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On the hydrostatic approximation of compressible anisotropic Navier-Stokes equations - rigorous justification

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Abstract

In this work, we obtain the hydrostatic approximation by taking the small aspect ratio limit to the Navier-Stokes equations. The aspect ratio (the ratio of the depth to horizontal width) is a geometrical constraint in general large scale geophysical motions that the vertical scale is significantly smaller than horizontal. We use the versatile relative entropy inequality to prove rigorously the limit from the compressible Navier-Stokes equations to the compressible Primitive Equations. This is the first work to use relative entropy inequality for proving hydrostatic approximation and derive the compressible Primitive Equations.

Key words: anisotropic Naiver-Stokes equations, aspect ratio limit, hydrostatic approximation.

2010 Mathematics Subject Classifications: 35Q30, 35Q86.

1 Introduction

The atmosphere and ocean have attracted considerable attention in the scientific research community, especially for the geophysics, as it has so many fluid dynamic properties and mysterious phenomena. One of the most interesting and physically important features of large-scale meteorology and oceanography is that vertical dimension of the

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domain is much smaller than the horizontal dimension of domain. Therefore, many scientists suggest the viscosity coefficients must be anisotropic, such as [11, 49, 54]. The anisotropic Navier-Stokes equations are widely used in geophysical fluid dynamics. In this paper, we consider the following compressible anisotropic Navier-Stokes equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) = \mu_x \Delta_x \mathbf{u} + \mu_z \partial_{zz} \mathbf{u}, \end{cases} \tag{1.1}$$

in the thin domain $(0,T) \times \Omega_\epsilon$. Here $\Omega_\epsilon = \{(x,z) | x \in \mathbb{T}^2, -\epsilon < z < \epsilon\}$, x denotes the horizontal direction and z denotes the vertical direction, while, μ_x and μ_z are given constant horizontal viscous coefficient and vertical viscous coefficient. The velocity $\mathbf{u} = (\mathbf{v}, w)$, where $\mathbf{v}(t,x,z) \in \mathbb{R}^2$ and $w(t,x,z) \in \mathbb{R}$ represent the horizonal velocity and vertical velocity respectively. Through out this paper, we use $\mathrm{div}\mathbf{u} = \mathrm{div_x}\mathbf{v} + \partial_\mathrm{z}\mathrm{w}$ and $\nabla = (\nabla_x, \partial_z)$ to denote the three-dimensional spatial divergence and gradient respectively, and Δ_x stands for horizontal Laplacian. As atmosphere and ocean are the thin layers, where the fluid layer depth is small compared to radius of sphere, Pedlosky [49] pointed out that "the pressure difference between any two points on the same vertical line depends only on the weight of the fluid between these points...". Here we neglect the gravity and suppose the pressure $p(\rho)$ satisfies the barotropic pressure law where the pressure and the density are related by the formula: $p(\rho) = \rho^\gamma$ $(\gamma > 1)$. **Therefore we assume the density ρ is independent of z** that is $\rho = \rho(t,x)$. This plausible assumption agrees well with experiment and is frequently taken as a hypothesis in geophysical fluid dynamics.

Similar to the assumption by [1, 38], we suppose $\mu_x = 1$ and $\mu_z = \epsilon^2$. As stressed by Azérad and Guillén [1], it is necessary to consider the above anisotropic viscosities scaling, which is fundamental for the derivation of Primitive Equations (PE). Under this assumption, the system is rewritten as the following

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\rho \mathbf{v}) + \partial_z(\rho w) = 0, \\ \rho \partial_t \mathbf{v} + \rho(\mathbf{u} \cdot \nabla)\mathbf{v} - \Delta_x \mathbf{v} - \epsilon^2 \partial_{zz} \mathbf{v} + \nabla_x p(\rho) = 0, \\ \rho \partial_t w + \rho \mathbf{u} \cdot \nabla w - \Delta_x \mathbf{w} - \epsilon^2 \partial_{zz} w + \partial_z p(\rho) = 0. \end{cases} \tag{1.2}$$

