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Gravitational Waves in Cosmology

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Contents

The dissertation summarizes our original results published in refereed journals. Most of them were written during my appointment at the Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University in Prague, and during my regular visits to the Department of Mathematical Sciences of Loughborough University, UK.

A common theme of these publications is the investigation of exact gravitational waves in Einstein's general theory of relativity. To achieve a synoptic organization, the results are classified according to their main topic, thus defining the basic structure of this dissertation:

- Part A: *Exact radiative spacetimes*
  Summarizes publications on properties of some important classes of exact solutions of Einstein's equations (including the cosmological constant) which represent gravitational waves, in particular general motion of test particles and global structure of the spacetimes.
- Part B: *Impulsive waves*
  Contributions related to studies of impulsive gravitational waves which may propagate in Minkowski, (anti-)de Sitter, or other universes. They contain a geometric classification, various methods of construction of such solutions, and their physical interpretation.
- Part C: *Asymptotic structure of radiation*
  Recent publications in which we analyzed the asymptotic directional properties of general gravitational or electromagnetic fields in spacetimes with a cosmological constant that admit a spacelike or timelike conformal infinity.

We label our contributions accordingly: the reference (in usual square brackets) is given by the letter A, B or C, respectively, followed by a specific number. The list of publications selected for the dissertation is given in the appendix (page 42).

Introduction

The idea of gravitational waves was mentioned by Albert Einstein before the formulation of general relativity already. This is illustrated in a discussion with Max Born which took place after Einstein's lecture in Vienna on 23 September 1913 [1]. Shortly after finishing his theory of gravitation in November 1915 [2] Einstein published papers [3] in which he analyzed gravitational waves: solutions of linearized field equations for weak perturbations of Minkowski spacetime propagating with the speed of light. In this work, Einstein was soon followed by Weyl [4], Eddington [5] and others.

Einstein's gravitational waves have similar properties to the well-known electromagnetic waves. Both propagate through an empty space with the speed of light, both are transverse, and both admit two polarisation modes. Nevertheless, there are some fundamental differences. Gravitational waves are periodic tidal deformations of space and time in the plane perpendicular to the direction of propagation. The magnitude of these deformations can be characterised by an amplitude h which is a dimensionless ratio $\Delta l/l$, where l is the original distance (or size of a test object) and Δl is the amount of deformation. The specific property of gravitational waves, in comparison with electromagnetic waves, is that *gravitational waves are extremely weak*. Typical astrophysical sources of gravitational waves are investigated including supernova explosions, binary systems, and collisions of black holes or neutron stars. Current estimates indicate that their frequencies are in the range 1–1000 Hz, and their amplitudes h here on Earth are smaller than 10^{-20}, almost unimaginably small number! This explains why gravitational waves have not yet been detected by an experimental device which would directly measure the deformations induced by these waves. The legendary attempts were done in the 1960s by Joseph Weber who invented resonant bar detectors [6]. During the past 40 years the sensitivity of such instruments has been steadily growing, but so far this effort has not been successful.

However, the situation now looks more optimistic. There exist strong *indirect* indications of the existence of gravitational waves based of the precise observations of compact systems. The first such system, the binary pulsar PSR 1913 + 16, was discovered in 1974 by R. A. Hulse and J. H. Taylor [7]. The two close neutron stars inspiral together because the emitted gravitational waves continuously carry away the binding energy. The measured shortening of the period of revolution 76.0 ± 0.5 μs per year excellently agrees with the value 75.8 μs predicted by general relativity. For this convincing (albeit indirect) proof of gravitational waves the discoverers were awarded the Nobel Prize for 1993.

Moreover, there is a growing expectation that feeble gravitational waves will be *directly* detected in near future — almost a century after the theoretical prediction of their existence — by means of the state-of-the-art large *interferometric detectors* $LIGO$, $VIRGO$ or GEO [8]. Their sensitivity in measuring relative spatial deformations is of the order $h \sim 10^{-21}$ and better.

The investigation of gravitational waves is thus nowadays a broad and active (even a hectic) field which joins the theory of general relativity with astronomy and other branches of physics and technology. On a theoretical side, numerous works have been devoted to specific approximate (analytic or numerical) analyses of gravitational radiation generated by various spatially isolated gravitating sources, in particular binary systems, collisions and mergers of black holes or neutron stars, supernova explosions, and other possible astrophysical sources. The purpose of this big effort is to provide a best possible "catalogue" of signals from the anticipated sources (namely the templates of possible frequencies, waveforms, polarisations, and their time dependence) which could be searched for using computers in the real-time data stream from the detectors to find the precious unique signal.

In the present dissertation, however, we do not investigate these interesting topics. To obtain a realistic description of gravitational waves generated by astrophysical sources, various *approximation* methods must necessarily be applied because the field equations and equations of motion are highly complicated. The subject of our interest is more theoretical: we investigate those gravitational waves which are *exact* solutions of the Einstein field equations. We concentrate on finding such solutions and on presenting their physical interpretation. We are interested in global properties of spacetimes which represent gravitational waves that may propagate in simple cosmological models.

Of course, such a study has a rather theoretical character. Its significance lies in the fact that — contrary to perturbation or numerical approaches — it enables one to investigate some principal problems. In particular, it is possible to analyze the global structure of the spacetimes, properties of horizons, character of singularities, behaviour of gravitational waves in cosmological models which are not asymptotically flat, to study nonperturbative effects of waves on particles and fields etc. The analysis of specific explicit exact solutions helps us to understand general features of gravitational radiation in the full nonlinear Einstein's theory, to penetrate deeper into the heart of general relativity.

1 (A) Exact radiative spacetimes

The first class of exact metrics representing gravitational waves was found in 1923 by Brinkman [9]. These were later rediscovered by several authors — these metrics are now known as *plane-fronted* gravitational waves with parallel rays, the so called pp-waves. In 1925, Beck [10] found *cylindrically symmetric* gravitational waves which were rediscovered and studied by Einstein and Rosen [11].

A deeper understanding of general relativity resulted in the 1950's from series of papers by Lichnerowicz and others [12] who investigated the Einstein field equations using the theory of characteristics. This revealed a close analogy between gravitation and electrodynamics. In the electromagnetic case, one may distinguish algebraically between radiative *null* fields and general *non-null* fields. A null field is characterized by $F_{\alpha\beta}F^{\alpha\beta} = 0 = {}^*F_{\alpha\beta}F^{\alpha\beta}$ and $F_{\alpha\beta}k^\beta = 0 = {}^*F_{\alpha\beta}k^\beta$, where k^β is the privileged principal null direction. In the case of gravitational fields one may distinguish *six algebraic types* denoted by *I*, *II*, *D*, *III*, N and 0 [13–17]. Type N gravitational fields (such as the pp-waves) satisfy strikingly similar conditions as the electromagnetic type N fields, namely $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 0 = {}^*R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ and $R_{\alpha\beta\gamma\delta}k^\delta = 0 = {}^*R_{\alpha\beta\gamma\delta}k^\delta$. These results led Pirani [18] to the first attempt at an invariant definition of gravitational radiation. However, such characterization based on the algebraically special structure of the field now appears to be oversimplified [19].

Another important tool for an investigation of gravitational radiation and for the construction of exact solutions has been the gravitational ray optics introduced by Sachs [20]. He investigated the way in which a family of null rays would deform the shadow of an opaque object. To characterize the corresponding changes of the size, orientation, and shape he introduced the *optical scalars* $\Theta = \frac{1}{2}k^{\alpha}{}_{;\alpha}$, $\omega = \sqrt{\frac{1}{2}k_{[\alpha;\beta]}k^{\alpha;\beta}}$, and $|\sigma| = \sqrt{\frac{1}{2}k_{(\alpha;\beta)}k^{\alpha;\beta} - \Theta^2}$, which represent respectively the expansion, rotation and shear. Further fundamental developments came in 1962 with the works of Bondi, van der Burg, Metzner and Sachs [21–23] in which it was demonstrated that the waves emitted from isolated sources transport mass to infinity. Subsequently, the method of spin coefficients was developed by Newman and Penrose [24]. The introduction of these geometrical concepts had a great influence in finding new exact radiative solutions which are now 'prototypes' of gravitational waves in general relativity, namely the Kundt class of *nonexpanding 'plane-fronted' waves* (see [25–27]), the Robinson–Trautman *expanding 'spherical' waves* [28], and other solutions. These are summarized in the compendium [16].

The next crucial step in the investigation of gravitational radiation was made in 1963 by Penrose [29, 30]: the *definition of an asymptotically flat spacetime* [17]. Penrose's concept of a smoothly asymptotically flat spacetime represents the geometrical framework for a rigorous discussion of gravitational radiation from spatially isolated sources. The concept of *conformal infinity* $\mathcal{I}$ became standard tool in relativity. Nevertheless, the only *explicitly* known solutions describing finite sources which are asymptotically flat with global $\mathcal{I}$ are the boost-rotational symmetric solutions representing fields of *'uniformly accelerated' objects* [31–35]. Many details and references can be found in [36–39].

Another interesting topic in the exact theory of radiation is the problem of *colliding plane gravitational waves* which propagate against each other, see [40] for a review. The study was initiated at the beginning of the 1970's by Kahn and Penrose and by Szekeres [41, 42]. There are also interesting models of *exact cosmological gravitational waves* [43,44], or models of exact shock and impulsive gravitational waves propagating into expanding Friedmann–Robertson–Walker universes [45, 46]. Some of these solutions serve as exact examples of possible primordial gravitational waves.

Our work is devoted to *exact* solutions of Einstein's equations representing radiative spacetimes and to their physical interpretation. In particular, we concentrate on algebraically special exact vacuum solutions which admit an arbitrary value of the *cosmological constant* Λ. The cosmological term Λ was introduced into the field equations by Einstein in his 1917 article [47], a landmark in the development of modern cosmology. Einstein suggested the model of a static, homogeneous and isotropic closed universe filled with a dust whose attraction was compensated by the repulsive effect of Λ. In the same year, Willem de Sitter found exact *vacuum* model with $\Lambda > 0$ describing an exponentially expanding universe [48]. Together with flat Minkowski spacetime ($\Lambda = 0$) and the so-called anti-de Sitter universe ($\Lambda < 0$) they form a trio of fundamental spaces in general relativity.

1.1 Spaces of constant curvature

These three spaces of constant curvature are the simplest, maximally symmetric exact solutions of Einstein's vacuum field equations [49–51]. Yet, they are important in contemporary theoretical physics. The de Sitter universe is the basic model of the inflationary phase of an exponential expansion in the early universe [52, 53]. In addition, according to the "cosmic no-hair" conjecture [54], it is locally the asymptotic state of many cosmological models with a positive Λ. Somewhat surprisingly, the anti-de Sitter spacetime has also recently become a subject of intensive studies thanks to the Maldacena's conjecture [55], the so-called AdS/CFT correspondence.

The de Sitter and anti-de Sitter spacetimes can be represented as the 4-dimensional hyperboloid

$$-{Z_0}^2 + {Z_1}^2 + {Z_2}^2 + {Z_3}^2 + \epsilon {Z_4}^2 = \epsilon a^2 \ , \qquad a = \sqrt{3/|\Lambda|} \ , \tag{1}$$

in the flat 5-dimensional space with metric $\mathrm{d}s^2 = -\mathrm{d}{Z_0}^2 + \mathrm{d}{Z_1}^2 + \mathrm{d}{Z_2}^2 + \mathrm{d}{Z_3}^2 + \epsilon \mathrm{d}{Z_4}^2$, in which $\epsilon = \mathrm{sign}\,\Lambda$. Various parameterizations of the hyperboloid (1) are known which introduce suitable coordinates for the (anti-)de Sitter spacetime. For example, the expressions

$$Z_0 = \tfrac{1}{\sqrt{2}}(\mathcal{U}+\mathcal{V})/\left[1+\tfrac{1}{6}\Lambda(\eta\bar\eta-\mathcal{U}\mathcal{V})\right] , \qquad Z_2 = \tfrac{1}{\sqrt{2}}(\eta+\bar\eta)/\left[1+\tfrac{1}{6}\Lambda(\eta\bar\eta-\mathcal{U}\mathcal{V})\right] ,$$
$$Z_1 = \tfrac{1}{\sqrt{2}}(\mathcal{U}-\mathcal{V})/\left[1+\tfrac{1}{6}\Lambda(\eta\bar\eta-\mathcal{U}\mathcal{V})\right] , \qquad Z_3 = \tfrac{-\mathrm{i}}{\sqrt{2}}(\eta-\bar\eta)/\left[1+\tfrac{1}{6}\Lambda(\eta\bar\eta-\mathcal{U}\mathcal{V})\right] , \tag{2}$$
$$Z_4 = a\left[1-\tfrac{1}{6}\Lambda(\eta\bar\eta-\mathcal{U}\mathcal{V})\right]/\left[1+\tfrac{1}{6}\Lambda(\eta\bar\eta-\mathcal{U}\mathcal{V})\right] ,$$

give the line element for all spaces of constant curvature in the conformally flat unified form

$$\mathrm{d}s^2 = \frac{2\,\mathrm{d}\eta\,\mathrm{d}\bar\eta - 2\,\mathrm{d}\mathcal{U}\,\mathrm{d}\mathcal{V}}{[\,1+\frac{1}{6}\Lambda(\eta\bar\eta-\mathcal{U}\mathcal{V})\,]^2} . \tag{3}$$

The global conformal structure of the de Sitter spacetime becomes obvious if we introduce the conformal time η_E, so that $\mathrm{d}s^2_E = \Omega^2\,\mathrm{d}s^2$, where $\Omega = a^{-1}\sin\eta_E$, and $\mathrm{d}s^2_E$ is the metric of the Einstein static universe. The Penrose conformal diagram is shown in figure 1: it covers the ranges $\chi \in [0,\pi]$, $\eta_E \in [0,\pi]$. The boundaries of η_E given by $\Omega = 0$ represent the conformal infinities $\mathcal{I}^\pm$. Obviously, these have a spacelike character. Analogously, for the anti-de Sitter spacetime $\Omega = a^{-1}\cos\chi$, so that $\chi \in [0,\frac{\pi}{2}]$, η_E is arbitrary. The conformal infinity $\mathcal{I}$ given by $\chi = \frac{\pi}{2}$ has a timelike character here.

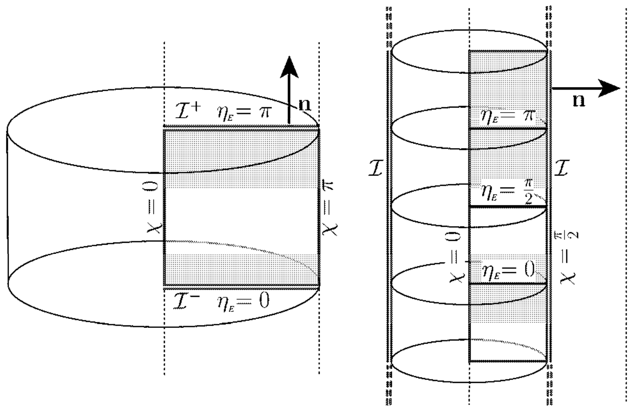


Figure 1: The conformal structure of de Sitter spacetime (left) and anti-de Sitter spacetime (right) .

1.2 Exact type N Einstein spaces

We study exact solutions of Einstein's equations representing radiative spacetimes which are not asymptotically flat. By considering $\Lambda \neq 0$ we deal with gravitational waves 'in' everywhere curved (anti-)de Sitter model. The theory of gravitational radiation should eventually be formulated with other boundary conditions than those corresponding to asymptotic flatness, and any exact explicit example of such a wave may give a useful insight. It may serve as a 'test-bed' for numerical simulations or for analytic formulations of radiative 'boundary conditions' in cosmological models.

In a series of papers [A1]–[A11] we studied *nontwisting type N solutions of vacuum Einstein equations with a cosmological constant.* For them, the Debever–Penrose vector k^α is quadruple and defines a privileged null congruence characterized by expansion Θ, twist ω, and shear σ. The Kundt–Thompson theorem implies $\sigma = 0$. Considering the nontwisting case, we are left with two classes:

- *Kundt* class of *nonexpanding* gravitational waves ($\Theta = 0$), cf. [16,26,27,56],
- *Robinson–Trautman* class of *expanding* gravitational waves ($\Theta \neq 0$), cf. [16,28].

We will denote the above Kundt class by $KN(\Lambda)$, and the Robinson–Trautman class by $RTN(\Lambda)$.

In [A1] we described all these spacetimes in detail. All solutions of the Kundt class $KN(\Lambda)$ can be written in suitable coordinates, first given in [56], as

$$\mathrm{d}s^2 = 2\,\frac{1}{p^2}\,\mathrm{d}\xi\mathrm{d}\bar\xi - 2\,\frac{q^2}{p^2}\,\mathrm{d}u\mathrm{d}v + \mathrm{F}\,\mathrm{d}u^2 , \tag{4}$$

where $p = 1 + \frac{1}{6}\Lambda\xi\bar{\xi}$, $q = (1 - \frac{1}{6}\Lambda\xi\bar{\xi})\,\alpha + \bar{\beta}\,\xi + \beta\,\bar{\xi}$, $\mathrm{F} = \kappa(q^2/p^2)v^2 - (q^2/p^2)_{,u}\,v + (q/p)\,\mathrm{H}$, $\kappa = \frac{1}{3}\Lambda\alpha^2 + 2\beta\bar{\beta}$, and $\mathrm{H} = \mathrm{H}(\xi,\bar{\xi},u)$. The vacuum field equation is $p^2\,\mathrm{H}_{\xi\bar{\xi}} + \frac{1}{3}\Lambda\,\mathrm{H} = 0$, which has an explicit general solution $\mathrm{H} = (f_{,\xi} + \bar{f}_{,\bar{\xi}}) - \frac{1}{3}\Lambda(\bar{\xi}f + \xi\bar{f})/p$, where $f(\xi,u)$ is an *arbitrary* function. As shown in [56], [A1], there exist distinct subclasses characterized by specific choices of the parameters α and β. For $\Lambda = 0$ there are two subclasses. The simpler one represents the well-known *pp*-waves, for which $p = 1 = q$, $\mathrm{F} = f_{,\xi} + \bar{f}_{,\bar{\xi}}$. This has been investigated by many authors (see section 24.5 in [16]). The second subclass was discovered by Kundt [26] (see Chapter 31 in [16]). There is an asymmetry for $\Lambda \neq 0$: there exist *three* subclasses of nonexpanding waves for $\Lambda < 0$, whereas there is only *one* subclass for $\Lambda > 0$. There exists a special subsubclass such that k^α is the *Killing* vector. It was explicitly demonstrated in [A1] that this is identical with a family of solutions found by Siklos [57].

The class of expanding Robinson–Trautman solutions $RTN(\Lambda)$ has been known for a long time [28]. In 1981, a more convenient coordinate parametrization was found [58],

$$\mathrm{d}s^2 = 2\,v^2\mathrm{d}\xi\mathrm{d}\bar{\xi} + 2\,v\bar{\mathrm{A}}\,\mathrm{d}\xi\mathrm{d}u + 2\,v\mathrm{A}\,\mathrm{d}\bar{\xi}\mathrm{d}u + 2\,\psi\,\mathrm{d}u\mathrm{d}v + 2\,(\mathrm{A}\bar{\mathrm{A}} + \psi\mathrm{B})\,\mathrm{d}u^2\ , \tag{5}$$

where $\mathrm{A} = \epsilon\xi - vf$, $\mathrm{B} = -\epsilon + \frac{1}{2}v\,(f_{,\xi} + \bar{f}_{,\bar{\xi}}) + \frac{1}{6}\Lambda v^2\psi$, $\psi = 1 + \epsilon\xi\bar{\xi}$, and $\epsilon = -1, 0, +1$. Again, the solutions depend on an arbitrary function $f(\xi,u)$. The parameter ϵ is proportional to the Gaussian curvature of the 2-surfaces $u = const.$ Thus, there are nine invariant subclasses of this class since the cases $\epsilon = +1, 0, -1$ and $\Lambda > 0$, $\Lambda = 0$, $\Lambda < 0$ are independent.

