, 2011.
- Ch. S. Hönig. *Volterra Stieltjes-Integral Equations*. North Holland & American Elsevier, Mathematics Studies 16, Amsterdam & New York, 1975.
- R.M. McLeod. *The generalized Riemann integral*. Carus Monograph, No.2, Mathematical Association of America, Washington, 1980.

References

- Š. Schwabik. *Generalized Ordinary Differential Equations*. World Scientific, 1992.
- Š. Schwabik. Abstract Perron-Stieltjes integral. *Math. Bohem.* **121**(1996), 425–447.
- Š. Schwabik. Linear Stieltjes integral equations in Banach spaces. *Math. Bohem.* **124** (1999), 433–457.
- Š. Schwabik. Linear Stieltjes integral equations in Banach spaces II; Operator valued solutions. *Math. Bohem.* **125** (2000), 431–454.
- R.M. Dudley and R. Norvaiša. *Concrete functional calculus*. Springer Monographs in Mathematics. Springer, New York, 2011.
- Ch. S. Hönig. *Volterra Stieltjes-Integral Equations*. North Holland & American Elsevier, Mathematics Studies 16, Amsterdam & New York, 1975.
- R.M. McLeod. *The generalized Riemann integral*. Carus Monograph, No.2, Mathematical Association of America, Washington, 1980.
- M. Bohner and A. Peterson. *Dynamic Equations on Time Scales: An Introduction with Applications*. Birkhäuser, Boston, 2001.

MANY THANKS to everybody for attention and, in particular, to all the organizers

MANY THANKS to everybody for attention and, in particular, to all the organizers