Inspired by [38], we introduce the following new unknowns,

$$\mathbf{u}_\epsilon = (\mathbf{v}_\epsilon, w_\epsilon),\ \mathbf{v}_\epsilon(x,z,t) = \mathbf{v}(x, \epsilon z, t),\ w_\epsilon = \frac{1}{\epsilon} w(x, \epsilon z, t),\ \rho_\epsilon = \rho(x,t),$$

for any $(x,z) \in \Omega := \mathbb{T}^2 \times (-1,1)$. Then the system (1.2) becomes the following compressible scaled Navier-Stokes equations (CNS):

$$\begin{cases} \partial_t \rho_\epsilon + \operatorname{div}_x(\rho_\epsilon \mathbf{v}_\epsilon) + \partial_z(\rho_\epsilon w_\epsilon) = 0, \\ \rho_\epsilon \partial_t \mathbf{v}_\epsilon + \rho_\epsilon(\mathbf{u}_\epsilon \cdot \nabla)\mathbf{v}_\epsilon - \Delta_x \mathbf{v}_\epsilon - \partial_{zz} \mathbf{v}_\epsilon + \nabla_x p(\rho_\epsilon) = 0, \\ \epsilon^2(\rho_\epsilon \partial_t w_\epsilon + \rho_\epsilon \mathbf{u}_\epsilon \cdot \nabla w_\epsilon - \Delta_x \mathbf{w}_\epsilon - \partial_{zz} w_\epsilon) + \partial_z p(\rho_\epsilon) = 0. \end{cases} \tag{1.3}$$

We supplement the CNS with the following boundary and initial conditions:

$$\begin{aligned} &\rho_\epsilon,\ \mathbf{u}_\epsilon \text{ are periodic in } x, y, z, \\ &(\rho_\epsilon, \mathbf{u}_\epsilon)|_{t=0} = (\rho_0, \mathbf{u}_0). \end{aligned} \tag{1.4}$$

The goal of this work is to investigate the limit process $\epsilon \to 0$ in the system of (1.3) converge in a certain sense to the following compressible Primitive Equations(CPE):

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\rho \mathbf{v}) + \partial_z(\rho w) = 0, \\ \partial_t(\rho \mathbf{v}) + \operatorname{div}_x(\rho \mathbf{v} \otimes \mathbf{v}) + \partial_z(\rho \mathbf{v} w) + \nabla_x p(\rho) = \Delta_x \mathbf{v} + \partial_{zz} \mathbf{v}, \\ \partial_z p(\rho) = 0. \end{cases} \tag{1.5}$$

The geophysical fluid dynamics is a fundamental subject to understand the atmosphere and ocean. Whereas, from the mathematical point of view and numerical perspective, it is very complicated to use the full hydrodynamical and thermodynamical equations to analyze and simulate atmospheric flows and oceanic flows. Therefore, scientists introduce the Primitive Equation (PE) model in the geophysical fluid dynamics. It was Richardson that derived originally PE model in 1920's for weather prediction. But lacking stability of calculations, this model was not so successful. Then, Bryan [11] improved PE model by applying the hydrostatic approximation in 1969. Compared with abundant successful results in simulation and application for PE at early stage, the mathematical research of PE was started very late. It was until 1990s that Lions, Teman and Wang [39, 40] were first to study the PE and received fundamental results in this field. Then PE has historically progressed by concentrated the mathematical arguments developed by the precise analysis of simpler models. There is a large literatures dedicated to PE model see [6, 7, 8, 13, 14, 15, 33, 34, 36, 41, 42, 51, 53] and references therein. Let us give a short retrospect and comment for some results. Guillén-González, Masmoudi and Rodríguez-Bellido [32] proved the local existence of strong solutions in the three dimension case. The celebrated breakthrough result was made by Cao and Titi [12]. They were first who proved the global well-posedness of PE in the three dimensional case. Then, by virtue of semigroup method, Hieber and Kashiwabara [35] extended this result relaxing the smoothness on the initial data. On the other hand, regarding to inviscid PE (hydrostatic incompressible Euler equations), the existence and uniqueness is an outstanding open problem. Brenier [4] proved the existence of smooth solutions in two-dimensions under the convex horizontal velocity assumptions. And he [5] suggested that the existence problem may be ill-posed in Sobolev spaces. Later, Masmoudi and Wong [47] extended Brenier's result, removing the convex horizontal velocity assumptions. Partly for historical reasons, the research of geophysical fluid concerns on PE model at incompressible case. However, it is well known that atmosphere and ocean have compressible property. Therefore, it is interesting and natural to consider the PE model at compressible case, that is CPE. With the constant viscosity coefficients, Gatapov and Kazhikhov [30], Ersoy and Ngom [20] proved the global existence of weak solutions in 2D case. Recently, Liu and Titi [43, 45] proved the local existence of strong solutions in 3D case and consider the zero Mach number limit of CPE. On the other hand, Ersoy et al. [19] used the dimensionless number and asymptotic analysis, obtaining the CPE in the case where the viscosity coefficients are depending on the density. Ersoy et al. [19], Tang and Gao [50] showed the stability of weak solutions. The stability means that a subsequence of weak solutions will converge to another weak solutions if it satisfies some uniform bounds. Recently, Liu and Titi [44] and independently Wang et al. [52] used the B-D entropy to prove the global existence of weak solutions.