In the subsequent paper [A2] we presented a physical interpretation of all these nontwisting type N spaces (4), (5) which we based on the study of the deviation of geodesics [18, 59]. There always exists an orthonormal frame tied to any timelike observer. We demonstrated that the equation of geodesic deviation with respect to this interpretation frame is given by

$$\begin{aligned}
\ddot{Z}^{(1)} &= \tfrac{\Lambda}{3}Z^{(1)} - \mathcal{A}_{+}Z^{(1)} + \mathcal{A}_{\times}Z^{(2)}\ ,\\
\ddot{Z}^{(2)} &= \tfrac{\Lambda}{3}Z^{(2)} + \mathcal{A}_{+}Z^{(2)} + \mathcal{A}_{\times}Z^{(1)}\ ,\\
\ddot{Z}^{(3)} &= \tfrac{\Lambda}{3}Z^{(3)}\ .
\end{aligned} \tag{6}$$

The amplitudes of the gravitational waves are given by $\mathcal{A}_{+} = \mathcal{R}e\,\{\mathcal{A}\}$, $\mathcal{A}_{\times} = \mathcal{I}m\,\{\mathcal{A}\}$ such that

$$\mathcal{A} = -\tfrac{1}{2}pq\,\dot{u}^2\,f_{,\xi\xi\xi} \tag{7}$$

for the $KN(\Lambda)$ spacetimes, and

$$\mathcal{A} = -\tfrac{1}{2}(\psi/v)\dot{u}^2\,f_{,\xi\xi\xi} \tag{8}$$

in the $RTN(\Lambda)$ spacetimes (evaluated along the geodesics). The equations (6) express relative accelerations of nearby test particles in terms of their actual positions. The components of the displacement vector $Z^{(i)} \equiv e^{(i)}_\mu Z^\mu$ determine the distance between the particles. Similarly, $\ddot{Z}^{(i)} \equiv e^{(i)}_\mu (D^2Z^\mu/d\tau^2)$ are relative accelerations. The system (6) enables us to draw the following conclusions:

- The particles move isotropically if $\mathcal{A} = 0$, in which case no gravitational wave is present. Indeed, both the $KN(\Lambda)$ and $RTN(\Lambda)$ spacetimes for $f_{,\xi\xi\xi} = 0$ are Minkowski ($\Lambda = 0$), de Sitter ($\Lambda > 0$), and anti-de Sitter ($\Lambda < 0$) universe. These maximally symmetric, homogeneous, and isotropic spacetimes are thus *natural backgrounds* for other "non-trivial" radiative solutions.

- If $f_{,\xi\xi\xi} \neq 0$ then the amplitudes $\mathcal{A}$ do not vanish, and the particles are influenced by the wave. For $\Lambda \neq 0$, the influence of the wave is added to the (anti-)de Sitter isotropic expansion/contraction. Therefore, the $KN(\Lambda)$ and $RTN(\Lambda)$ metrics can be interpreted as *exact gravitational waves* which propagate in spaces of constant curvature.

- The wave has a *transverse character* as only motions in the plane $(\mathbf{e}_{(1)}, \mathbf{e}_{(2)})$ perpendicular to $\mathbf{e}_{(3)}$ are affected. The propagation direction is $\mathbf{e}_{(3)}$.

- There are *two polarization modes*, "+" and "×", with the amplitudes $\mathcal{A}_+$ and $\mathcal{A}_\times$.
- *Curvature singularities* occur where $\mathcal{A}$ diverges. The singularities can be characterized by the invariant constructed in [60] from the second derivatives of the Riemann tensor.
- Special classes of explicit timelike geodesics were also discussed in [A2]. For a $\Lambda > 0$ the *waves are exponentially damped* and the spacetimes locally approach the de Sitter universe, which is an explicit demonstration of the cosmic no-hair conjecture [54] under the presence of waves.

We devoted several works to a analysis of members from the family $KN(\Lambda)$ of nonexpanding spacetimes, in particular to investigation of geodesics and aspects of their global structure (subsections 1.2.1–1.2.3). In subsection 1.2.4 we present our results concerning the $RTN(\Lambda)$ expanding spacetimes.

1.2.1 Geodesics in the *pp*-waves: chaotic behaviour

In [A3] we introduced new specific sandwich *pp*-waves constructed from *homogeneous* solutions

$$\mathrm{d}s^2 = 2\,\mathrm{d}\xi\mathrm{d}\bar\xi - 2\,\mathrm{d}u\mathrm{d}v + (f_{,\xi} + \bar f_{,\bar\xi})\,\mathrm{d}u^2 \ , \tag{9}$$

where $f_{,\xi} = h(u)\xi^2$. Now, the amplitude given by (7) is $\mathcal{A} = -h(u)\,\dot u^2$. It is given by the profile function $h(u)$ of the retarded time u. We considered several non-standard wave-profiles h, for which we calculated the motions of test particles explicitly demonstrating the caustic property of the sandwich waves. Moreover, we showed that the behaviour of particles in the impulsive limit of these solutions is *independent* of the particular wave-profile of the initial sequence of sandwich waves. Our results inspired rigorous works of Steinbauer and Kunzinger [61, 62]. Using the involved technique of Colombaeu algebras of generalised nonlinear functions [63, 64] they proved that the distributional limit of geodesic motions in the impulsive *pp*-waves is indeed independent of regularisation.

In the subsequent works [A4]–[A6] we concentrated on studies of geodesics in *nonhomogeneous* *pp*-waves which are given by the structural function

$$f_{,\xi} = \tfrac{2}{n}\,h(u)\,\xi^n \ , \quad \text{for} \quad n = 3, 4, 5, \cdots \ . \tag{10}$$

Quite surprisingly, we demonstrated in [A4], [A5] by both rigorous analytic and numerical methods that geodesics in these spacetimes with a constant profile h exhibit the *chaotic behaviour*. This seems to be the first known example of a chaotic motion in an *exact radiative* spacetime. (The chaotic behaviour of geodesics in the two fixed-centres problem was previously examined in [65–67]. Chaotic geodesic motion were also studied for spinning particle in Schwarzschild spacetime [68], in some static axisymmetric spacetimes [69, 70], or in FRW spacetimes [71]. Chaotic interaction of particles with *linearized* gravitational waves was described in [72, 73].)

In [A4] we announced that geodesic motion in the class of *pp*-waves, which are the simplest exact gravitational wave spacetimes, can also be chaotic. The geodesic equations for (9) reduce to

$$\ddot\xi - \tfrac{1}{2}\bar f_{,\bar\xi\bar\xi}\,U^2 = 0 \ , \tag{11}$$

$u(\tau) = U\tau + \tilde U$, $v(\tau) = \frac{1}{2}U^{-1}\int[\,2\dot\xi\dot{\bar\xi} + (f_{,\xi} + \bar f_{,\bar\xi})\,U^2 - \epsilon\,]\,\mathrm{d}\tau + \tilde V$, where τ is an affine parameter, and $U, \tilde U, \tilde V$ are constants. We demonstrated that, introducing real coordinates x and y by $\xi = x + \mathrm{i}y$, the system (11) follows from the Hamiltonian

$$H = \tfrac{1}{2}\left(p_x^2 + p_y^2\right) + V(x, y, u) \ , \qquad \text{where} \qquad V(x, y, u) = \tfrac{1}{n}U^2 h(u)\,\mathcal{R}e\,\{\xi^n\} \ . \tag{12}$$

It was shown previously by Rod, Churchill and Pecelli in a series of mathematical papers [74–76] that motion in the Hamiltonian system (12) where $h(u) = \text{const.}$ is chaotic in a rigourous sense. In the simplest case $n = 3$ the corresponding 'monkey saddle' potential is $V(x, y) = \frac{1}{3}x^3 - xy^2$, that is the particular case of the famous Hénon–Heiles Hamiltonian [77] (with missing quadratic terms) which

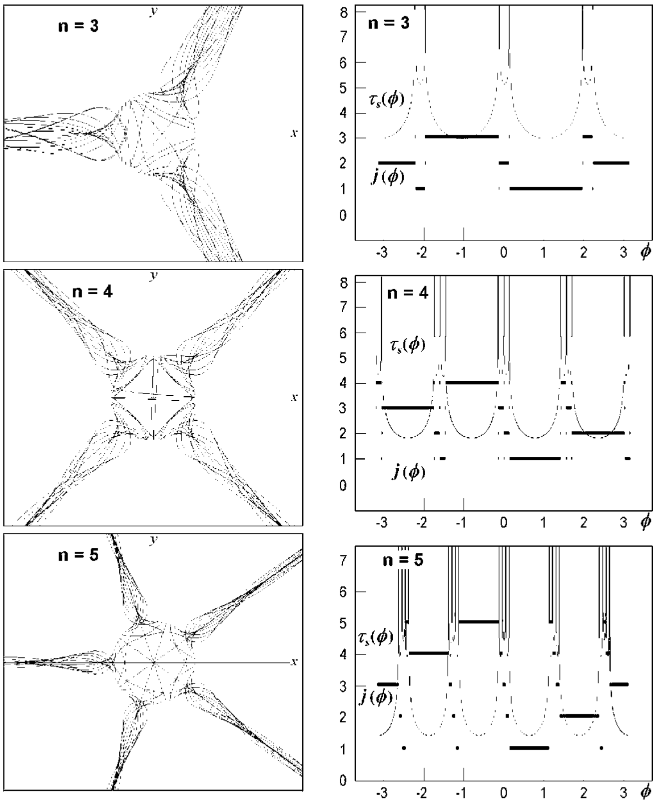


Figure 2: Geodesics starting from a unit circle in non-homogeneous pp-waves escape to infinity only along one of the n channels (left). The functions $j(\phi)$ and $\tau_s(\phi)$ indicate that basin boundaries separating different outcomes are fractal (right) which demonstrates the presence of chaos.

is a 'textbook' example of a chaotic system. Therefore, geodesic motion in all non-homogeneous *pp*-wave spacetimes with the function (10) having a constant profile h is chaotic.

In order to support these arguments we investigated in [A4] and in a more detailed work [A5] the structure of motion by an invariant *fractal method*. Complementary to the analysis described above, we studied *unbounded* geodesics. The fractal method (see, e.g. [65–67]) starts with a definition of distinct asymptotic outcomes given here by "types of ends" of all trajectories. Subsequently, a set of initial conditions is evolved until one of the outcome states is reached. Chaos is established if the basin boundaries which separate initial conditions leading to different outcomes are fractal. Such fractal partitions measure an extremely sensitive dependence of the evolution on the choice of initial conditions. We have observed exactly these structures in the system studied.

We integrated numerically the equations of motion given by (11). The geodesics started from a unit circle in the (x, y)-plane. The initial positions were parametrized by an angle $\phi \in [-\pi, \pi)$. In figure 2 we present typical trajectories for $n = 3, 4, 5$. Each unbounded geodesic escapes to infinity where the curvature singularity is located *only along* one of the *n distinct outcome channels* in the potential V with the radial axis $\phi_j = (2j-1)\pi/n$, $j = 1, \cdots, n$ (in fact, it oscillates around the axis with frequency growing to infinity and amplitude approaching zero [A4]). These channels represent possible outcomes of our system and we label them by the corresponding values of j. In certain regions the function $j(\phi)$ representing a portrait of the basin structure depends sensitively on initial position given by ϕ — see the right part of figure 2. The boundaries between the outcomes are fractal which we confirmed in [A5] on the enlarged detail, on the detail of the detail etc. up to the sixth level. At *each level* the structure has the property that between two sets of geodesics with channels j_1 and $j_2 \neq j_1$ there is always a smaller set of geodesics with channel j_3 such that $j_3 \neq j_1$ and $j_3 \neq j_2$.

We generalized these results to more realistic situations in which the profiles $h(u)$ of the gravitational waves are *not* constant. In [78] we investigated geodesics in *nonhomogeneous shock and sand-*

wich pp-waves. We demonstrated that a shorter duration of the wave implies a less chaotic geodesic motions. In the limit of impulsive waves the geodesics are integrable, and the ring of test particles is deformed into hypotrochoidal curves with n loops. The box-counting dimension approaches $d = 1$ so that the initially fractal basin boundaries become regular, and the motion is non-chaotic.

Another generalization was presented in our contribution [A6]. We investigated the behaviour of geodesics in vacuum *pp*-waves (9) which contain both homogeneous and nonhomogeneous terms, i.e. $f_{,\xi} = \alpha\xi^2 + \frac{2}{3}\beta\xi^3$. The equation of motion (11) corresponds to the Hamiltonian with the potential

$$V(x, y) = \tfrac{1}{2}(x^2 - y^2) + \tfrac{1}{3}(x^3 - 3xy^2)\ . \tag{13}$$

This may be called a "modified Hénon–Heiles potential": it has almost the form of the Hénon–Heiles potential up to the negative sign of the y^2 term. In [A6] we classified and described all possible energy manifolds $H = E$ for various values of E.

For $E < 0$ the motion is not chaotic. For $E > \frac{1}{6}$ there are three exits to infinity, and the geodesics exhibit a chaotic behaviour as in [A4], [A5]. Naturally, the region $0 < E < \frac{1}{6}$ is the most interesting one. With an increasing energy, one expects that chaos has to appear, similarly as in the classical Hénon–Heiles system. We investigated this *onset of chaos* by the fractal method based on numerical simulations (note that the study of standard Hénon–Heiles system by this method was performed only very recently [79], and this was directly inspired by our contributions [A4], [A5]).

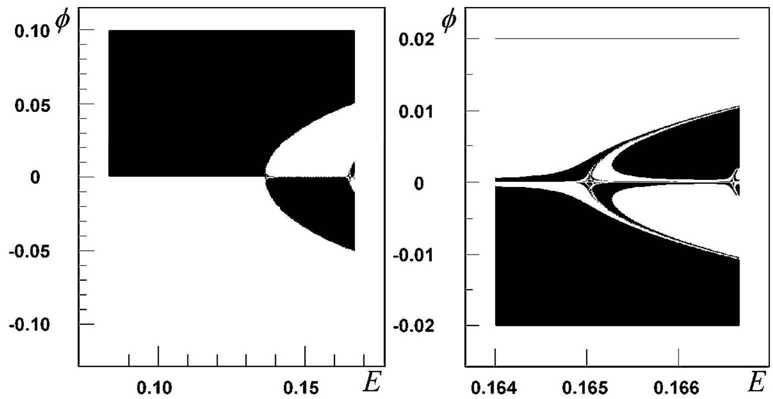


Figure 3: The basins of the two exits, 'up' (black) and 'down' (white), in the initial conditions characterized by energy E and starting angle ϕ. The picture on right is an enlargement of the one on left. The basin boundaries are fractal.

We have parametrized the initial position by the angle ϕ. By extensive numerical simulations we have calculated the function $j(\phi, E)$, which is an analog of the scattering function used in the studies of a chaotic scattering in open Hamiltonian systems [80]. This is indicated in figure 3 where the black colour denotes the basin for the exit "up" (with $y \to \infty$) while the white colour denotes the basin for the exit "down" (with $y \to -\infty$). We demonstrated that for energies $E < E_0 = 0,1364 \pm 0,0001$ the only basin boundary in $j(\phi)$ is $\phi = 0$. For E_0 this basin boundary bifurcates into three branches, and for $E > E_0$ the boundaries have *fractal structure*, which is an evidence of chaotic behaviour.

1.2.2 Geodesics in the Kundt waves: envelope singularity

In [A7], [A8] we concentrated on physical interpretation of the subfamily of the Kundt gravitational waves with $\Lambda = 0$ [26,27], which are different from the *pp*-waves. They can be written in the form (4) for the parameters $\alpha = 0, \beta = 1$, i.e. $p = 1$, $q = \xi + \bar{\xi}$, $\mathrm{F} = 2q^2v^2 + q\,\mathrm{H}$. It was convenient to rewrite these solutions in real spatial coordinates x and y such that $\xi = \frac{1}{\sqrt{2}}(x + \mathrm{i}\,y)$. The metric has the form

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - 4x^2\mathrm{d}u\,\mathrm{d}v + 4(x^2v^2 + xG)\,\mathrm{d}u^2\ . \tag{14}$$

The family of conformally flat pure radiation metrics (14) attracted attention recently [81–83]. They contain no invariants or Killing vectors, and thus provide an interesting exceptional case for invariant classification of exact solutions. To distinguish the Wils [81] metric within the more general Edgar–Ludwig solution [83], it is necessary to go as far as the fourth derivative of the curvature tensor. Even more interestingly, it has been demonstrated [60, 84] that for these spacetimes *all curvature invariants* of all orders *identically vanish*. They may thus play an important role in string theory and quantum gravity since there are no quantum corrections to all perturbative orders [85].

In [A7] we found that the sequence of plane wave-fronts $u = u_0 = const.$ in the Kundt spacetimes are tangent hyperplanes rotated around an expanding cylinder. Indeed, the transformation

$$x = \sqrt{X^2 + Z^2 - T^2}\ , \quad y = Y\ , \quad u = \frac{X - \sqrt{X^2 + Z^2 - T^2}}{T + Z}\ , \quad v = \frac{T + Z}{2\sqrt{X^2 + Z^2 - T^2}}\ , \tag{15}$$

puts the metric (14) with $G = 0$ to standard form of flat space. Each surface u_0 corresponds to a hyperplane $-T + \sin\alpha\ X + \cos\alpha\ Z = 0$, where $\alpha = 2\arctan u_0$. As u_0 increases from $-\infty$ to $+\infty$ then these wave surfaces form a family of null hyperplanes which rolls all around the cylinder $x = 0$, i.e. $X^2 + Z^2 = T^2$. To obtain a unique foliation of the region outside this expanding cylinder it is necessary to restrict u_0 to half-hyperplanes, as indicated in figure 4.

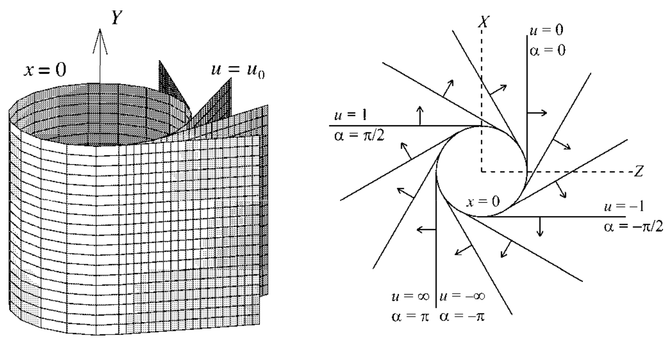


Figure 4: The wave surfaces $u = u_0$ are tangent half-planes to expanding cylinder (for T = const.; Left), or circle (for T, Y = const.; Right) in Minkowski background coordinates. For different values of α, the successive surfaces are rotated around the cylinder which expands with the speed of light.

Is the *envelope singularity* $x = 0$ only a coordinate one or is it a "physical" singularity? We studied this in [A8] by investigating the behaviour of test particles. The simplest radiative Kundt spacetimes (14) are $G^{(n)} = \mathcal{R}e\,\{(x + \mathrm{i}\,y)^n\}$, $n = 2, 3, 4, \ldots$. The geodesic equations simplify for *geodesics in the hypersurface* $y = 0$, and we found that there are *exact* geodesics of the "power-law" form

$$x = \alpha(\tau - \tau_0)^p\ , \quad u = \beta(\tau - \tau_0)^q + u_0\ , \quad v = \gamma(\tau - \tau_0)^r\ . \tag{16}$$

There are three families of them denoted as $\mathrm{A}^{(n)}$, $\mathrm{B}^{(n)}$ and $\mathrm{C}^{(n)}$. In [A8] we also found frames that are parallelly transported along *any* geodesic, and we demonstrated that the spatial direction of propagation of the wave *always rotates*. Finally, we calculated projections of the curvature tensor onto these frames. It turned out, that all the Riemann tensor components evaluated along timelike geodesics are proportional to $\mathcal{A}_+ = -2x\dot{u}^2 G_{,xx}$, whereas along null geodesics the relevant NP coefficients are $\Psi^{||}_j = (L_1 + \sqrt{2}/\dot{x})^j\,\mathcal{A}_+$, where $\dot{L}_1 = (n + 2a + 2)/\sqrt{2}x$. For exact geodesics $\mathrm{A}^{(n)}$, $\mathrm{B}^{(n)}$, $\mathrm{C}^{(n)}$ we obtain:

Class	ϵ	x	$\mathcal{A}_+$	$\Psi^{\|\|}_j$
$\mathrm{A}^{(n)}$	0	$\frac{2}{n+3}$	-2	$\frac{(j-2)\,n+(j-6)}{n+3}$
$\mathrm{B}^{(n)}$	0	$\frac{2(1-n)}{n^2-6n+1}$	-2	$\frac{(j-2)\,n^2-4(j-3)\,n-(j+2)}{n^2-6n+1}$
$\mathrm{C}^{(n)}$	-1	1	-2	—

Table 1: The powers of $(\tau - \tau_0)$ for quantities which enter the components of the curvature tensor in frames parallelly propagated along the classes of exact geodesics $y = 0$.

Interestingly, although the dependence of $x \sim (\tau-\tau_0)^p$ on the affine parameter τ varies for different classes $\mathrm{A}^{(n)}$, $\mathrm{B}^{(n)}$, and $\mathrm{C}^{(n)}$, the dependence of $\mathcal{A}_+$ is *the same*, namely $\mathcal{A}_+ \sim (\tau-\tau_0)^{-2}$. For the explicit class of *timelike geodesics* $\mathrm{C}^{(n)}$ we obtain $x \sim (\tau-\tau_0)$, so that the amplitude $\mathcal{A}_+$ behaves as

$$\lim_{x\to 0} \mathcal{A}_+ = \infty \ , \qquad \lim_{x\to\infty} \mathcal{A}_+ = 0 \ . \tag{17}$$

The curvature tensor in parallel frames along these geodesics approaching $x=0$ thus *diverge*. The envelope singularity of the Kundt spacetimes is a *non-scalar curvature singularity*, according to the scheme introduced in [86]. Similar results follow from the class of exact *null* geodesics $\mathrm{A}^{(n)}$ and $\mathrm{B}^{(n)}$.

1.2.3 Geodesics in the Kundt waves with Λ: the Siklos family

In the comprehensive publication [A9] we analyzed the Siklos class of solutions [57],

$$\mathrm{d}s^2 = (-3/\Lambda)\ x^{-2}\,(\mathrm{d}x^2+\mathrm{d}y^2+2\,\mathrm{d}u\,\mathrm{d}v+H\,\mathrm{d}u^2)\ , \tag{18}$$

which is identical to the subclass of the Kundt waves (4) with a negative cosmological constant Λ, and parameters $\alpha=1$, $\beta=\sqrt{-\Lambda/6}$. Interestingly, in this particular case the interpretation orthonormal frame is *uniformly rotating* with respect to a parallelly propagated tetrad with angular velocity $\Omega=\sqrt{-\Lambda/3}$. The direction of propagation of transverse gravitational waves in the anti-de Sitter universe is thus twisting.

In [A9] we analyzed in more detail properties of the Kaigorodovov spacetime [16, 87] which is the simplest representative of the Siklos class given by $H=x^3$. It is a vacuum homogeneous spacetime of type N with $\Lambda<0$, admitting five Killing vectors. We have found an explicit form of all geodesics and a general solution of the equation of geodesic deviation. This enabled us to investigate main features of the global structure of the Kaigorodov spacetime. In particular, $x=\infty$ represents a curvature singularity, in the region $x=0$ the Kaigorodov spacetime is *weakly asymptotically anti-de Sitter*. Our results [A9] concerning the Kaigorodov spacetime were recently used in [88, 89] and elsewhere for construction of specific cosmological *brane models*, motivated by the AdS/CFT conjecture.

In the subsequent work [A10] we studied geometric properties of *sandwich* gravitational waves constructed from the Siklos family of spacetimes by considering the function H in (18) to be of the form $H=h(x,y)d(u)$, where the wave-profile function $d(u)$ is non-vanishing on a finite interval only, say $u\in[u_1,u_2]$. In front on the propagating wave (for $u<u_1$) and also behind it (for $u>u_2$) there are two anti-de Sitter regions, separated by the sandwich wave of a finite duration. The situation is analogous to the well-known plane waves in Minkowski space [25, 27] but the geometry is different. The "background" is everywhere curved, and the wave surfaces are hyperboloidal surfaces. We concentrated on the generalized Defrise solution characterized by the function of the form $H=d(u)\,x^{-2}$. An interesting property of these solutions is that they are *non-singular* since the components of the curvature tensor are only proportional to one of the following functions:

$$\mathcal{A}_+ = -4C^2\,d(u)\ , \qquad \mathcal{A}_\times = 0\ , \qquad \mathcal{M} = C^2\,d(u)\ , \tag{19}$$

where C is a constant. Therefore, for an arbitrary bounded profile function $d(u)$, the frame components of the curvature tensor *remains finite*, as seen by *any* timelike observer.

1.2.4 Type N Robinson–Trautman sandwich waves

Recently [A11], we also concentrated on construction of *expanding* sandwich gravitational waves which belong to the Robinson–Trautman family $RTN(\Lambda)$ of vacuum solutions of type N. Instead of using the line element (5) we employed the original coordinates from [28],

$$\mathrm{d}s^2 = 2\,r^2P^{-2}\,\mathrm{d}\zeta\,\mathrm{d}\bar\zeta - 2\,\mathrm{d}u\,\mathrm{d}r - \left[\,2\epsilon - 2r(\ln P)_{,u} - \tfrac{\Lambda}{3}r^2\,\right]\,\mathrm{d}u^2\ , \tag{20}$$

where $\epsilon = +1, 0$ or -1 fixes the Gaussian curvature of the 2-surfaces of constant u and r, and the function $P(\zeta, \bar{\zeta}, u)$ has the form $P = (1 + \epsilon F\bar{F})(F_{,\zeta}\bar{F}_{,\bar{\zeta}})^{-1/2}$, where $F(\zeta, u)$ is an arbitrary complex function. We introduced a specific family of $RTN(\Lambda)$ solutions by considering the simple choice

$$F(\zeta, u) = \zeta^{g(u)} \ , \qquad (21)$$

where g is an arbitrary positive function. Then $\Psi_4 = -\frac{g'\zeta}{2gr\bar{\zeta}}(|\zeta|^{-g} + \epsilon|\zeta|^{g})^2$ so that the solution is conformally flat if and only if g is constant. Provided that $g' \neq 0$ there are curvature singularities at $r = 0$ and at $\zeta = 0$ or ∞. Any wave surface has the spatial geometry of a sphere, plane or hyperboloid for $\epsilon = 1, 0, -1$, respectively, and if $\arg\zeta \in [0, 2\pi)$ then the angle covers the range $[0, 2\pi g)$. When $g < 1$ this indicates that *the wave surface includes a deficit angle* $2\pi(1 - g)$, i.e. a cosmic string.

The particular case where $\Lambda = 0$, $\epsilon = 1$ and the function g consists of two constant pieces joined by a segment linear in u has previously been studied in [90], and this can be interpreted as a "disintegrating" cosmic string. More generally, if $g(u) = 1$ for $u < 0$ and g decreases continuously until it reaches a positive constant value when $u > 0$, the metric again describes a sandwich Robinson–Trautman wave representing the disintegration of a cosmic string, see figure 5. If $\epsilon = 1$ the wave surfaces (at constant time t) are concentric spheres, with strings of equal tension at $\rho = 0$ and π in the background spacetime ahead of the surfaces.

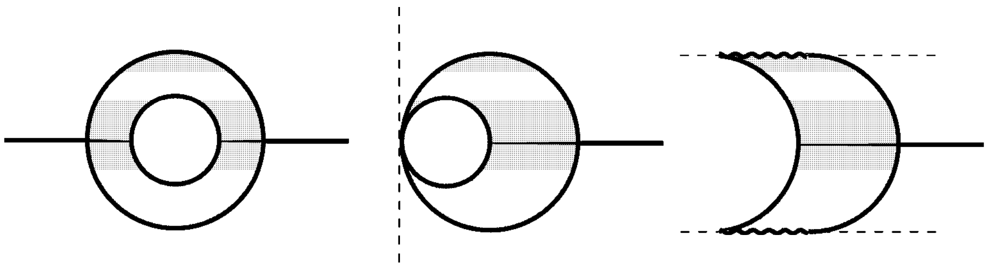


Figure 5: The shaded regions represent Robinson–Trautman sandwich waves at some fixed time for different values of ϵ. The expanding (hemi)spherical wave surfaces are given by $u = const.$ The region behind the wave is Minkowski or (anti-)de Sitter, while the region ahead of the wave is Minkowski or (anti-)de Sitter with a cosmic string.

1.3 More general radiative Kundt spacetimes

In preceding section we summarized our contributions to analysis of type N Einstein spaces. Recently we have begun investigations of more general radiative spacetimes, both nonexpanding and expanding.

In the work [A12] we presented the complete new family of type III Kundt spacetimes for which Λ is nonzero. The presence of a pure radiation field was also assumed. We identified three subclasses, namely generalized pp-waves, generalized Kundt waves, and generalized Siklos waves (see [A12] for more details). The wave surfaces $u = const.$ are plane, spherical or hyperboloidal in Minkowski, de Sitter or anti-de Sitter backgrounds respectively, when Λ is respectively zero, positive and negative.

The singularities in these spacetimes are *envelopes of the wave surfaces*, similarly as in the case $\Lambda = 0$ described in 1.2.2, see figure 4. Using the 5-dimensional parametrization of the (anti-)de Sitter spacetime (1), we demonstrated that the singularity $\xi + \bar{\xi} = 0$ is located on ${Z_3}^2 + \epsilon {Z_4}^2 = \epsilon a^2$, and ${Z_1}^2 + {Z_2}^2 = {Z_0}^2$. For the de Sitter background this is an expanding torus, for the anti-de Sitter background it is an expanding hyperboloid. For both cases, the wave surfaces u_0 are intersection of (1) with the plane $-Z_0 + \cos\alpha\, Z_1 + \sin\alpha\, Z_2 = 0$. These rotated planes cut the hyperboloid at $(\sin\alpha\, Z_1 - \cos\alpha\, Z_2)^2 + {Z_3}^2 + \epsilon {Z_4}^2 = \epsilon a^2$. For the de Sitter background, these wave surfaces are a family of spheres with constant area $4\pi a^2$. The plane cuts are tangent to the expanding torus, so that the singularity can be interpreted as a caustic envelope of wave surfaces, see figure 6.

Another possible generalization of nonexpanding Kundt waves $KN(\Lambda)$ was presented in [A13]. We found a particular yet rather large class of explicit solutions of algebraic type II and N which

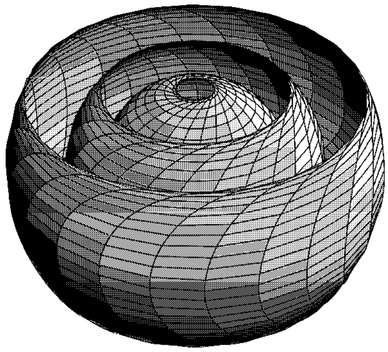


Figure 6: Portions of the de Sitter universe covered by the coordinates of the generalized Kundt waves. The wave surfaces $u = u_0$ are represented by semicircles of constant radius a which are all tangent to the two expanding circles at $Z_3 = \pm a$ which are particular sections of the expanding torus corresponding to the singularity $x = 0$.

represent gravitational waves in various type D and O background universes. These can be written in a general unified form (4) with

$$p = 1 + \alpha\xi\bar{\xi}\ , \qquad q = (1+\beta\xi\bar{\xi})\,\varepsilon + C\,\xi + \bar{C}\,\bar{\xi}\ , \qquad \mathrm{F} = D\,\frac{q^2}{p^2}\,v^2 - \frac{(q^2)_{,u}}{p^2}\,v + \frac{q}{p}\,\mathrm{H}\ , \tag{22}$$

where α, β, and ε are constants, and $C(u)$, $D(u)$, $\mathrm{H}(\xi,\bar{\xi},u)$ are arbitrary functions. For specific choices of the parameters and structural functions one obtains a variety of type II Kundt solutions to Einstein's equations which admit a cosmological constant, electromagnetic field, pure radiation and/or general non-null matter component. When $D = -2\varepsilon\beta + 2C\bar{C}$ these are type N spacetimes, when $\mathrm{H} = [A_0(u) + A_1(u)\zeta + \bar{A}_1(u)\bar{\zeta} + A_2(u)\zeta\bar{\zeta}]/p$ they are of type D, and if both choices are made simultaneously, we obtain conformally flat (type O) spacetimes. The later spacetimes represent backgrounds, such as Minkowski, (anti-)de Sitter, pp-waves [9, 25, 91, 92], Edgar–Ludwig [56, 83], Bertotti–Robinson [93–95], and type D Nariai [96] or Plebański–Hacyan [97] spaces.

By considering a non-trivial function $\mathrm{H}(\xi,\bar{\xi},u)$ it is possible to introduce gravitational waves of arbitrary profiles into the above background universes. In [A13] we discussed and interpreted all such possibilities and we studied the effect on test particles. Sandwich and impulsive waves propagating in these spaces with different geometries and matter content can also easily be constructed.

1.4 More general radiative Robinson–Trautman spacetimes

In [A14] and [A15] we analyzed properties of more general radiative Robinson–Trautman spacetimes which are of algebraic type II, in particular those with a cosmological constant Λ which may serve as an exact model of gravitational radiation in asymptotically (anti-)de Sitter universe generated under the black-hole formation. Vacuum Robinson–Trautman solutions of this type with $\Lambda = 0$ have attracted an increased attention in the last two decades. The most rigorous approach was taken by Chruściel [98, 99] who demonstrated that metrics of the form

$$\mathrm{d}s^2 = 2\,r^2P^{-2}\,\mathrm{d}\zeta\,\mathrm{d}\bar{\zeta} - 2\,\mathrm{d}u\,\mathrm{d}r - \left[2P^2(\ln P)_{,\zeta\bar{\zeta}} - 2r(\ln P)_{,u} - 2m/r - \tfrac{\Lambda}{3}r^2\right]\,\mathrm{d}u^2\ , \tag{23}$$

with $\Lambda = 0$ *exist globally* for all values $u \geq u_0$ for arbitrary smooth initial data prescribed on u_0. Moreover, the solutions converge asymptotically to the Schwarzschild metric with the corresponding mass m. Introducing $P = f(u,\zeta,\bar{\zeta})\,(1+\frac{1}{2}\zeta\bar{\zeta})$, for $u \to \infty$ there is

$$f = 1 + f_{1,0}\,e^{-2u/m} + f_{2,0}\,e^{-4u/m} + \cdots + f_{14,0}\,e^{-28u/m} + f_{15,1}\,u\,e^{-30u/m} + f_{15,0}\,e^{-30u/m} + \cdots\ , \tag{24}$$

where $f_{i,j}$ are smooth functions of the spatial coordinates $\zeta,\bar{\zeta}$, i.e. $P \to (1+\frac{1}{2}\zeta\bar{\zeta})$ which describes the spherically symmetric Schwarzschild solution. It can be shown, however, that the continuation across the horizon $u=+\infty$ of the formed black hole is *not analytic.* In fact, it has only a finite degree of smoothness: the metric can be maximally of the class C^{117}. In [A14] we generalized these results to the case $\Lambda>0$: the solutions (23) with $0<9\Lambda m^2<1$ again exist and they asymptotically approach a spherically symmetric Schwarzschild–de Sitter spacetime as $u\to+\infty$. However, the presence of Λ changes the global structure. Future conformal infinity $\mathcal{I}^+$ does exist but it has a spacelike character. In addition, these Robinson–Tratutman solutions (23) may serve as explicit models exhibiting the cosmic no-hair conjecture [54] because close to $\mathcal{I}$ they locally approach the de Sitter spacetime.

The presence of $\Lambda>0$ also influences the degree of smoothness of a possible extension across the black hole horizon localized at $u=+\infty$. The function f is again given by the expansion (24), but for $0<9\Lambda m^2<1$ it is necessary to employ a different transformation to Kruskal null coordinates. Consequently, extension across the horizon can be *smoother* than in the $\Lambda=0$ case [A14]. The horizon can even be made "arbitrarily smooth" by letting Λ to approach its extreme value, $\Lambda\to 1/9m^2$.

This interesting effect motivated us to analyze in detail the properties of the solutions (23) in the *"extreme" case* $9\Lambda m^2=1$, for which the black hole horizon and the cosmological horizon coincide. In subsequent publication [A15] we demonstrated that for such an extreme value $\Lambda=1/9m^2$ the expansion (24) takes, in appropriate Kruskal null coordinates $\hat u$, $\hat v$, the form

$$f=1+f_{1,0}\,e^{-(2\delta/m)\cot\hat u}+\cdots+f_{14,0}\,e^{-(28\delta/m)\cot\hat u}+\delta f_{15,1}\cot\hat u\,e^{-(30\delta/m)\cot\hat u}+\cdots\ , \tag{25}$$

where $\delta=-(3-2\ln 2)m<0$. It follows that the spacetimes can be extended across the horizon $\hat u=0_-$ *in a smooth C^∞ way* and they can be joined with the solutions representing an extreme Schwarzschild–de Sitter spacetime with the same values of Λ a m. The extension is smooth, but *not analytic* because it is not unique. This result has been used by Chruściel [100] as an argument against a "natural" assumption of analyticity which is usually considered in proofs of the "rigidity" theorem. Besides this extreme Robinson–Trautman solutions we investigated in [A15] also the case $9\Lambda m^2>1$ which represents a formation of a naked singularity in the de Sitter universe, and solutions with $\Lambda<0$. With decreasing Λ the smoothness of the extension across the horizon decreases.

In several papers we also studied another important representative of the expanding Robinson–Trautman family of spacetimes, the so-called *C-metric.* This is a well-known type D solution of the Einstein(–Maxwell) equations representing *uniformly accelerated sources.* It was discovered in 1917 by Levi-Civita [93] and Weyl [101]. The principal physical understanding of the C-metric with $\Lambda=0$ was obtained in [33]. Subsequently, a great number of works analyzed various aspects of this solution [102–105]. A generalization of the C-metric to $\Lambda\neq0$ was discovered in [106], see also [107].

In our works we concentrated on the physical interpretation of these spacetimes. We explicitly put the C-metric in the Robinson-Trautman form (23). This is useful for identifying the curvature singularity at $r=0$ and also of the conformal infinity $\mathcal{I}$ given by $r=\infty$, whose character is determined by the sign of Λ. To achieve an interpretation of the constant parameters entering the metric — in particular to identify the acceleration of the sources — we introduced in [A16] a new set of coordinates which are best adapted to the accelerated observers in the de Sitter universe, namely

$$\begin{aligned}
\mathrm{d}s^2 &= \Big[\sqrt{1+\tfrac{3}{\Lambda}A^2}-A\,R\,\zeta(\theta)\Big]^{-2}\Big\{-F(R)\,\mathrm{d}T^2+\frac{\mathrm{d}R^2}{F(R)}+R^2\big(\mathrm{d}\theta^2+G^2(\theta)\,c^2\mathrm{d}\Phi^2\big)\Big\}\,,\\
\text{where}\quad F(R) &= 1-\tfrac{\Lambda}{3}R^2-\sqrt{1+\tfrac{3}{\Lambda}A^2}\,\frac{2m}{R}+(1+\tfrac{3}{\Lambda}A^2)\frac{e^2}{R^2}\ ,\\
G^2(\theta) &= 1-\zeta^2(\theta)+2mA\zeta^3(\theta)-e^2A^2\zeta^4(\theta)\ ,
\end{aligned} \tag{26}$$

and $\zeta(\theta)$ is the inverse of $\theta(\zeta)=\int(1-\zeta^2+2mA\zeta^3-e^2A^2\zeta^4)^{-\frac12}\,\mathrm{d}\zeta$. It may be seen that, either when $A=0$ or when both $m=0$ and $e=0$, we have $\zeta(\theta)=-\cos\theta$ so that $G(\theta)=\sin\theta$. Obviously, when $A=0$, $c=1$, the metric (26) reduces to the familiar form of the Reissner–Nordström–de Sitter black hole solution in which the parameters m and e have the usual interpretation as mass and charge. Moreover, when $A\neq0$, $c=1$ and $m=0=e$, *the origin $R=0$ is accelerating in a de Sitter universe with uniform acceleration given by the parameter A.* When m and e are small, (26) can naturally be regarded as a perturbation and thus can be interpreted as describing *a pair of charged black holes uniformly accelerating in a de Sitter universe.*

Similar analysis can be done also for the C-metric spacetime with for $\Lambda < 0$. However, as we demontrated in [A17], there are some fundamental differences concerning the global structure. In the de Sitter case, there are *two* accelerating black holes connected to each other by finite string(s). In the anti-de Sitter background, the metric form (26) admits just a *single* accelerating black hole attached to *semi-infinite* open string(s). These have a "hyperboloidal" character and connect the black hole to $\mathcal{I}$. The maximal possible value of acceleration is $A = \sqrt{-\Lambda/3}$, with the corresponding privileged observers at the origin $R = 0$ localized on the lines of constant value of χ in figure 1.

Nevertheless, there exist alternative metric forms of the "anti-de Sitter C-metric" which admit an *arbitrarily large* value of the acceleration A of the black holes. Such spacetimes describe *a pair* of uniformly accelerated black holes in the asymptotically anti-de Sitter universe [107]. The "limiting case" $A = \sqrt{-\Lambda/3}$ plays a special role in the context of the Randall–Sundrum model. We will return to the C-metric in section 3.8.

2 (B) Impulsive waves

In recent years, many aspects of exact solutions of Einstein's equations which describe impulsive gravitational waves in spaces of constant curvature have been thoroughly investigated. These impulses may be either nonexpanding or expanding. We provide here a brief but comprehensive review of all impulsive waves which may propagate in a Minkowski, de Sitter, or anti-de Sitter universe. From a unified point of view we describe all the main methods for their construction: the Penrose "cut and paste" method, explicit construction of continuous coordinates, distributional limits of sandwich waves, embedding from higher dimensions, and boosts of sources or limits of infinite accelerations. This framework enables us to elucidate the significance of our original publications [B1]–[B16] which we devoted to this topic. We will also describe our studies of physically important particular solutions, namely nonexpanding impulses generated by multipole particles, and expanding impulses generated by snapping or colliding strings, including the behaviour of geodesics in these spacetimes.

2.1 On various methods of construction of impulsive waves

The history of studies of impulsive waves can be divided (roughly speaking) into three periods. During the first epoch, which culminated in classic works by Lichnerowicz and others [12, 111], principal mathematical properties of possible shock or impulsive waves in general relativity were found and investigated. It was demonstrated that these must necessarily be localized on null hypersurfaces, across which the derivatives of the metric (the second or the first) are discontinuous. Nevertheless, explicit constructions of impulsive solutions and further investigation of their properties had not been performed until the fundamental contributions by Penrose [108, 110] and Aichelburg and Sexl [112] appeared at the beginning of 1970's. During this "Golden Era" most of the different methods of construction occured almost simultaneously. The last era, which can be called the "Renaissance", started at the beginning of 1990's with a fundamental paper by Hotta and Tanaka [113]. By boosting the Schwarzschild–(anti-)de Sitter black hole to the speed of light they constructed a nonexpanding impulse in the de Sitter and anti-de Sitter universes. Simultaneously, an interesting expanding impulsive-wave solution generated by a "snapping" cosmic string in Minkowski space was discovered and discussed by Gleiser and Pullin [114], Bičák [115], Nutku and Penrose [116], and Hogan [117, 118].

As a result of these and subsequent systematic investigations a complete picture has now emerged. This comprises both nonexpanding and expanding impulses in constant-curvature spaces with an arbitrary value of the cosmological constant. We will summarize here the classic and also more recent results here. We shall present these in the context of a unifying description of the main methods by which complete families of impulsive gravitational waves can be constructed.

2.2 The "cut and paste" method

Let us start with an elegant geometrical method for constructing general nonexpanding (plane) and expanding (spherical) impulsive gravitational waves in Minkowski background. This was presented by Penrose in now classic works [108, 110]. His "cut and paste" approach, which is in some respects similar to that of Israel [119–121], is based on cutting the spacetime manifold $\mathcal{M}$ along a suitable null hypersurface and then re-attaching the two pieces with a specific "warp". The first case leads to impulsive *pp*-waves, the second case gives expanding impulsive waves. Recently, this approach has also been extended to backgrounds with a nonzero cosmological constant Λ [122], [B7]–[B10].

Nonexpanding impulsive waves

In this case, the Penrose method is based on the removal of the null hypersurface $\mathcal{U} = 0$ from spacetime in the metric form (3) and re-attaching the halves $\mathcal{M}_-(\mathcal{U} < 0)$ and $\mathcal{M}_+(\mathcal{U} > 0)$ by making the identification with a "warp" in the coordinate $\mathcal{V}$ such that

$$\left[\,\eta,\,\bar\eta,\,\mathcal{V},\,\mathcal{U}=0_-\,\right]_{\mathcal{M}_-} \equiv \left[\,\eta,\,\bar\eta,\,\mathcal{V}-H(\eta,\bar\eta),\,\mathcal{U}=0_+\,\right]_{\mathcal{M}_+}\,, \tag{27}$$

where $H(\eta,\bar\eta)$ is an arbitrary function of η and $\bar\eta$ alone, see the left part of figure 7. It was shown in [110] that the condition (27) automatically guarantees that the Einstein field equations are satisfied everywhere including on $\mathcal{U} = 0$. However, impulsive components are introduced into the curvature tensor proportional to $\delta(\mathcal{U})$. These represent gravitational (plus possibly null-matter) waves.

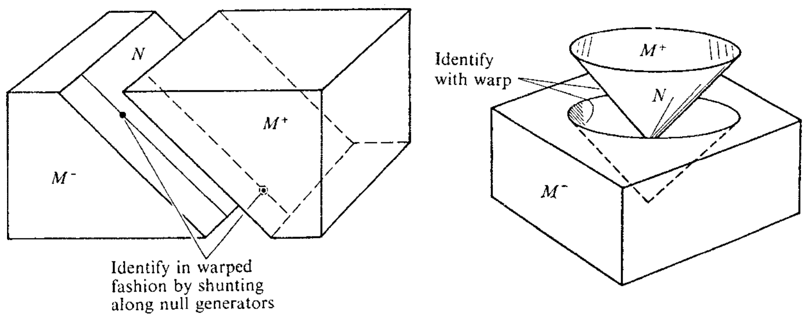


Figure 7: The Penrose "cut and paste" construction of nonexpanding (left) and expanding (right) impulsive gravitational waves in Minkowski background (reproduced from the classic work [110]).

In [110] Penrose considered only the Minkowski background space (3) with $\Lambda = 0$, in which case the impulsive surface $\mathcal{U} = 0$ is a plane. Impulsive *pp*-waves are thus obtained. However, the above method can easily be generalized to $\Lambda \neq 0$ cases. We demonstrated in [B7] that exactly the same junction conditions (27) applied to a general background spacetime introduce impulsive waves also in the de Sitter ($\Lambda > 0$) or anti-de Sitter ($\Lambda < 0$) universes. The wave surface is a *sphere* or a *hyperboloid*, respectively. For cases $\Lambda \neq 0$, the geometry of these impulsive spherical and hyperboloidal waves was described in detail in [B5] using various convenient coordinate representations. It was also demonstrated that the wave surfaces are indeed nonexpanding.

Expanding impulsive waves

In [110] Penrose demonstrated that his "cut and paste" method can be also used to construct expanding spherical impulsive (purely) gravitational waves. However, the junction conditions are somewhat more complicated than in the nonexpanding case. Instead of starting with the form of spacetime (3), it is necessary to first perform the transformation

$$\mathcal{U} = \frac{Z\bar Z}{p}V - U\,, \qquad \mathcal{V} = \frac{V}{p} - \epsilon U\,, \qquad \eta = \frac{Z}{p}V\,, \qquad p = 1+\epsilon Z\bar Z\,, \qquad \epsilon = -1, 0, +1\,. \tag{28}$$

With this, the constant curvature spacetimes take the following form

$$\mathrm{d}s_0^2 = \frac{2\,(V/p)^2\,\mathrm{d}Z\,\mathrm{d}\bar{Z} + 2\,\mathrm{d}U\,\mathrm{d}V - 2\epsilon\,\mathrm{d}U^2}{\left[\,1+\frac{1}{6}\Lambda U(V-\epsilon U)\,\right]^2}\ . \tag{29}$$

The hypersurface $U=0$ is a *null cone*. The spacetime can be divided into $\mathcal{M}_-(U<0)$ inside the null cone, and $\mathcal{M}_+(U>0)$ outside this. Now, the Penrose junction conditions prescribe the identification

$$\left[\,Z,\,\bar{Z},\,V,\,U=0_-\,\right]_{\mathcal{M}_-} \equiv \left[\,h(Z),\,\bar{h}(\bar{Z}),\,\frac{(1+\epsilon h\bar{h}\,)\,V}{(1+\epsilon Z\bar{Z})|h'|},\,U=0_+\,\right]_{\mathcal{M}_+} \tag{30}$$

of the points from the two re-attached parts across the impulsive sphere $U=0$, see the right part of figure 7. In (30) an arbitrary function $h(Z)$ introduces a specific "warp" corresponding to one of all possible impulsive *vacuum* spacetimes of this type.

Penrose described in [110] the above "cut and paste" construction for the case $\Lambda=0,\ \epsilon=0$. Generalization to impulsive spherical waves in the de Sitter and anti-de Sitter universe (with $\epsilon=0$) was later found by Hogan [122]. In [118] he also introduced impulses in Minkowski spacetime with $\epsilon=+1$. The completely general form (29), (30) of the Penrose junction conditions has been recently presented in our contributions [B9], [B10].

2.3 Continuous coordinates

The Penrose "cut and paste" approach described in the previous section is a general method. Nevertheless, the formal junction condition identifications (27) or (30) do not immediately provide explicit metric forms of the complete spacetimes. It is thus of interest to find a suitable coordinate system for the above solutions in which the metric is explicitly continuous everywhere, including on the impulse.

Nonexpanding impulsive waves

The metric can be written as

$$\mathrm{d}s^2 = \frac{2\,|\mathrm{d}Z + U\Theta(U)(H_{,Z\bar{Z}}\mathrm{d}Z + H_{,\bar{Z}\bar{Z}}\mathrm{d}\bar{Z})|^2 - 2\,\mathrm{d}U\mathrm{d}V}{[\,1+\frac{1}{6}\Lambda(Z\bar{Z}-UV-U\Theta(U)G)\,]^2}\ , \tag{31}$$

where $\Theta(U)$ is the Heaviside step function. This is continuous across the null hypersurface $U=0$ but the discontinuity in the derivatives of the metric introduces impulsive components in the Weyl tensor proportional to the Dirac distribution [B7], $\Psi_4=(1+\frac{1}{6}\Lambda Z\bar{Z})^2 H_{,ZZ}\,\delta(U)$. The metric (31) explicitly describes impulsive waves in de Sitter, anti-de Sitter or Minkowski backgrounds. For $\Lambda=0$, the line element (31) reduces to the well-known Rosen form of impulsive *pp*-waves [62, 110], [B2]. Note also that the continuous coordinate system for the particular Aichelburg–Sexl solution [112], see (41) further in the text, was found by D'Eath [124] and used for analytic investigation of ultrarelativistic black-hole encounters. This was used for studies of high-energy scattering in quantum gravity on the Planck scale [125].

Expanding impulsive waves

In this case, the line element can be written in the form

$$\mathrm{d}s^2 = \frac{2\left|(V/p)\,\mathrm{d}Z + U\Theta(U)\,p\,\bar{H}\,\mathrm{d}\bar{Z}\right|^2 + 2\,\mathrm{d}U\,\mathrm{d}V - 2\epsilon\,\mathrm{d}U^2}{\left[\,1+\frac{1}{6}\Lambda U(V-\epsilon U)\,\right]^2}\ , \tag{32}$$

where $H(Z)=\frac{1}{2}[\,(h'''/h')-\frac{3}{2}(h''/h')^2]$, and $h=h(Z)$ is an arbitrary function This metric which was presented for a Minkowski background in [116–118], with a cosmological constant in [122], and in the most general form in [B9], [B10] is explicitly continuous everywhere, including across the null hypersurface $U=0$. Again, the discontinuity in the derivatives of the metric yields impulsive components in the Weyl tensor, $\Psi_4=(p^2H/V)\,\delta(U)$.

2.4 Limits of sandwich waves

Impulsive waves can also be understood as distributional limits of appropriate sequence of sandwich waves in a suitable family of exact radiative spacetimes. In fact, this seems to be the most intuitive way of construction of impulsive waves, although mathematical difficulties occur with this approach in the case of expanding impulses of the Robinson–Trautman type.

Nonexpanding impulsive waves

We demonstrated in [B1] that *all* nonexpanding impulses in Minkowski or (anti-)de Sitter universes can simply be constructed from the Kundt class $KN(\Lambda)$ of exact type N solutions (4) by considering the distributional form $\mathrm{H}(\xi,\bar\xi)\,\delta(u)$ of the structural function H. The metrics can thus be written as

$$\mathrm{d}s^2 = 2\,\frac{1}{p^2}\,\mathrm{d}\xi\mathrm{d}\bar\xi - 2\,\frac{q^2}{p^2}\,\mathrm{d}u\mathrm{d}v + \left[\kappa\frac{q^2}{p^2}\,v^2 - \frac{(q^2)_{,u}}{p^2}\,v + \frac{q}{p}\,\mathrm{H}(\xi,\bar\xi)\,\delta(u)\right]\mathrm{d}u^2\ , \tag{33}$$

As indicated in section 1.2, for a general wave-profile of the function H there exist various distinct canonical subclasses of (4) characterized by specific choices of the parameters α and β. However, *impulsive* limits of these subclasses become (locally) *equivalent*. For $\Lambda = 0$, the only non-trivial impulsive gravitational waves of the form (33) in Minkowski space are thus well-known impulsive *pp*-waves. Similar results hold also for the $\Lambda \neq 0$ cases.

Interestingly, there exists yet another coordinate form of the complete family of nonexpanding impulses of the Kundt class $KN(\Lambda)$ which also contains a single term with the Dirac delta,

$$\mathrm{d}s^2 = \frac{2\,\mathrm{d}\eta\,\mathrm{d}\bar\eta - 2\,\mathrm{d}\mathcal{U}\,\mathrm{d}\mathcal{V} + 2H(\eta,\bar\eta)\,\delta(\mathcal{U})\,\mathrm{d}\mathcal{U}^2}{[\,1+\frac{1}{6}\Lambda(\eta\bar\eta-\mathcal{U}\mathcal{V})\,]^2}\ , \tag{34}$$

which explicitly includes the impulse located on the wavefront $\mathcal{U} = 0$ (see [B7], [127] for details). Again, in Minkowski background this is just the Brinkmann form of a general impulsive *pp*-wave.

Expanding impulsive waves

As was argued in [B9], the family of solutions for expanding impulsive spherical gravitational waves can be considered to be an impulsive limit of the class $RTN(\Lambda)$ of vacuum Robinson–Trautman type N solutions with a cosmological constant. It is convenient to consider these solutions in the García–Plebański [58] coordinates (5). Introducing a new coordinate w by $w = \psi v$, and assuming the impulsive limit $f \equiv f(\xi)\delta(u)$, we express the above family of expanding impulses as

$$\begin{aligned}\mathrm{d}s^2 = {}& 2\frac{w^2}{\psi^2}\left|\mathrm{d}\xi - f\,\delta(u)\,\mathrm{d}u\right|^2 + 2\,\mathrm{d}u\,\mathrm{d}w + \left(\tfrac{1}{3}\Lambda w^2 - 2\epsilon\right)\mathrm{d}u^2 \\ & + w\left[(f_{,\xi} + \bar f_{,\bar\xi}) - \frac{2\epsilon}{\psi}(f\bar\xi + \bar f\xi)\right]\delta(u)\mathrm{d}u^2\ ,\end{aligned} \tag{35}$$

where $f(\xi)$ is an arbitrary function. Similarly as in the nonexpanding case (33), the spherical impulse in the form (35) is explicitly located on the null surface $u = 0$. For $u \neq 0$ the spacetime again reduces to the background of constant curvature. However, a difficult mathematical problem occurs in this case: the metric (35) contains the *product* of the Dirac distributions in the first term. This is not a well-defined concept in the linear theory of distributions, so that the metric (35) is only formal. Nevertheless, it demonstrates that impulsive limits of Robinson–Trautman type N vacuum spacetimes are equivalent to expanding impulsive gravitational waves (32).

2.5 Embedding from higher dimensions

The complete class of *nonexpanding* impulsive waves in spaces of constant curvature with a *nonvanishing* Λ can alternatively be introduced using a convenient 5-dimensional formalism as metrics

$$\mathrm{d}s^2 = -\mathrm{d}Z_0{}^2 + \mathrm{d}Z_1{}^2 + \mathrm{d}Z_2{}^2 + \mathrm{d}Z_3{}^2 + \epsilon\,\mathrm{d}Z_4{}^2 + \mathrm{H}(Z_2, Z_3, Z_4)\,\delta(Z_0 + Z_1)\,(\mathrm{d}Z_0 + \mathrm{d}Z_1)^2\ , \qquad (36)$$

restricted by the constraint (1). Physically, this is an embedding of impulsive *pp*-waves (36) which propagate in 5-dimensional flat space onto the (anti-)de Sitter hyperboloid. The wave is absent when $\mathrm{H} = 0$, in which case (36) with the constraint (1) reduces to the standard form of the de Sitter or the anti-de Sitter spacetime (1). We demonstrated in [B6] that for a nontrivial H the metric represents impulsive waves propagating in the (anti-)de Sitter universe. Each impulse is located on the null hypersurface $Z_0 + Z_1 = 0$. Considering (1), this is given by

$$Z_2{}^2 + Z_3{}^2 + \epsilon Z_4{}^2 = \epsilon a^2\ , \qquad (37)$$

which is a nonexpanding 2-sphere in the de Sitter universe or a hyperboloidal 2-surface in the anti-de Sitter universe. The wave surfaces can naturally be parametrized as

$$Z_2 = a\sqrt{\epsilon(1-z^2)}\,\cos\phi\ , \quad Z_3 = a\sqrt{\epsilon(1-z^2)}\,\sin\phi\ , \quad Z_4 = a\,z\ . \qquad (38)$$

Purely gravitational waves occur when the vacuum field equation

$$(\Delta + \tfrac{2}{3}\Lambda)\,\mathrm{H} = 0 \qquad (39)$$

is satisfied [127,128], [B7], in which Δ is the Laplacian on the impulsive surface.

2.6 Boosts and limits of infinite acceleration

Let us finally outline yet another fundamental method for the construction of impulses in spaces of constant curvature. It is based on boosting suitable, initially static sources to the speed of light which yields specific nonexpanding impulsive waves. Similarly, limits of infinite acceleration of specific sources give special expanding impulsive solutions.

Nonexpanding impulsive waves

It was first demonstrated in 1971 by Aichelburg and Sexl in a classic paper [112] that a specific impulsive gravitational *pp*-wave solution can be obtained by boosting the Schwarzschild black hole to the speed of light, while its mass is reduced to zero in an appropriate way. In particular, this leads to an impulsive *pp*-wave metric,

$$\mathrm{d}s^2 = -\mathrm{d}\tilde{t}^2 + \mathrm{d}\tilde{x}^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{H}\,\delta(\tilde{t}+\tilde{x})(\mathrm{d}\tilde{t}+\mathrm{d}\tilde{x})^2\ , \qquad (40)$$

where

$$\mathrm{H} = -\tfrac{1}{2}b_0\,\log(y^2+z^2)\ . \qquad (41)$$

This is the famous Aichelburg–Sexl solution [112] which represents an axially-symmetric impulsive gravitational wave in Minkowski space generated by a single null monopole particle located on $y = 0 = z$. Similarly, numbers of other specific impulsive waves in flat space have been obtained by boosting various spacetimes of the Kerr–Newman [129–131] or the Weyl family [B4].

The method can be generalized to obtain impulses in spacetimes with $\Lambda \neq 0$. This was first done in 1993 by Hotta and Tanaka [113] who boosted the Schwarzschild–de Sitter solution to obtain a nonexpanding spherical impulsive gravitational wave generated by a pair of null monopole particles in the de Sitter background. They also described an analogous solution in the anti-de Sitter universe.

Their main "trick" was to consider the boost in the 5-dimensional representation of the (anti-)de Sitter spacetime (1) where the boost can explicitly be performed. The metric is (36) where

$$\mathrm{H} = b_0 \left(\frac{z}{2} \log \left| \frac{1+z}{1-z} \right| - 1 \right) . \tag{42}$$

This is the axially-symmetric Hotta–Tanaka solution [113]. Further details on boosting monopole (and also multipole) particles to the speed of light in the (anti-)de Sitter universe, the geometry of the nonexpanding wave surfaces, and discussion of various useful coordinates can be found in our works [B5], [B4]. In particular, we demonstrated that although the impulsive wave surface is nonexpanding, for $\Lambda > 0$ this coincides with the horizon of the closed de Sitter universe. The background space contracts to a minimum size and then reexpand in such a way that the nonexpanding impulsive wave in fact propagates with the speed of light from the "north pole" of the universe across the equator to its "south pole".

Expanding impulsive waves

Interestingly, specific expanding impulsive spherical gravitational waves can be obtained by considering exact solutions representing accelerating sources and taking these to the limit in which the acceleration becomes unbounded. It was first realized by Bičák [115] with recent generalizations given in [B12], [B13]) that expanding impulses can be obtained from boost-rotation symmetric solutions which represent gravitational field of uniformly accelerating objects.

The boost-rotation symmetric spacetimes can be described by the line element

$$\mathrm{d}s^2 = e^{\lambda}\mathrm{d}\rho^2 + \rho^2 e^{-\mu}\mathrm{d}\phi^2 + (\zeta^2 - \tau^2)^{-1}\left[e^{\lambda}(\zeta\mathrm{d}\zeta - \tau\mathrm{d}\tau)^2 - e^{\mu}(\zeta\mathrm{d}\tau - \tau\mathrm{d}\zeta)^2\right] , \tag{43}$$

where specific metric functions μ and λ depend only on $\zeta^2 - \tau^2$ and ρ^2. In particular, the Bonnor–Swaminarayan solution [31, 38] generally contains five arbitrary constants m_1, m_2, A_1, A_2, and B which determine the masses and uniform accelerations of two pairs of particles, and the singularity structure on the axis of symmetry $\rho = 0$ (see e.g. [31, 36], [B12]). Of particular interest are special cases described by Bičák, Hoenselaers and Schmidt [34] which represent only *two* (Curzon–Chazy) particles of mass m which are accelerating in opposite directions with acceleration A. For these solutions the metric functions in (43) can be written in a simple form

$$\mu = -\frac{2m}{AR} + 4mA + B , \qquad \lambda = -\frac{m^2}{A^2R^4}\,\rho^2(\zeta^2 - \tau^2) + \frac{2mA}{R}\,(\rho^2 + \zeta^2 - \tau^2) + B , \tag{44}$$

where $R = \frac{1}{2}\sqrt{(\rho^2 + \zeta^2 - \tau^2 - A^{-2})^2 + 4A^{-2}\rho^2}$. For $B = 0$, the axis $\rho = 0$ is regular between the symmetrically located particles but these are connected to infinity by two semi-infinite strings which cause their acceleration. For $B = -4mA$ there is a finite strut between the particles.

The limit of infinite acceleration $A \to \infty$ of these solutions was investigated in [115]. It is necessary to scale the mass parameter m to zero in such a way that the "monopole moment" $M_0 = -4mA$ remains constant. In this limit we obtain

$$\mu = B - M_0 , \qquad \lambda = B - \mathrm{sign}\,(\,\rho^2 + \zeta^2 - \tau^2\,)\,M_0 . \tag{45}$$

The resulting spacetime is locally flat everywhere except on the sphere $\rho^2 + \zeta^2 = \tau^2$. It therefore describes an expanding spherical impulsive gravitational wave generated by two particles which move apart at the speed of light in the Minkowski background and are connected to infinity by two semi-infinite strings ($B = 0$), or each other by an expanding strut ($B = M_0$). Performing a suitable transformation we put the solution (43), (45) for *two receding strings* into the form

$$\mathrm{d}s^2 = \left(V + \tfrac{1}{2}PU\right)^2\mathrm{d}\phi^2 + \left(V - \tfrac{1}{2}PU\right)^2\mathrm{d}\psi^2 + 2\,\mathrm{d}U\,\mathrm{d}V , \tag{46}$$

where $P = \Theta(-U) + \beta^2\,\Theta(U)$ with $\beta = \exp(M_0)$. The metric (46) is exactly that constructed by Gleiser and Pullin [114] by their "cut and paste" method. It represents an impulsive spherical gravitational wave propagating in the Minkowski universe. Outside the wave ($U > 0$), there are two receding strings characterized by a deficit angle $(1 - \beta)2\pi$, which can be interpreted as remnants of one cosmic string which "snapped". However, as pointed out by Bičák [115, 132], the complete solution rather describes two semi-infinite strings approaching at the speed of light and separating again at the instant at which they collide. The complementary solution for an *expanding strut* is given by $P = \beta^{-2}\,\Theta(-U) + \Theta(U)$.

We analyzed in [B12] the limit of infinite acceleration of much larger class of boost-rotationally symmetric spacetimes. These explicit solutions were found by Bičák, Hoenselaers and Schmidt [34] and represent two uniformly accelerating particles with an *arbitrary multipole structure* attached to conical singularities (as in the two cases above). These solutions can be written as [B12]

$$\begin{aligned} \mu &= 2\sum_{n=0}^{\infty} M_n\,\frac{P_n}{(x-y)^{n+1}} - \sum_{n=0}^{\infty}\frac{M_n}{2^n} + B\,, \qquad (47)\\ \lambda &= -2\sum_{k,l=0}^{\infty} M_k M_l\,\frac{(k+1)(l+1)}{(k+l+2)}\,\frac{(P_kP_l - P_{k+1}P_{l+1})}{(x-y)^{k+l+2}} - \left(\frac{x+y}{x-y}\right)\sum_{n=0}^{\infty}\frac{M_n}{2^n}\sum_{l=0}^{n}\left(\frac{2}{x-y}\right)^l P_l + B\,, \end{aligned}$$

where the constants M_n are the multipole moments, $x - y = 4A^2R$, $x + y = 2A^2(\rho^2 + \zeta^2 - \tau^2)$, the argument of the Legendre polynomials P_n is $\alpha = \frac{1}{2}(\rho^2 - \zeta^2 + \tau^2 + A^{-2})/R$, and B is a constant. Now, we consider the null limit $A \to \infty$ in which all the multipole moments M_n are kept constant. Interestingly, in this limit we again obtain (45), only the parameter M_0 is now replaced by a more general parameter $M = \sum_{n=0}^{\infty} 2^{-n}M_n$. In the limit, the multipole structure of initial particles *disappears* and the solution is characterised by the single constant M only.

In the following papers [B13], [B14] we also investigated possible null limits $A \to \infty$ of another important explicit class of boost-rotation solutions, namely the well-know C-metric. In [B13] we investigated the limits $A \to \infty$ for $\Lambda = 0$. It was demonstrated that (scaling again the mass parameter m to zero such that mA remains constant) this limit is *identical* to the metric (46) of a spherical impulsive gravitational wave with

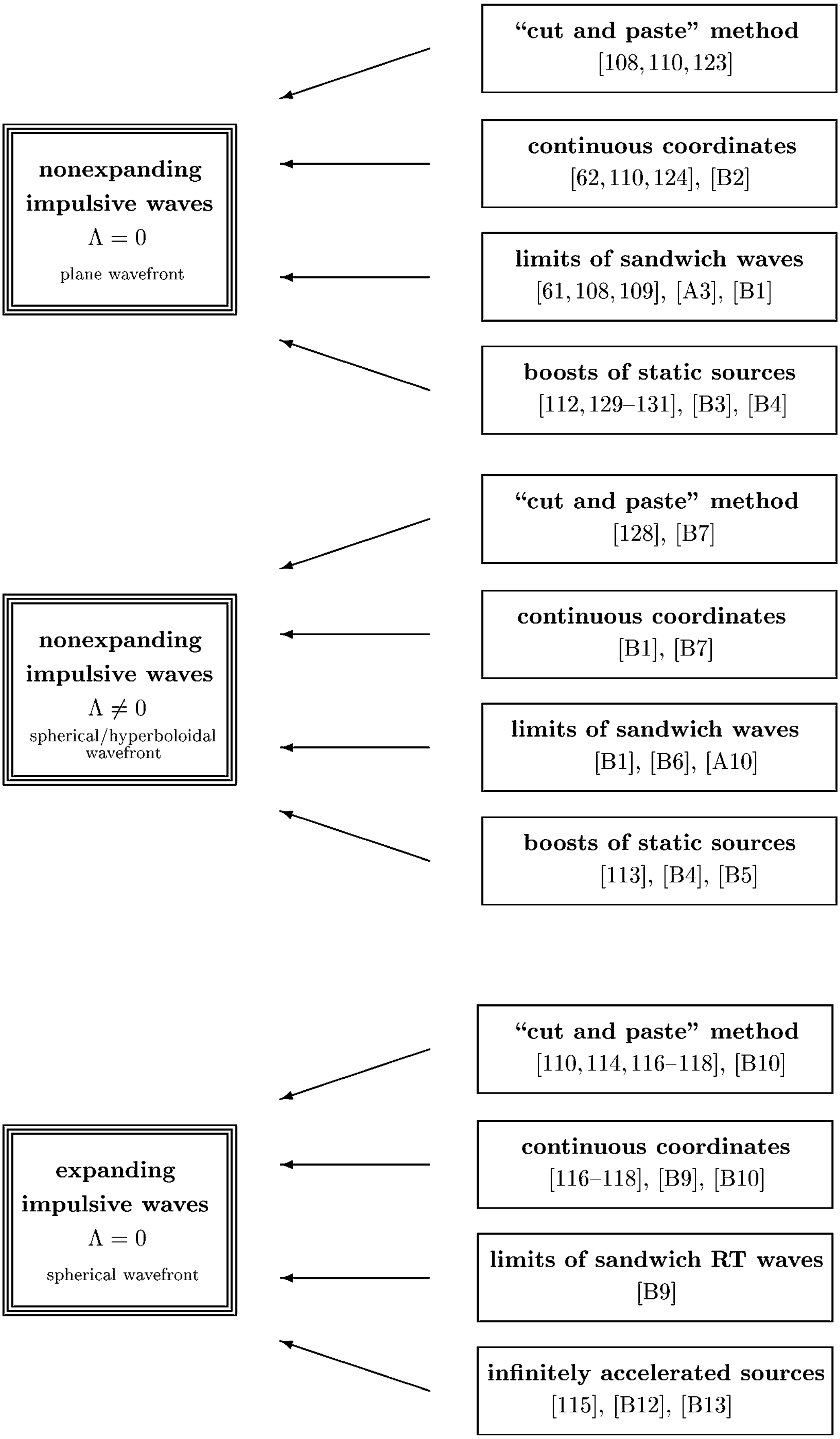
$$\beta = \tfrac{1}{2}\left[\,1 + \sqrt{3}\cot(\varphi + \tfrac{1}{3}\pi)\,\right] \in (0,1]\,, \quad \text{where} \quad \varphi = \tfrac{1}{3}\arccos(1 - 54\,m^2A^2)\,. \qquad (48)$$

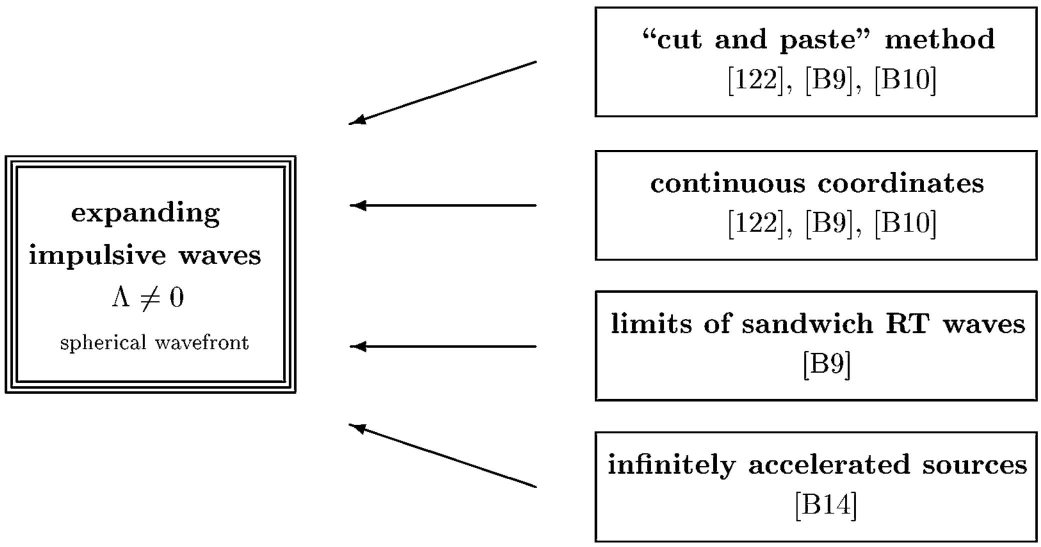
Subsequently, in [B14] we performed the limit $A \to \infty$ of a more general C-metric which admits a nonvanishing value of the cosmological constant Λ [106, 107], starting from the Robinson–Trautman form (23) of the C-metric. By a careful application of the coordinate freedom we obtained the explicit form of P in the limit $A \to \infty$, with mA being constant, namely

$$P^2(\zeta,\bar\zeta,u) = \begin{cases} 1 & \text{for} \quad u > 0 \\ g^{-2}\,(\zeta\bar\zeta)^{1-g} & \text{for} \quad u < 0 \end{cases} \qquad (49)$$

where $g = -\alpha/\beta \in (0,1]$ in which $\alpha = 6mA/[\cos(2\varphi - \frac{2\pi}{3}) + 1]$, $\beta = -6mA/[2\cos(2\varphi - \frac{\pi}{3}) - 1]$. This is exactly the form of P generated by (21) using $F = \zeta^{g(u)}$ such that $\epsilon = 0$ and $g(u) = e^{-c\,\Theta(-u)}$, c =const. Because $g'/g = c\,\delta(u)$, the Weyl tensor component is non-trivial only on $u = 0$. The limit of the C-metric thus may be interpreted as the Robinson–Trautman impulse, i.e. as a natural limit of the Robinson–Trautman sandwich waves described in [A11], see subsection 1.2.4. It is an impulsive spherical wave generated by the snapping of a cosmic string with deficit angle $2\pi(1 - g)$ in a Minkowski, de Sitter or anti-de Sitter background according to the value of Λ. It reduces to the corresponding complete background with no string and no impulse when $mA = 0$, in which case $P = 1$ everywhere. It also agrees with the null limit of the C-metric that was previously obtained in [B13] for the Minkowski background only.

2.7 Summary of construction methods





2.8 Particular solutions and other properties

Now, we briefly mention some other properties of impulsive waves in spaces of constant curvature.

2.8.1 Nonexpanding impulses generated by null multipole particles

It has been shown in section 2.6 that the simplest nonexpanding impulsive gravitational waves can be generated by boosting the Schwarzschild–(anti-)de Sitter black hole solutions. This result can be generalized: it has been shown [B4] that it is possible to consider particular nonexpanding impulsive waves generated by null *multipole* particles as limits of boosted static multipole particles.

It is well-known [16, 133] that static, axially symmetric and asymptotically flat vacuum solutions which represent field of sources with a multipole structure can be written in the Weyl coordinates

$$\mathrm{d}s^2 = -e^{2\psi}\mathrm{d}t^2 + e^{-2\psi}\left[e^{2\gamma}(\mathrm{d}\varrho^2 + \mathrm{d}z^2) + \varrho^2\mathrm{d}\varphi^2\right] , \tag{50}$$

where

$$\psi = \sum_{m=0}^{\infty} \frac{a_m}{r^{m+1}}\, P_m(\cos\theta) , \qquad \gamma = \sum_{m,n=0}^{\infty} \frac{(m+1)(n+1)}{m+n+2}\, \frac{a_m a_n}{r^{m+n+2}}\big(P_{m+1}P_{n+1} - P_m P_n\big) , \tag{51}$$

$r = \sqrt{\varrho^2 + z^2}$, $\cos\theta = z/r$, and P_m are the Legendre polynomials with argument $\cos\theta$. Arbitrary constants a_m determine the m^{th} multipole moments of the source. Performing the boost in the x-direction where $x = \varrho\cos\varphi$, $y = \varrho\sin\varphi$, the line element (50) in the limit $\mathrm{v} \to 1$ gives the impulsive *pp*-wave metric (40) where

$$\mathrm{H} = \sum_{m=0}^{\infty} b_m \mathrm{H}_m = \sum_{m=0}^{\infty} \tfrac{1}{2}m\, b_m \int_{-\infty}^{+\infty} \frac{1}{(\rho^2 + x^2)^{(m+1)/2}}\, P_m\!\left(\frac{z}{\sqrt{\rho^2 + x^2}}\right) \mathrm{d}x . \tag{52}$$

For the simplest case $a_0 \neq 0$, $a_m = 0$ for $m \geq 1$ (which corresponds to the boosted Curzon–Chazy solution) the integral gives exactly the same result (41) as the Aichelburg–Sexl solution [112] generated by a single null monopole particle. For the higher multipole components $m \geq 1$ the integral (52) can also be explicitly evaluated. As shown in [B4], we obtain

$$\mathrm{H}_m(\rho, \phi) = \rho^{-m} \cos[m(\phi - \phi_m)] , \tag{53}$$

where $\rho^2 = y^2 + z^2$, $\cos(\phi - \phi_m) = z/\rho$, and ϕ_m is a constant. This term represents the m^{th} multipole component of the exact impulsive purely gravitational *pp*-wave generated by a source of an arbitrary multipole structure [B3], see figure 8.

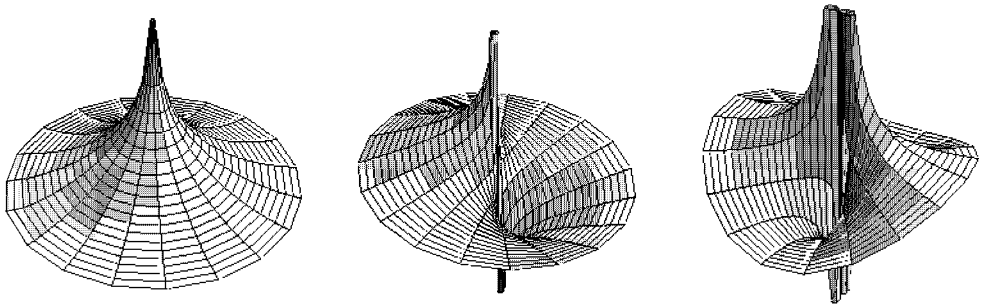


Figure 8: These pictures illustrate the monopole, dipole and quadrupole terms showing the dependence of the functions H_0, H_1 and H_2 near the singular point representing the source of the impulsive waves.

Interestingly, there are analogous impulsive solutions also in the case $\Lambda \neq 0$. It was demonstrated in [B6] that nontrivial solutions of the vacuum Einstein equation (39) for the metric (36) on (1), expressed in terms of the parameters (38), can be written as

$$\mathrm{H}(z,\phi) = \sum_{m=0}^{\infty} b_m \mathrm{H}_m = \sum_{m=0}^{\infty} b_m Q_1^m(z)\cos[m(\phi-\phi_m)]\ , \tag{54}$$

where $Q_1^m(z)$ are associated Legendre functions of the second kind generated by the relation

$$Q_1^m(z) = (-\epsilon)^m |1-z^2|^{m/2} \frac{\mathrm{d}^m Q_1(z)}{\mathrm{d}z^m}\ . \tag{55}$$

The first term for $m=0$, exactly corresponds to the simplest axisymmetric Hotta–Tanaka solution (42). The higher components H_m describe nonexpanding impulsive gravitational waves in (anti-) de Sitter universe generated by null point sources with an m-pole structure, localized on the wave-front (37) at the singularities $z = \pm 1$, see [B6].

2.8.2 Expanding impulses generated by snapping and colliding strings

A physically most interesting expanding spherical impulsive gravitational wave is probably that generated by a *snapping cosmic string*. This explicit solution, described in detail above in section 2.6, can be written in various forms — for example using the coordinates (46) or in the form (32) with the function H given by

$$H = \tfrac{1}{2}\delta(1-\tfrac{1}{2}\delta)\, Z^{-2}\ , \tag{56}$$

where $\delta > 0$. This is generated from the function $h(Z) = Z^{1-\delta}$. As shown in [116], [B10], the generating function h is closely related to a geometrical interpretation of the Penrose junction conditions (30). On the impulse $U=0$,

$$\frac{\eta}{\mathcal{V}} = \begin{cases} Z & \text{for} \quad U = 0_-\ , \\ h(Z) & \text{for} \quad U = 0_+\ . \end{cases} \tag{57}$$

This is exactly the relation for a stereographic correspondence between a sphere and an Argand plane by projection from the North pole onto a plane through the equator [17]. This permits us to represent the wave surface $U=0$ either as a Riemann sphere, or as its associated complex plane parametrized by the coordinate Z. The Penrose junction condition (30) across the wave surface then can be understood as a mapping on the complex Argand plane $Z \to h(Z)$. According to (57), this is equivalent to mapping points P_- on the "inside" of the wave surface to the identified points P_+ on the "outside", as illustrated on figure 9.

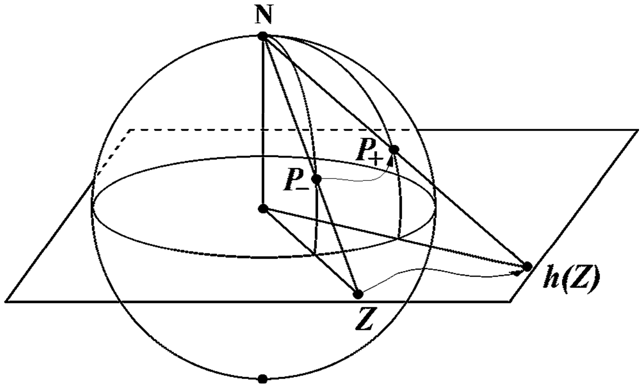


Figure 9: The stereographic correspondence between the Riemann sphere and the complex Argand plane enables a geometrical description of the Penrose junction conditions. Mapping in the complex plane $Z \to h(Z)$ is equivalent to mapping points P_- inside the impulsive spherical surface to the corresponding points P_+ outside.

We may assume that the spacetime $U < 0$ *inside* the impulse represented by $Z = |Z|e^{i\phi}$ covers the *complete* sphere, $\phi \in [-\pi, \pi)$. However, the range of the function $h(Z)$ does not cover the entire sphere outside the spherical impulse for $U > 0$. In particular, the complex mapping covers the plane minus a wedge as $\arg h(Z) \in [-(1-\delta)\pi, (1-\delta)\pi)$. This represents Minkowski, de Sitter, or anti-de Sitter space with a deficit angle $2\pi\delta$ which may be considered to describe a snapped cosmic *string* in the region *outside* the spherical wave. The strings are located along the axis $\eta = 0$.

With the help of the above geometrical insight we were able to construct even more general explicit solutions, namely expanding impulsive waves generated by two *colliding cosmic strings* [B10]. In such a situation, as first suggested by Nutku and Penrose [116], two non-aligned strings initially approach each other and snap at their common point at the instant at which they collide. The remnants are four semi-infinite strings which recede from the point of interaction with the speed of light, generating a specific impulse. A picture of this situation is presented on figure 10, see [B10].

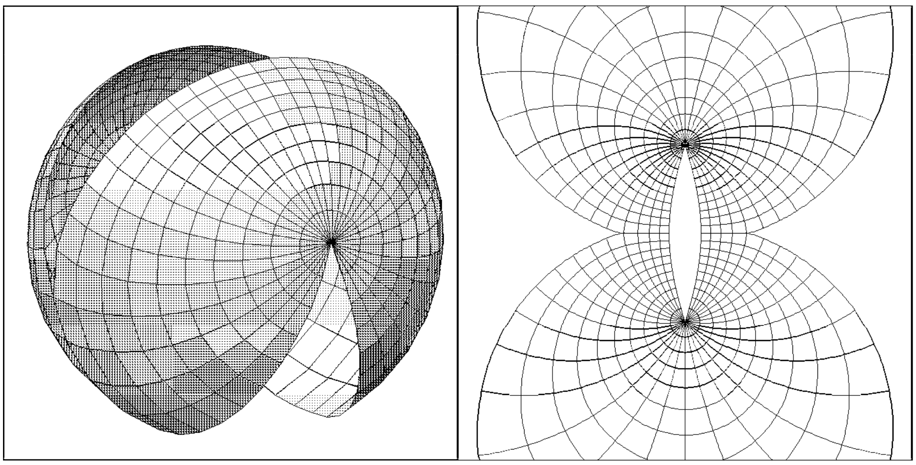


Figure 10: Representation on the Riemann sphere (left) and in the complex Argand plane (right) of the solution for two cosmic strings parallel to the x and y-axes which are moving apart in the positive and negative z-directions.

2.8.3 Geodesics in nonexpanding impulsive waves

Geodesics in Minkowski space with plane-fronted impulsive *pp*-waves were discussed in many works, e.g. in [123,128]. However, as the corresponding geodesic and geodesic deviation equations in standard coordinates contain highly singular products of distributions, an advanced framework of Colombeau algebras of generalized functions had to be employed to solve these equations in a mathematically rigorous sense [61].

In [B8] we studied the behaviour of geodesics in spacetimes (36) which describe nonexpanding impulsive waves in the (anti-)de Sitter background. The solution is

$$U=\tau\ , \qquad U=a\,\dot U^0\sinh(\tau/a)\ , \qquad U=a\,\dot U^0\sin(\tau/a)\ , \tag{58}$$

These relations allow us to take $U=\frac{1}{2}(Z_0+Z_1)$ as the geodesic parameter. Then a general solution of the remaining functions Z_p and $V=\frac{1}{2}(Z_0-Z_1)$ can be written

$$\begin{aligned} Z_p(U) &= Z_p^0\sqrt{1-\tfrac{1}{3}\Lambda\,e\,(\dot U^0)^{-2}U^2}\ +\ (\dot Z_p^0/\dot U^0)\,U\ +\ A_p\,\Theta(U)\,U\ , \\ V(U) &= V^0\sqrt{1-\tfrac{1}{3}\Lambda\,e\,(\dot U^0)^{-2}U^2}\ +\ (\dot V^0/\dot U^0)\,U\ +\ B\,\Theta(U)\sqrt{1-\tfrac{1}{3}\Lambda\,e\,(\dot U^0)^{-2}U^2} \\ &\quad +\ (\dot U^0)^{-1}(\dot Z_i^0 A_i+\epsilon\dot Z_4^0 A_4)\,\Theta(U)\,U\ +\ C\,\Theta(U)\,U\ . \end{aligned} \tag{59}$$

The coefficients A_p, B, C are suitable constants. It follows from (59) that the geodesics are *continuous but refracted* by the impulse in the transverse directions Z_p . However, there is a discontinuity in the longitudinal coordinate V (and its derivative) on the impulse.

2.8.4 Geodesics in expanding impulsive waves

In our work [B11] we presented an explicit form for geodesics in spacetimes with expanding spherical gravitational impulses. It is possible to derive such geodesics assuming these are C^1 in the continuous coordinate system (32). The effect of the impulse on free test particles can be characterised by the "refraction" and the "shift" of geodesic trajectories which can conveniently be described by angles α and β. They represent the *position* of the particle at the instant τ_i of interaction with the impulse resp. the direction of its *velocity* (inclination of the trajectory) in the x,z-plane. These parameters are defined as $\cot\alpha=z_0/x_0$, $\cot\beta=\dot z_0/\dot x_0$. Straightforward calculations give

$$2\cot\alpha^- = \cot^{q}\left(\tfrac{1}{2}\alpha^+\right)-\cot^{-q}\left(\tfrac{1}{2}\alpha^+\right)\ , \tag{60}$$

where $q=1/(1-\delta)$, which is the relation $\alpha^-(\alpha^+)$ that *identifies the points* on both sides of the impulse in Minkowskian coordinate systems. Analogously we derive the relation for the velocities,

$$\cot\beta^- - \cot\alpha^- = \mathcal{N}\left(\cot\beta^+ - \cot\alpha^+\right)\ , \tag{61}$$

where

$$\mathcal{N}=\mathcal{N}(\alpha^+,\beta^+)=\frac{1-\epsilon}{(1-\delta)(E-D)-(1-\delta)\,2F\cot\beta^+}\ . \tag{62}$$

with

$$\begin{aligned} D &= \frac{Z_0^{\delta-2}}{1-\delta}\left[(\tfrac{1}{2}\delta)^2+(1-\tfrac{1}{2}\delta)^2\epsilon Z_0^2\right]\ , \qquad & E &= \frac{Z_0^{-\delta}}{1-\delta}\left[(1-\tfrac{1}{2}\delta)^2+(\tfrac{1}{2}\delta)^2\epsilon Z_0^2\right]\ , \\ F &= \frac{\tfrac{1}{2}\delta(1-\tfrac{1}{2}\delta)p}{(1-\delta)Z_0}\ , & Z_0^{1-\delta} &= \cot\left(\tfrac{1}{2}\alpha^+\right)\ , \end{aligned} \tag{63}$$

This is the *refraction formula* for trajectories of free test particles which cross the spherical impulse.

The typical behaviour of geodesics affected by the impulsive gravitational wave (32), (56), as described by the refraction formula (61), is shown in figure 11 for various choices of α^+,β^+, and ϵ. Each test particle follows in the region $U>0$ a trajectory with the inclination angle β^+ until it reaches the spherical impulse at the point represented by α^+. The impulse influences the particle in such a way that it emerges in the region $U<0$ at the point given by α^- and continues to move uniformly along the straight trajectory with inclination β^-. From figure 11 it is obvious that the dependence of β^- on the data α^+, β^+, and the parameter ϵ is rather delicate. In [B11] we analyzed all details of this dependence, including the change in the *speed* of the corresponding test particles due to the interaction with the impulsive gravitational wave.

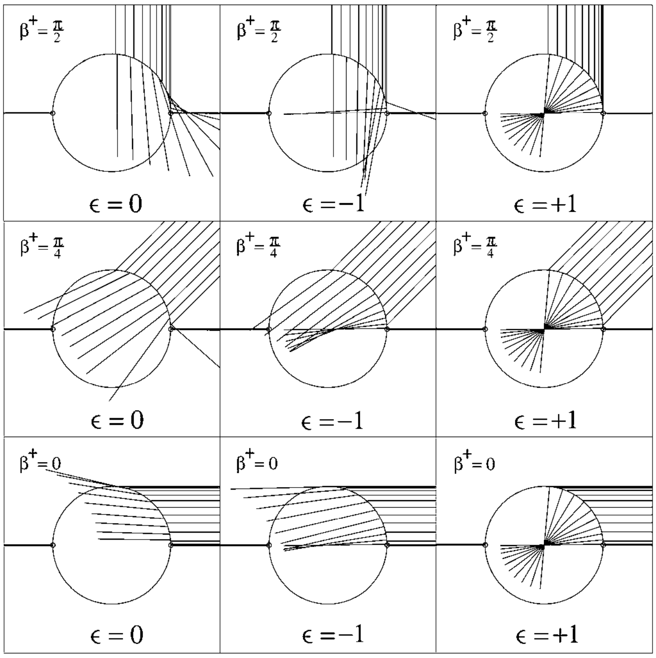


Figure 11: Typical behavior of geodesic trajectories, which are refracted and shifted by the expanding spherical impulse, for various values of α^+, β^+ and the parameter ϵ.

2.9 Final remarks

We have presented a review of exact solutions of Einstein's equations which describe impulsive waves in spaces of constant curvature. These are either gravitational and/or null matter nonexpanding impulses, or expanding spherical impulsive purely gravitational waves (attached to cosmic strings) which propagate in Minkowski (when $\Lambda = 0$), de Sitter ($\Lambda > 0$), or anti-de Sitter ($\Lambda < 0$) universes. A summary with the corresponding references, including our contributions [B1]–[B8] on nonexpanding impulses, and [B9]–[B14] on expanding impulses, is presented in diagrams in section 2.7.

Let us note that there also exist exact solutions which represent impulses in other background spacetimes that are more general than the maximally symmetric spaces of constant curvature that were considered above. For example, Dray and 't Hooft [123] suggested in 1985 a "shift method" which in a sense extends the Penrose "cut and paste" idea. As a particular example they constructed impulsive waves localized on the horizon of the Schwarzschild black hole which is of algebraic type D, see also [125, 128]. Recently, nonexpanding impulsive gravitational waves were discovered in other type D universes. In [134] Ortaggio found an explicit impulse which propagates in the Nariai universe [96]. Subsequently, in [B16] we generalized this result to impulsive solutions in a large family of direct product backgrounds (those spacetimes which are the direct product of two constant curvature 2-spaces) which also includes the Bertotti–Robinson [94, 95] and Plebanski–Hacyan [97] universes. Undoubtedly, there exist other spacetimes of type D, and possibly of a more general algebraic type, that admit specific impulsive waves. This poses an interesting problem for future investigations.

3 (C) Asymptotic structure of radiation

In this final part, we are going to describe yet another contribution to studies of gravitational and electromagnetic radiation — an analysis of the asymptotic directional behaviour of fields in spacetimes which admit any value of the cosmological constant. This was published in a series of recent papers [C1]–[C6] in which we investigated the properties of general gravitational and electromagnetic fields near conformal infinity $\mathcal{I}$ of any type. We evaluated the fields in suitable tetrads which are parallelly propagated along null geodesics which approach a point P of $\mathcal{I}$. When the (local) character of the conformal infinity is null, such as in asymptotically flat spacetimes, the dominant term which is identified with radiation is unique. However, for spacetimes with a non-vanishing cosmological constant the conformal infinity is spacelike (for $\Lambda > 0$) or timelike (for $\Lambda < 0$), and the radiative component of each field depends substantially on the null direction along which P is approached.

We explicitly found and described the directional dependence of asymptotic fields near such de Sitter-like or anti-de Sitter-like $\mathcal{I}$. In fact, we demonstrated that the corresponding directional structure of radiation has a universal character that is determined by the algebraic type of the field. The directional structure of radiation near (anti-)de Sitter-like infinities supplements the standard peeling-off property of fields. This characterization offers a better understanding of the asymptotic behaviour of the fields under the presence of a cosmological constant.

3.1 On studies of asymptotic behaviour of radiative fields

A fundamental technique for the investigation of radiative properties of gravitational and electromagnetic fields at 'large distance' from a spatially bounded source is based on introducing a suitable *Bondi-Sachs coordinate system* adapted to *null hypersurfaces*, and expanding the metric functions in inverse powers of the luminosity distance r which plays the role of an appropriate 'radial' coordinate parameterizing outgoing null geodesics [21, 22]. In the case of asymptotically flat spacetimes this framework allows one to introduce the Bondi mass and momentum, and characterize the time evolution including radiation in terms of the news functions which are the analogue of the radiative part of the Poynting vector in electrodynamics. Using these concepts, it is possible to formulate a balance between the amount of energy radiated by gravitational waves and the decrease of the Bondi mass of an isolated system. These pioneering contributions were refined and generalized [23] after the development of the complex null tetrad formalism and the associated spin coefficient formalism [24] which lead to great simplifications in the expressions, see e.g. [36, 39, 135] for reviews. Nevertheless, in these works the analysis of radiative fields assumed that spacetime is asymptotically flat.

Alternatively, information about the character of radiation can be extracted from the tetrad components of fields measured along a family of null geodesics approaching $\mathcal{I}$. The gravitational or electromagnetic field represents outgoing radiation if the dominant component of the Weyl or Maxwell tensor, expressed in the Newman-Penrose formalism [24] as quantities Ψ_4 or Φ_2, respectively, is non-vanishing. This component manifests itself through typical transverse effects on nearby test particles [22,59], [A2]. Such a characterization of the radiative field remains valid also in more general spacetimes because the peeling-off property holds even for a non-vanishing Λ.

Another step made by Penrose [29,30,49] (see [17] for a comprehensive overview) was his *coordinate-independent* approach to the definition of radiation for massless fields based on the *conformal treatment of infinity*. The Penrose technique enables one to apply methods of local differential geometry near conformal infinity $\mathcal{I}$ which is defined as the boundary $\Omega = 0$ of the physical spacetime manifold $(\mathcal{M}, \mathbf{g})$ in the conformally related 'unphysical' spacetime manifold $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$, $\tilde{\mathbf{g}} = \Omega^2 \mathbf{g}$. Properties of radiation fields in $\mathcal{M}$ can thus be studied by analyzing conformally rescaled fields on $\mathcal{I}$ in the compactified manifold $\tilde{\mathcal{M}}$. In particular, one can evaluate gravitational radiation propagating along a given null geodesic which is described by the Ψ_4 component of the Weyl tensor projected on a parallelly transported complex null tetrad. The crucial point is that such a tetrad is (essentially) *determined uniquely* by the conformal geometry, see [17]. Moreover, the Penrose covariant approach can be naturally applied also to spacetimes which include the cosmological constant [17, 30, 51].

Let us now concentrate on the main *differences* between the asymptotically flat spacetimes and those with a non-vanishing cosmological constant Λ. New features appear in these solutions for which the conformal infinity $\mathcal{I}$ is spacelike or timelike. As Penrose observed and repeatedly emphasized in his early works [30, 138], the concept of radiation for massless fields turns out to be 'less invariant' in cases when $\mathcal{I}$ does not have a null character, see section 9.7 of [17]. Namely, it emerges as necessarily *direction dependent* since the choice of the appropriate null tetrad, and thus the radiative component Ψ_4 of the field, may differ for different null geodesics reaching the same point on $\mathcal{I}$. Such a directional dependence was explicitly described for the first time in the context of the *test electromagnetic field* generated by a pair of uniformly accelerated point-like charges in the de Sitter background [139]. In particular, there always exist two special directions — those opposite to the direction from the sources — along which the radiation vanishes.

Subsequently, we analyzed the *exact* solution of the Einstein-Maxwell equations which generalizes the classic C-metric to admit a cosmological constant [106,107], [A16], [A17]. For $\Lambda > 0$ it represents a pair of accelerated charged black holes in de Sitter-like universe. In [C1] we demonstrated that the electromagnetic field exhibits exactly the *same* asymptotic radiative behaviour at the spacelike conformal infinity $\mathcal{I}$ as for the test fields [139] of accelerated charges. Moreover, we found and described the specific directional structure of the *gravitational radiation field*, and we proved that the directional pattern of radiation is adapted to the principal null directions of this type D spacetime.

In [C2] we investigated the asymptotic behaviour of fields corresponding to the C-metric with $\Lambda < 0$, i.e. the directional dependence of radiation generated by accelerated black holes in an anti-de Sitter universe. Some fundamental differences from the case $\Lambda > 0$ occur since $\mathcal{I}$ now has a timelike character. In fact, the whole structure of the spacetime is more complex and new phenomena also arise: $\mathcal{I}$ is divided by Killing horizons into several domains with a different structure of principal null directions — in these domains the directional structure of radiation is thus different. The field vanishes along directions which are mirror images of the principal null directions with respect to $\mathcal{I}$. Also, ingoing and outgoing radiation has to be treated separately.

These studies of particular exact models with $\Lambda \neq 0$ gave us a sufficient insight necessary to understand the behaviour of *general* fields near spacelike or timelike conformal infinities. The directional dependence of gravitational and electromagnetic radiation is given by specific orientation and degeneracy of principal null directions at $\mathcal{I}$. In [C3] we showed that the directional structure of radiation close to a de Sitter-like infinity has a *universal character* that is determined by the *algebraic type* of the fields. For example, the radiation vanishes along spatial directions on $\mathcal{I}$ which are antipodal to principal null directions. In the following work [C4] we investigated the complementary situation when $\Lambda < 0$. Although the idea is similar, the behaviour of fields turns out to be more complicated because $\mathcal{I}$ is timelike, and thus admits a 'richer structure' of possible patterns.

We summarize here these results on the asymptotic directional structure of general fields near conformal infinity $\mathcal{I}$ of any type in a synoptic form which is based on our review article [C5].

3.2 Conformal infinity $\mathcal{I}$ and null geodesics

According to standard formalism [17,30,136], a manifold $\mathcal{M}$ with physical metric $\mathbf{g}$ can be embedded into a larger *conformal manifold* $\tilde{\mathcal{M}}$ with *conformal metric* $\tilde{\mathbf{g}}$ via a conformal transformation

$$\tilde{\mathbf{g}} = \Omega^2 \mathbf{g} \,. \tag{64}$$

Obviously, the spacetimes $(\mathcal{M}, \mathbf{g})$ and $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$ have identical local causal structure. A boundary $\Omega = 0$ is *conformal infinity* $\mathcal{I}$. Its character is determined by the gradient $\mathbf{d}\Omega$ on $\mathcal{I}$. We introduce a normalized vector $\tilde{\mathbf{n}}$ *normal to the conformal infinity* $\mathcal{I}$,

$$\tilde{\mathbf{n}}^a = \tilde{N}\, \tilde{\mathbf{g}}^{ab}\, \mathbf{d}_b \Omega \,, \qquad \tilde{\mathbf{g}}_{ab} \tilde{\mathbf{n}}^a \tilde{\mathbf{n}}^b = \sigma \,. \tag{65}$$

The normalization factor σ determines the character of the conformal infinity, namely

$$\sigma = \begin{cases} -1 : & \mathcal{I} \text{ is } \textit{spacelike}, \\ \ \ 0 : & \mathcal{I} \text{ is } \textit{null}, \\ +1 : & \mathcal{I} \text{ is } \textit{timelike}. \end{cases} \tag{66}$$

Assuming a vanishing trace of the energy-momentum tensor, which is valid in vacuum, pure radiation, or electrovacuum spacetimes, Einstein field equations imply $\sigma = -\operatorname{sign}\Lambda$. The character of the conformal infinity is thus correlated with the sign of the cosmological constant.

We consider null geodesics $z(\eta)$ in $(\mathcal{M},\mathbf{g})$ and we relate them to null geodesics $\tilde{z}(\tilde{\eta})$ in $(\tilde{\mathcal{M}},\tilde{\mathbf{g}})$. *They are conformally invariant*, $z(\eta)$ coincide with $\tilde{z}(\tilde{\eta})$, with the affine parameters related by

$$\frac{d\tilde{\eta}}{d\eta}=\Omega^2\,,\qquad \text{i.e.,}\quad \frac{Dz}{d\eta}=\Omega^2\frac{D\tilde{z}}{d\tilde{\eta}}\,. \tag{67}$$

We may assume that $\tilde{\eta}=0$ at conformal infinity $\mathcal{I}$. Therefore, as $\tilde{\eta}\to 0$ the null geodesic $\tilde{z}(\tilde{\eta})$ approaches a specific point $P\in\mathcal{I}$. Such a geodesic can be either *outgoing* or *ingoing*:

$$\frac{D\tilde{z}^a}{d\tilde{\eta}}\,\mathbf{d}_a\Omega\bigg|_{\mathcal{I}}\equiv\frac{d\,\Omega}{d\,\tilde{\eta}}\bigg|_{\mathcal{I}}=-\epsilon\,, \tag{68}$$

where

$$\epsilon=\begin{cases}+1: & \text{for } \textit{outgoing geodesics}, \quad \tilde{\eta}<0 \ \text{ in } \mathcal{M},\\ -1: & \text{for } \textit{ingoing geodesics}, \quad \tilde{\eta}>0 \ \text{ in } \mathcal{M}.\end{cases} \tag{69}$$

Assuming smoothness of the conformal factor along $\tilde{z}(\tilde{\eta})$ we obtain

$$\Omega=-\epsilon\,\tilde{\eta}+\Omega_2\,\tilde{\eta}^2+\ldots\,, \tag{70}$$

with Ω_2 constant. Substituting into equation (67), straightforward integration leads to the relation between the physical and conformal affine parameters, $\eta=-\tilde{\eta}^{-1}\big(1-2\epsilon\,\Omega_2\,\tilde{\eta}\ln|\tilde{\eta}|-\eta_0\,\tilde{\eta}+\ldots\big)$. Near $\mathcal{I}$ we thus obtain in the leading order that $\tilde{\eta}\approx-\eta^{-1}$ and $\Omega\approx\epsilon\,\eta^{-1}$. The null geodesic $z(\eta)$ thus reaches the point $P\in\mathcal{I}$ for an *infinite* value of the affine parameter η, namely $z(\epsilon\infty)=P$.

3.3 Interpretation tetrad and reference tetrad

In order to study the behaviour of fields near $\mathcal{I}$ we introduce the normalized 'interpretation' tetrad which is parallelly transported along null geodesics $z(\eta)$ approaching $\mathcal{I}$. To achieve this we employ various orthonormal and null tetrads, distinguished by specific subscripts. We denote the vectors of an *orthonormal tetrad* as $\mathbf{t}$, $\mathbf{q}$, $\mathbf{r}$, $\mathbf{s}$, were $\mathbf{t}$ is a future oriented unit timelike vector, and the remaining three are spacelike unit vectors. With this tetrad we associate a *null tetrad* of null vectors $\mathbf{k}$, $\mathbf{l}$, $\mathbf{m}$, $\bar{\mathbf{m}}$, such that

$$\mathbf{k}=\tfrac{1}{\sqrt{2}}(\mathbf{t}+\mathbf{q})\,,\quad \mathbf{l}=\tfrac{1}{\sqrt{2}}(\mathbf{t}-\mathbf{q})\,,\quad \mathbf{m}=\tfrac{1}{\sqrt{2}}(\mathbf{r}-i\,\mathbf{s})\,,\quad \bar{\mathbf{m}}=\tfrac{1}{\sqrt{2}}(\mathbf{r}+i\,\mathbf{s})\,, \tag{71}$$

where the only non-vanishing scalar products are $\mathbf{g}_{ab}\mathbf{k}^a\mathbf{l}^b=-1$, $\mathbf{g}_{ab}\mathbf{m}^a\bar{\mathbf{m}}^b=1$. Lorentz transformations between various tetrads are null rotation with $\mathbf{k}$ fixed, null rotation with $\mathbf{l}$ fixed, boost in the $\mathbf{k}$-$\mathbf{l}$ plane, and a spatial rotation in the $\mathbf{m}$-$\bar{\mathbf{m}}$ plane [16].

Now, an *interpretation* null tetrad $\mathbf{k}_{\rm i}$, $\mathbf{l}_{\rm i}$, $\mathbf{m}_{\rm i}$, $\bar{\mathbf{m}}_{\rm i}$ is any tetrad *parallelly transported along a null geodesic* $z(\eta)$ *in the physical spacetime* $\mathcal{M}$, with $\mathbf{k}_{\rm i}$ *tangent* to $z(\eta)$. We thus require

$$\mathbf{k}_{\rm i}=\frac{1}{\sqrt{2}\tilde{N}|_{\mathcal{I}}}\frac{Dz}{d\eta}\,,\quad \mathbf{k}_{\rm i}^a\nabla_a\mathbf{k}_{\rm i}^b=0\,,\quad \mathbf{k}_{\rm i}^a\nabla_a\mathbf{l}_{\rm i}^b=0\,,\quad \mathbf{k}_{\rm i}^a\nabla_a\mathbf{m}_{\rm i}^b=0\,,\quad \mathbf{k}_{\rm i}^a\nabla_a\bar{\mathbf{m}}_{\rm i}^b=0\,. \tag{72}$$

In the following, we will need to identify the direction $\mathbf{k}_{\rm i}$ of the null geodesic near conformal infinity using suitable directional parameters. For this purpose we set up a *reference tetrad* $\mathbf{k}_{\rm o}$, $\mathbf{l}_{\rm o}$, $\mathbf{m}_{\rm o}$, $\bar{\mathbf{m}}_{\rm o}$. It is any tetrad adjusted to $\mathcal{I}$ which satisfies the condition

$$\mathbf{n}=\epsilon_{\rm o}\tfrac{1}{\sqrt{2}}(-\sigma\mathbf{k}_{\rm o}+\mathbf{l}_{\rm o})\,, \tag{73}$$

where $\mathbf{n}=\Omega\,\tilde{\mathbf{n}}$. Otherwise, the reference tetrad can be chosen *arbitrarily*, ergo conveniently. It may either respect the symmetry of the spacetime (by adapting the reference tetrad to the Killing vectors) or its specific algebraic structure (it can be oriented along the principal null directions). The parameter $\epsilon_o = \pm 1$ are chosen in such a way that $\mathbf{k}_o$ and $\mathbf{l}_o$ are *future oriented.*

We use the given reference tetrad $\mathbf{k}_o$, $\mathbf{l}_o$, $\mathbf{m}_o$, $\bar{\mathbf{m}}_o$ as a fixed basis with respect to which we define other directions, for example asymptotic directions along which various null geodesics approach a point P at $\mathcal{I}$. It is natural to characterize such a general null direction $\mathbf{k}$ by a *complex parameter* R: the direction $\mathbf{k}$ is obtained from $\mathbf{k}_o$ by the null rotation with the parameter R,

$$\mathbf{k} \propto \mathbf{k}_o + \bar{R}\,\mathbf{m}_o + R\,\bar{\mathbf{m}}_o + R\bar{R}\,\mathbf{l}_o\,. \tag{74}$$

We can now project a null vector $\mathbf{k}$, whose direction is represented by the parameter R by (74), onto the corresponding conformal infinity.

In spacetimes with $\Lambda > 0$, for which $\mathcal{I}$ is spacelike, we perform a normalized spatial projection to a three-dimensional space orthogonal to $\mathbf{t}_o$, $\mathbf{q} = [\mathbf{k} + (\mathbf{k}\cdot\mathbf{t}_o)\,\mathbf{t}_o]\,|\mathbf{k}\cdot\mathbf{t}_o|^{-1}$, where $\mathbf{k}\cdot\mathbf{t}_o = \mathbf{g}_{ab}\mathbf{k}^a\mathbf{t}_o^b$. The unit spatial direction $\mathbf{q}$ corresponding to $\mathbf{k}$ can be expressed in standard *spherical angles* θ, ϕ,

$$\mathbf{q} = \cos\theta\,\mathbf{q}_o + \sin\theta\,(\cos\phi\,\mathbf{r}_o + \sin\phi\,\mathbf{s}_o)\,. \tag{75}$$

Comparing (74) with (75) we obtain

$$R = \tan\frac{\theta}{2}\,\exp(-\mathrm{i}\phi)\,. \tag{76}$$

Therefore, R is exactly the *stereographic representation* of the angles θ, ϕ.

Alternatively, in spacetimes with $\Lambda < 0$ for which $\mathcal{I}$ is timelike the projection of $\mathbf{k}$ onto $\mathcal{I}$ is $\mathbf{t} = [\mathbf{k} - (\mathbf{k}\cdot\mathbf{q}_o)\,\mathbf{q}_o]\;|\mathbf{k}\cdot\mathbf{q}_o|^{-1}$. The resulting unit timelike vector $\mathbf{t}$ is tangent to the Lorentzian (1+2) conformal infinity. We can analogously characterize $\mathbf{t}$ (and thus $\mathbf{k}$) by

$$\mathbf{t} = \cosh\psi\,\mathbf{t}_o + \sinh\psi\,(\cos\phi\,\mathbf{r}_o + \sin\phi\,\mathbf{s}_o)\,. \tag{77}$$

The parameters ψ, ϕ are *pseudo-spherical* parameters, $\psi \in (0,\infty)$ corresponding to a boost, and $\phi \in (-\pi, +\pi)$ being an angle. However, these parameters do not specify the null direction $\mathbf{k}$ uniquely — there always exist *one ingoing* and *one outgoing* null directions with the same parameters ψ and ϕ, which are distinguished by $\epsilon = \pm 1$. By comparing (74) and (77) we obtain

$$R = \begin{cases} \tanh\dfrac{\psi}{2}\,\exp(-\mathrm{i}\phi) & \text{for}\quad \epsilon = +\epsilon_o\,, \\[2ex] \coth\dfrac{\psi}{2}\,\exp(-\mathrm{i}\phi) & \text{for}\quad \epsilon = -\epsilon_o\,. \end{cases} \tag{78}$$

We also allow an infinite value $R = \infty$ which corresponds to $\psi = 0$, $\epsilon = -\epsilon_o$, i.e., $\mathbf{k} \propto (\mathbf{t}_o - \mathbf{q}_o)/\sqrt{2}$.

3.4 Fields and their asymptotic structure

Following the notation of [16], gravitational field characterized by the Weyl tensor $\mathbf{C}_{abcd}$ can be parametrized by five complex coefficients Ψ_j, whereas electromagnetic field is described by the tensor $\mathbf{F}_{ab}$ parametrized by three coefficients Φ_j. A field of general spin s can be characterized by $(2s+1)$ complex components Υ_j, $j = 0, 1, \ldots, 2s$.

For any field there exist *principal null directions* (PNDs) which are privileged null directions $\mathbf{k}$ such that $\Upsilon_0 = 0$ in a null tetrad $\mathbf{k}$, $\mathbf{l}$, $\mathbf{m}$, $\bar{\mathbf{m}}$. The PND $\mathbf{k}$ can be obtained from a reference tetrad $\mathbf{k}_o$, $\mathbf{l}_o$, $\mathbf{m}_o$, $\bar{\mathbf{m}}_o$ by a null rotation (74) given by a directional parameter $R \in \mathbb{C}$. The condition $\Upsilon_0 = 0$ thus takes the form of an algebraic equation of the order $2s$ for R,

$$R^{2s}\,\Upsilon^{o}_{2s} + \binom{2s}{2s-1} R^{2s-1}\,\Upsilon^{o}_{2s-1} + \ldots + \binom{2s}{1} R\,\Upsilon^{o}_{1} + \Upsilon^{o}_{0} = 0\,. \tag{79}$$

The complex roots R_n, $n = 1, 2, \dots, 2s$, of equation (79) parametrize PNDs $\mathbf{k}_n$ with respect to the reference tetrad $\mathbf{k}_o$, $\mathbf{l}_o$, $\mathbf{m}_o$, $\bar{\mathbf{m}}_o$. The situation when $\Upsilon^o_{2s} = 0$ formally corresponds to an infinite value of one of the roots, say $R_1 = \infty$, in which case $\mathbf{k}_1 = \mathbf{l}_o$. There are 4 principal null directions for gravitational field, 2 for eletromagnetic field, and $2s$ for spin-s field. According to whether some of these PNDs coincide, the fields are algebraically special and can be classified to various (Petrov) algebraic types [15–17].

Using the above quantities we derive the *asymptotic behaviour of general fields* with respect to the interpretation tetrad near conformal infinity. Assuming standard behaviour $\Upsilon^o_j \approx \Upsilon^o_{j*}\,\eta^{-(s+1)}$, $\Upsilon^o_{j*} = \text{const}$ (see [17,30,51,137]), after straightforward but somewhat lengthy calculations we obtain

$$\Upsilon^i_{2s} \approx \frac{1}{\eta}\,\epsilon_o^s \Upsilon^o_{2s*}\,\frac{\big(1-\sigma R_1\bar{R}\big)\big(1-\sigma R_2\bar{R}\big)\dots\big(1-\sigma R_{2s}\bar{R}\big)}{\big(1-\sigma R\bar{R}\big)^s}\;. \tag{80}$$

This expression fully characterizes the asymptotic behaviour on $\mathcal{I}$ of the dominant radiative component of any massless field of spin s in the normalized interpretation tetrad $\mathbf{k}_i$, $\mathbf{l}_i$, $\mathbf{m}_i$, $\bar{\mathbf{m}}_i$ which is parallelly propagated along a null geodesic $z(\eta)$. The complex parameter R represents the *direction* of the null geodesic along which a given point $P \in \mathcal{I}$ is approached as $\eta \to \epsilon\infty$. The constants R_n characterize the *principal null directions*, i.e. the algebraic structure of the field at P. The directional structure of radiation is thus completely determined by the algebraic type of the field. However, the dependence of Υ^i_{2s} on the direction R along which $P \in \mathcal{I}$ is approached occurs only if $\sigma \neq 0$, i.e., at a 'de Sitter-like' or 'anti-de Sitter-like' conformal infinity. For $\mathcal{I}$ of 'Minkowskian' type which has a null character this directional dependence completely vanishes.

3.5 Radiation on null $\mathcal{I}$

For 'Minkowskian' conformal infinity we have $\sigma = 0$, $\mathbf{l}_o \propto \mathbf{n}$, and the field has *no directional structure.* The radiative parts of gravitational and electromagnetic fields (80) are uniquely given by

$$|\Psi^i_4| \approx \frac{|\Psi^o_{4*}|}{|\,\eta\,|}\,, \qquad |\Phi^i_2| \approx \frac{|\Phi^o_{2*}|}{|\,\eta\,|}\,, \tag{81}$$

i.e., they are *the same for all null geodesics approaching a given point* $P \in \mathcal{I}$. For (locally) asymptotically flat spacetimes it is thus possible to distinguish between the radiative and non-radiative fields. Radiation is absent at those points of null conformal infinity where the constants Ψ^o_{4*} or Φ^o_{2*} vanish. As we discussed, this occurs when *the principal null direction is oriented along the vector* $\mathbf{l}_o \propto \mathbf{n}$. This can be viewed as an invariant characterization of the absence of radiation near $\mathcal{I}$.

3.6 Radiation on spacelike $\mathcal{I}$

The asymptotic structure of gravitational and electromagnetic fields near a *de Sitter-like conformal infinity*, $\sigma = -1$, is given by (80) for $s = 2$ and $s = 1$, respectively,

$$\Psi^i_4 \approx \frac{\Psi^o_{4*}}{\eta}\,\big(1+|R|^2\big)^{-2}\left(1-\frac{R_1}{R_\mathfrak{a}}\right)\left(1-\frac{R_2}{R_\mathfrak{a}}\right)\left(1-\frac{R_3}{R_\mathfrak{a}}\right)\left(1-\frac{R_4}{R_\mathfrak{a}}\right), \tag{82}$$

$$\Phi^i_2 \approx \epsilon_0\frac{\Phi^o_{2*}}{\eta}\,\big(1+|R|^2\big)^{-1}\left(1-\frac{R_1}{R_\mathfrak{a}}\right)\left(1-\frac{R_2}{R_\mathfrak{a}}\right), \tag{83}$$

see [C3], where $R_\mathfrak{a} = -\bar{R}^{-1}$ characterizes a spatial direction *opposite* to the direction given by R, i.e., the *antipodal* direction with $\theta_\mathfrak{a} = \pi - \theta$ and $\phi_\mathfrak{a} = \phi + \pi$. The remaining freedom in the choice of $\mathbf{m}_i$, $\bar{\mathbf{m}}_i$ changes just a phase of the field components: only $|\Psi^i_4|$ or $|\Phi^i_2|$ has an invariant meaning.

In a general spacetime there exist *four* spatial directions at $P \in \mathcal{I}$ along which the radiative component of the gravitational field (82) *vanishes*, namely the directions satisfying $R_\mathfrak{a} = R_n$, $n = 1, 2, 3, 4$ (or *two* such directions for electromagnetic field (83)). Spatial parts of them are thus exactly *opposite to the projections of the principal null directions onto* $\mathcal{I}$.

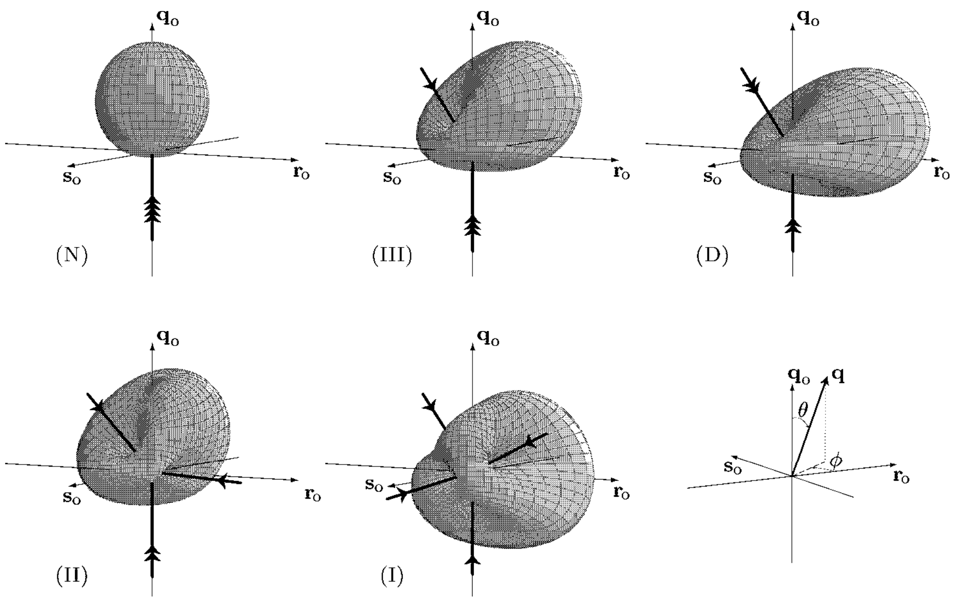


Figure 12: Specific directional structure of radiation for spacetimes of Petrov types N, III, D, II and I. Directions in the diagrams are spatial directions tangent to a spacelike $\mathcal{I}$. For each type, the radiative component $|\Psi^{\rm i}_4|$ along a null geodesic is depicted in the corresponding spatial direction $\mathbf{q}$ parametrized by spherical angles θ, ϕ, see (75). [Degenerate] principal null directions (PNDs) are indicated by [multiple] bold arrows. Thick lines represent spatial directions (opposite to PNDs) along which the radiation vanishes.

For a completely general choice of the reference tetrad near a de Sitter-like conformal infinity, the dominant radiative term (82) of any gravitational field can asymptotically be written in terms of spherical angles θ, ϕ given by (76) as

$$|\Psi^{\rm i}_4| \approx \frac{|\Psi^{\rm o}_{4*}|}{|\eta|}\,\cos^4\tfrac{\theta}{2}\prod_{n=1,2,3,4}\left|1+\tan\tfrac{\theta_n}{2}\,\tan\tfrac{\theta}{2}\,e^{{\rm i}(\phi-\phi_n)}\right|\,, \tag{84}$$

where θ_n, ϕ_n identify the PNDs $\mathbf{k}_n$ with respect to the reference tetrad.

In *algebraically special* spacetimes some PNDs coincide, and (82), (83) simplify. Moreover, it is always possible to choose the *canonical reference tetrad* aligned to the algebraic structure:

1. the vector $\mathbf{q}_{\rm o}$ is oriented along the spatial projection of the *degenerate* (multiple) PND onto $\mathcal{I}$, say $\mathbf{k}_4$, i.e. $\mathbf{k}_{\rm o} = \mathbf{k}_4$,
2. the $\mathbf{q}_{\rm o}$-$\mathbf{r}_{\rm o}$ plane is oriented so that it contains the spatial projection of one of the remaining PNDs, say $\mathbf{k}_1$ (for type *N* spacetimes this choice is arbitrary).

Using such a canonical reference tetrad, the degenerate PND $\mathbf{k}_4$ is parametrized by $\theta_4 = 0$, i.e. $R_4 = 0$, see equations (75), (76). The PND $\mathbf{k}_1$ has $\phi_1 = 0$, i.e. $R_1 = \tan(\theta_1/2)$ is a real constant.

For the *type N* spacetimes (which have a quadruply degenerate PND), in the canonical reference tetrad there is $R_1 = R_2 = R_3 = R_4 = 0$, so that the asymptotic behaviour of field (82) becomes

$$|\Psi^{\rm i}_4| \approx |\Psi^{\rm o}_{4*}|\,|\eta|^{-1}\cos^4\tfrac{\theta}{2}\,. \tag{85}$$

The corresponding directional structure of radiation is illustrated in figure 12(N). It is axisymmetric, with maximum value at $\theta = 0$ along the spatial projection of the quadruple PND onto $\mathcal{I}$. Along the opposite direction, $\theta = \pi$, the field vanishes.

Similarly, the *type D* spacetimes admit two double degenerate PNDs, $R_1 = R_2 = \tan\frac{\theta_1}{2}$ and $R_3 = R_4 = 0$. The gravitational field near spacelike $\mathcal{I}$ thus takes the form

$$|\Psi^{\rm i}_4| \approx |\Psi^{\rm o}_{4*}|\,|\eta|^{-1}\cos^4\tfrac{\theta}{2}\,\left|1+\tan\tfrac{\theta_1}{2}\,\tan\tfrac{\theta}{2}\,e^{{\rm i}\phi}\right|^2\,, \tag{86}$$

with two planes of symmetry, see figure 12(D). This directional dependence agrees with that for the C-metric spacetime with $\Lambda > 0$ first derived in our work [C1].

An analogous discussion also applies to electromagnetic field (83). Moreover, it turns out that the square of $\Phi_2^{\rm i}$ is the magnitude of the Poynting vector, $|\mathbf{S}_{\rm i}| \approx \frac{1}{4\pi}\left|\Phi_2^{\rm i}\right|^2$. If the two PNDs of the electromagnetic field coincide ($R_1 = R_2 = 0$) the directional dependence of the Poynting vector at $\mathcal{I}$ with respect to the canonical reference tetrad is the same as in Eq. (85), figure 12(N). If they differ ($R_1 = \tan\frac{\theta_1}{2}$, $R_2 = 0$), the asymptotic directional structure of $|\mathbf{S}_{\rm i}|$ is given by Eq. (86), illustrated in figure 12(D). The latter result was first obtained for the test field of uniformly accelerated charges in de Sitter spacetime [139] and then recovered in the context of the charged C-metric spacetime [C1].

3.7 Radiation on timelike $\mathcal{I}$

Finally, we shall describe the radiation near the 'anti-de Sitter-like' conformal infinity [C4]. With respect to a suitable reference tetrad $\mathbf{t}_{\rm o}$, $\mathbf{q}_{\rm o}$, $\mathbf{r}_{\rm o}$, $\mathbf{s}_{\rm o}$ these directions are parametrized by the complex parameter R, or its 'Lorentzian angles' ψ, ϕ and the orientation ϵ, related to R by pseudo-stereographic representation, see (78). The directional structure of radiation is given by expression (80) for $\sigma = +1$,

$$\Psi_4^{\rm i} \approx \frac{\Psi_{4*}^{\rm o}}{\eta}\left(1-|R|^2\right)^{-2}\left(1-\frac{R_1}{R_{\rm m}}\right)\left(1-\frac{R_2}{R_{\rm m}}\right)\left(1-\frac{R_3}{R_{\rm m}}\right)\left(1-\frac{R_4}{R_{\rm m}}\right), \tag{87}$$

$$\Phi_2^{\rm i} \approx \epsilon_{\rm o}\,\frac{\Phi_{2*}^{\rm o}}{\eta}\left(1-|R|^2\right)^{-1}\left(1-\frac{R_1}{R_{\rm m}}\right)\left(1-\frac{R_2}{R_{\rm m}}\right). \tag{88}$$

Here, the complex number $R_{\rm m} = \bar{R}^{-1}$ characterizes a direction obtained from the direction R by a *reflection with respect to* $\mathcal{I}$, i.e., the *mirrored* direction with $\psi_{\rm m} = \psi$, $\phi_{\rm m} = \phi$ but opposite orientation $\epsilon_{\rm m} = -\epsilon$. A generic gravitational field thus takes the asymptotic form

$$\left|\Psi_4^{\rm i}\right| \approx \frac{|\Psi_{4*}^{\rm o}|}{|\eta|}\left(\frac{\cosh\psi+\epsilon\epsilon_{\rm o}}{2}\right)^2\prod_{n=1,2,3,4}\left|1-\tanh^{\epsilon_n\epsilon_{\rm o}}\!\left(\tfrac{\psi_n}{2}\right)\tanh^{\epsilon\epsilon_{\rm o}}\!\left(\tfrac{\psi}{2}\right)e^{{\rm i}(\phi-\phi_n)}\right|, \tag{89}$$

where $\psi_n, \phi_n, \epsilon_n$ identify the PNDs $\mathbf{k}_n$, including their orientation with respect to $\mathcal{I}$.

We observe from (87) that the radiation 'diverges' for directions with $|R|=1$ (i.e., $\psi\to\infty$) which are null directions *tangent* to $\mathcal{I}$. This divergence at $|R|=1$ splits the radiation pattern into two components — the pattern for *outgoing* geodesics ($\epsilon = +1$) and that for *ingoing* geodesics ($\epsilon = -1$). These pairs of different patterns are separately depicted in figures 13 and 14.

There are, in general, *four* directions along which the radiation *vanishes*, namely PNDs *reflected with respect to* $\mathcal{I}$, given by $R=(R_n)_{\rm m}$. Outgoing PNDs give zeros in the radiation pattern for ingoing null geodesics, and vice versa. A qualitative shape of the radiation pattern thus depends on

1. *degeneracy* of the PNDs (algebraic type of the spacetime),
2. *orientation* of these PNDs with respect to $\mathcal{I}$ (the number of outgoing/tangent/ingoing PNDs).

There are *51 qualitatively different shapes of the radiation patterns* (3 for Petrov type N spacetimes, 9 for type III, 6 for D, 18 for II, and 15 for type I spacetimes); 21 pairs of them are related by the duality, see [C5]. The corresponding directional patterns with PNDs not tangent to $\mathcal{I}$ are shown in figure 13, some examples of those with PNDs tangent to $\mathcal{I}$ can be found in figure 14.

In the case of timelike conformal infinity the choice of canonical reference tetrads adjusted to PNDs is not straightforward — it splits to a lengthy discussion of separate cases depending on orientation of the PNDs with respect to $\mathcal{I}$. In the case of type N fields we can align the vector $\mathbf{k}_{\rm o}$ along this algebraically special direction, i.e., $\mathbf{k}_{\rm o} = \mathbf{k}_1 (= \mathbf{k}_2 = \mathbf{k}_3 = \mathbf{k}_4)$. The vector $\mathbf{l}_{\rm o}$ is fixed by the condition (73). The PNDs are then given by $R_n = 0$, i.e., $\psi_n = 0$ with orientations $\epsilon_n = \epsilon_{\rm o}$. The directional dependence of radiation (89) thus reduces to

$$\left|\Psi_4^{\rm i}\right| \approx \tfrac{1}{4}\,|\Psi_{4*}^{\rm o}|\,|\eta|^{-1}\left(\cosh\psi+\epsilon\epsilon_{\rm o}\right)^2, \tag{90}$$

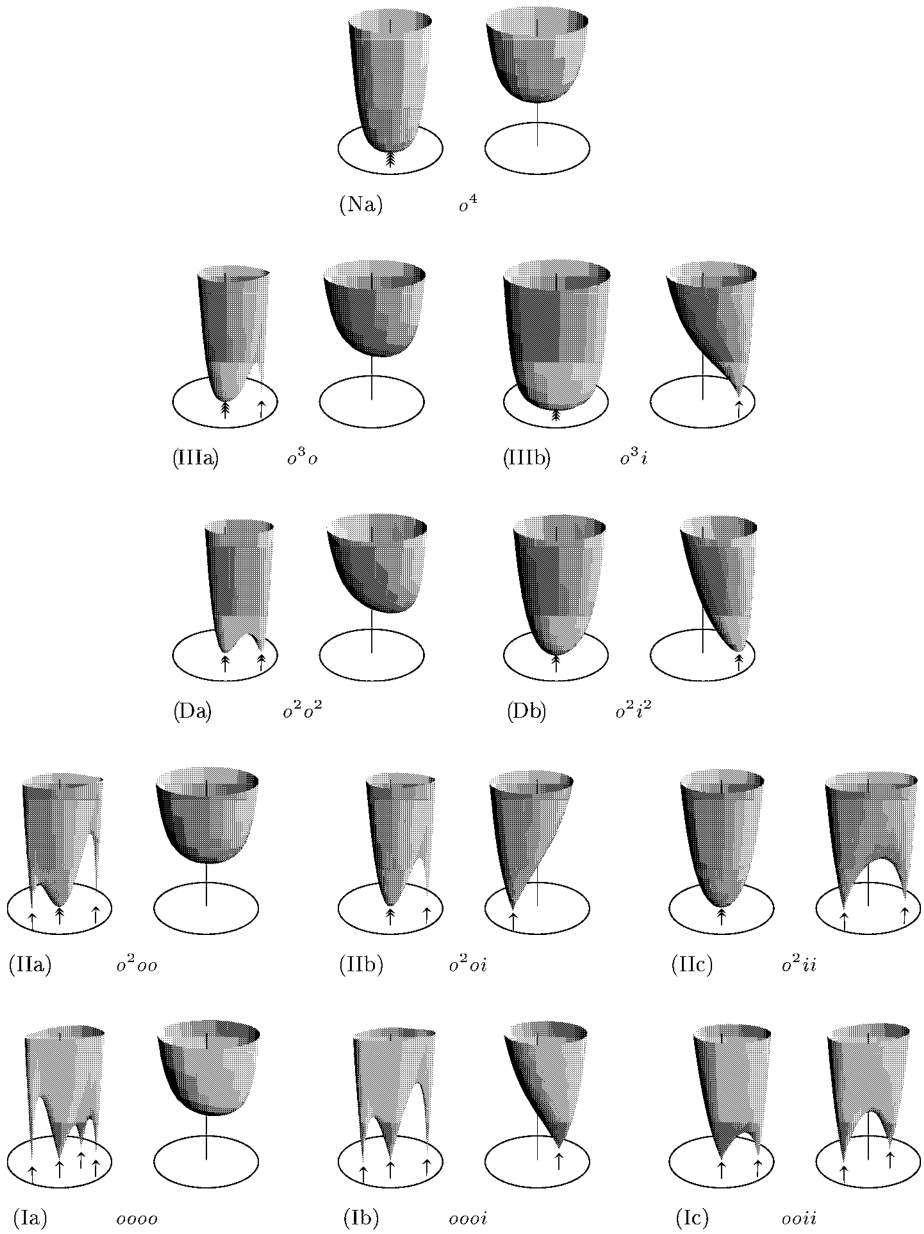


Figure 13: Directional structure of radiation near a timelike $\mathcal{I}$. All 11 qualitatively different shapes of the pattern when PNDs are not tangent to $\mathcal{I}$ are shown (remaining 9 are related by a reflection with respect to $\mathcal{I}$). Each diagram consists of patterns for ingoing (left) and outgoing geodesics (right). $|\Psi_4^{\rm i}|$ is drawn on the vertical axis, directions of geodesics are represented on the horizontal disc. Reflected *[degenerated] PNDs are indicated by [multiple] arrows under the discs. For PNDs that are not tangent to $\mathcal{I}$ these are directions of vanishing radiation. The algebraic Petrov types (N, III, D, II, I) corresponding to the degeneracy of PNDs are indicated by labels of diagrams, number of ingoing and outgoing PNDs is also displayed.*

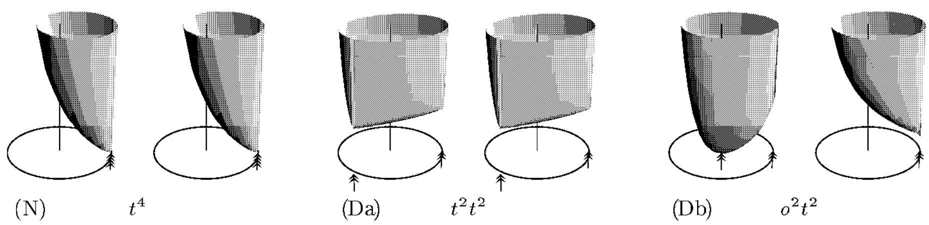


Figure 14: Examples of directional structure of radiation near a timelike $\mathcal{I}$ when PNDs are tangent to $\mathcal{I}$.

illustrated in figure 13(Na). It is possible to introduce natural reference tetrads adjusted to the algebraic structure for type D gravitational fields. A detailed discussion can be found in [C6].

To summarize, as in the $\Lambda > 0$ case [C3] the radiation pattern for $\Lambda < 0$ has a universal character determined by the algebraic type of the fields [C4]. However, new features occur when $\Lambda < 0$: both outgoing and ingoing patterns have to be studied, their shapes depend also on the orientation of the PNDs with respect to $\mathcal{I}$, and an interesting possibility of PNDs tangent to $\mathcal{I}$ appears. Radiation vanishes only along directions which are reflections of PNDs with respect to $\mathcal{I}$. The absence of η^{-1} term cannot be used to distinguish nonradiative sources: near an anti-de Sitter-like infinity the radiative component reflects not only properties of sources but also their relation to the observer.

3.8 Explicit illustration: the C-metric spacetimes

To demonstrate our approach explicitly and to exemplify the general results presented above, let us consider the specific example, namely the behaviour of gravitational and electromagnetic radiation generated by accelerating black holes. Such a situation is described by the C-metric spacetimes that admit an arbitrary value of Λ, see [C1], [C2]. In this case the fields have the asymptotic form

$$|\Psi_4^{\rm i}| \approx \frac{3A^2(m+2e^2Ax_\infty)}{\gamma\,\eta\,P_\infty^2}\,\frac{|1\pm R_2\bar{R}|^2}{|1\pm R^2\,|^2}\,, \tag{91}$$

$$|\Phi_2^{\rm i}| \approx \frac{|e|A}{\sqrt{2}\,\gamma\,\eta\,P_\infty}\,\frac{|1\pm R_2\bar{R}|}{|1\pm R^2\,|}\,, \tag{92}$$

where $\pm 1 = \operatorname{sign}\Lambda$, m and e characterize the mass and charge, A is the acceleration parameter, and

$$R_1 = R_2 = \sqrt{\frac{|\Lambda|}{3}}\,\frac{P_\infty}{A}\,, \qquad\qquad R_3 = R_4 = 0\,, \tag{93}$$

se [C2]. The fields (91), (92) are of the general form (82), (83), and (87), (88), where the directional parameters R_i characterize the double degenerate PNDs of the C-metric spacetime. The directions on $\mathcal{I}$ of these PNDs are, in fact, *directions toward the pair of accelerated black holes.* Radiation vanishes only along the directions $\bar{R} = \mp R_2^{-1}$ at $\mathcal{I}$ which points exactly 'away from the sources' (antipodal to R_2 when $\Lambda > 0$, or the mirror image of R_2 with respect to $\mathcal{I}$ when $\Lambda < 0$).

Of course, the corresponding directional pattern of radiation are of the form typical for type D spacetimes, see figures 12, 13. To be more specific, in the case $\Lambda > 0$, which we investigated in [C1], the pattern has the form shown in figure 12 (D). For $\Lambda < 0$ the situation is more complicated, see [C2], as illustrated in figure 13 (Da) and (Db). There is an additional complication, namely a significant dependence on the acceleration. For *small acceleration*, $A < \sqrt{-\Lambda/3}$, the C-metric represents *a single* accelerated black hole whereas for *large acceleration*, $A > \sqrt{-\Lambda/3}$, it describes *pairs of* accelerated black holes. The global structure and other properties are thus completely different.

In the case of a single black hole uniformly accelerating in asymptotically anti-de Sitter universe the global structure has the form indicated in figure 15. Because in the region near $\mathcal{I}$ the first PND $\mathbf{k}_1$ is everywhere oriented outside the universe, whereas the second one $\mathbf{k}_2$ inside it, the directional structure of radiation has the form indicated in the part (Db) of figure 13.

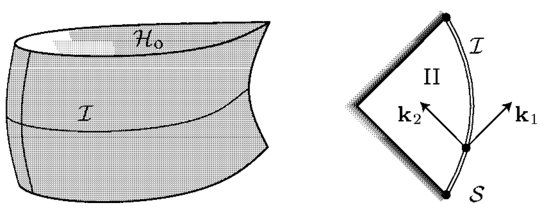


Figure 15: The C-metric spacetime when $A < \sqrt{-\frac{1}{3}\Lambda}$. The region near $\mathcal{I}$ is everywhere static since $R_2 > 1$. The first PND $\mathbf{k}_1$ is oriented outside the universe, whereas the second one $\mathbf{k}_2$ inside it.

For a large value of acceleration A the C-metric with $\Lambda < 0$ describes pairs of accelerated black holes in anti-de Sitter universe. In such a case there are additional Killing horizons outside the black holes, namely the acceleration horizons $\mathcal{H}_a$ separating two black holes, and cosmological horizons $\mathcal{H}_c$ separating different pairs of holes. These horizons divide spacetime into several regions: static domains O and II, non-static domains I^+ and I^-, and domains under the black hole horizons $\mathcal{H}_o$. Consequently, the conformal infinity $\mathcal{I}$, which can be visualized as an outer cylindrical 1+2 boundary, is also divided into several distinct domains with different structure of PNDs, in which radiation patterns differ. These domains can be denoted as $\mathcal{I}_O, \mathcal{I}_{II}$, and $\mathcal{I}_{I^+}, \mathcal{I}_{I^-}$, see figure 16. Consequently,

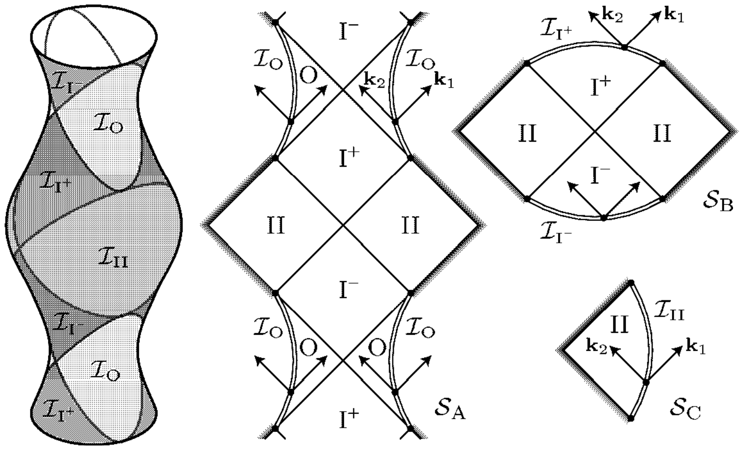


Figure 16: The conformal infinity $\mathcal{I}$ of the C-metric with $A > \sqrt{-\frac{1}{3}\Lambda}$ is divided by horizons $\mathcal{H}_c$ and $\mathcal{H}_a$ into distinct domains: $\mathcal{I}_O$ and $\mathcal{I}_{II}$ are static, $\mathcal{I}_{I^+}$ and $\mathcal{I}_{I^-}$ are non-static. In the regions $\mathcal{I}_O$ and $\mathcal{I}_{II}$ the PND $\mathbf{k}_1$ is oriented outwards, whereas the PND $\mathbf{k}_2$ inwards. In $\mathcal{I}_{I^+}$ both $\mathbf{k}_1$ and $\mathbf{k}_2$ point outside the universe, in $\mathcal{I}_{I^-}$ both the PNDs point inside it.

the structure of outgoing radiation ($|R| < 1$) in the domains $\mathcal{I}_O, \mathcal{I}_{I^+}, \mathcal{I}_{I^-}$, and $\mathcal{I}_{II}$ of conformal infinity are given by the right (Db), right (Da), left (Da), and right (Db) parts of figure 13, respectively; for ingoing radiation ($|R| > 1$) by left (Db), left (Da), right (Da), and left (Db) parts, respectively.

Résumé

This dissertation has been entirely devoted to rigorous treatments of gravitational radiation in the full Einstein theory. Our original contributions to the theory of cosmological gravitational waves have been presented in three separate parts, concentrating on specific topics:

Part A concerns articles on *explicit exact radiative spacetimes*, in particular the class of nonexpanding Kundt waves and expanding waves of the Robinson–Trautman family, both of which admit the cosmological constant Λ. We have studied the geometry of these solutions, and we gave their physical interpretation based on the motion of test particles and on the analysis of wave surfaces. We have also investigated some aspects of the global structure of these radiative spacetimes.

Part B summarizes the results of our systematic study of *impulsive gravitational waves* — both nonexpanding and expanding — which propagate in a Minkowski, de Sitter or anti-de Sitter universe. We have presented a complete picture that includes the classification of the corresponding solutions, description of their geometry, various methods of construction, and analysis of their physical sources.

Part C presents our recent investigations of the *general asymptotic behaviour* of gravitational, electromagnetic and other fields in spacetimes with an arbitrary value of Λ. We have demonstrated that the directional structure of radiation at conformal infinity is essentially determined by the algebraic structure of the fields whereas for $\Lambda = 0$ it degenerates to a unique value.

There are natural motivations for our work. Although important classes of exact radiative solutions were found and investigated in the early 1960s, and almost simultaneously general frameworks which allow one to study properties of radiative fields in asymptotically flat spacetimes were also developed, there still remain open fundamental problems concerning the concept of gravitational radiation in the complete nonlinear Einstein theory. No rigorous statements are available which would relate the properties of sufficiently general strong sources to the radiation fields produced. The theory of gravitational radiation should be applicable to "cosmological" situations which are not asymptotically flat. It is not easy to formulate a sufficiently general covariant radiative boundary conditions. For example, the presence of a nonvanishing cosmological constant Λ is not compatible with the asymptotic flatness that is naturally assumed in many of the existing analyses. Exact explicit examples of waves propagating in a spacetime which is not asymptotically flat, and investigations of the asymptotic behaviour of the fields in general settings may give useful insights. This is the motivation of our investigations of radiation in spacetimes with a general value of the cosmological constant Λ.

For $\Lambda \neq 0$ we deal with gravitational waves "in" everywhere curved de Sitter (for $\Lambda > 0$) or anti-de Sitter (for $\Lambda < 0$) spacetimes which are, on the other hand, sufficiently simple vacuum models, so that the analysis is feasible. Indeed, the de Sitter and anti-de Sitter universes are the simplest cosmological "background" spacetimes because they posses the same number of symmetries as flat Minkowski spacetime. In addition, the de Sitter universe has become popular as the model of the initial inflationary stage in the evolution of our universe. With the inflationary ideas, the "cosmic no-hair" conjecture has emerged: the de Sitter universe should be (locally) reached by a general cosmological model with $\Lambda > 0$ under rather general conditions. The spacetimes discussed in our work enable us to analyze the conjecture in the case of exact solutions with gravitational radiation present. Also the anti-de Sitter spacetime has recently become a subject of intensive studies thanks to the AdS/CFT correspondence which relates string theory in asymptotically anti-de Sitter space to a non-gravitational conformal field theory on the boundary. The spacetimes with a nonvanishing cosmological constant have thus been widely used in various branches of theoretical research, e.g. in inflationary models, brane cosmologies, supergravity and string theories.

Even more interestingly, there appears a growing observational evidence supporting a nonvanishing cosmological constant at the present epoch. The history of discussions of a possible presence of Λ in our real universe is rather complicated [140]. Astronomical observations of supernovae and of cosmic background radiation now indicate the existence of $\Lambda > 0$ in the form of "dark energy" [141–144].

The importance of theoretical investigations of exact radiative spacetimes lies in the possibility to study properties of gravitational waves in cosmological models, their influence on particles and fields, to investigate such principal problems as the global structure, the character of horizons, singularities, and other aspects. Specific exact explicit examples of gravitational waves in cosmological models — such as those presented in this dissertation — may lead to useful insights. Some of these have already been applied to investigation of conjectures recently formulated in the context of higher-dimensional brane cosmologies, quantum gravity or string theories. They may also serve as test-beds for numerical codes simulating more realistic situations in which gravitational waves are generated, and for analytic formulations of radiative boundary conditions in relativistic cosmology.

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