As stressed by [1, 38], the hydrostatic approximation is one of the important feature of PE model. A rigorous justification of the limit passage from anisotropic Navier-Stokes equations to its hydrostatic approximation via the small aspect limit seems to be of obvious practical importance. There are numerous studies of the incompressible convergence. For example, Azéard and Guillén [1] proved the weak solutions of anisotropic Navier-Stokes converges to weak solutions of PE. Li and Titi [38] used the method of weak-strong uniqueness to prove the aspect ratio limit of incompressible anisotropic Navier-Stokes equations, that is from weak solutions of anisotropic Navier-Stokes equations to strong solutions of incompressible PE model. Then Giga, Hieber and Kashiwabara et al. [26, 27] extended the results into maximal regularity spaces. Recently, Donatella and Nora [17] proved the convergence in downwind-matching coordinates. For the stationary case, readers can refer to [3, 8]. On the other hand, based on a revised global Cauchy-Kowalewski theorem, Paicu, Zhang and Zhang [48] proved the incompressible anisotropic Navier-Stokes equations converge to the Prandtl equation in Besov spaces for 2D case. However, for the compressible fluids flows, to the best of authors' knowledge, there are no results concerning the convergence from compressible Navier-Stokes system (CNS) to compressible Primitive Equations (CPE).

Our goal is to rigorously justify the limit in the framework of weak solutions of CNS. Recently, Bella, Feireisl and Novotný [2], Maltese and Novotný [46] proved the limit passage from 3D compressible Navier-Stokes equations to 1D and 2D compressible Navier-Stokes equations in thin domain. See also result by Ducomet et al. [18]. Heuristically, inspired by their works, we develop and adapt the corresponding idea of relative entropy inequality for compressible Navier-Stokes equations. There are huge differences at mathematical structure between Navier-Stokes equations and CPE model. Due to the hydrostatic approximation, there is no information for the vertical velocity in the momentum equation of CPE model, and the vertical velocity is determined by the horizontal velocity via the continuity equation, so it is very difficult to analyze the CPE model. Therefore, the classical method used in Navier-Stokes system can not be applied straightforwardly to CPE. Luckily, based on our previous work [29] of weak-strong uniqueness to CPE, we prove the aspect ratio limit of compressible anisotropic Navier-Stokes equations. Compared with the previous results [29], there are some delicate differences in the process of using relative energy inequality. We should emphasize that we obtain the weak-strong uniqueness that is from weak solutions of CPE to strong solutions of CPE in [29]. Here, our convergence is between two different systems and is from 3D to 2.5D. The role of weak solutions is played by the solutions of CNS, and the strong solutions is played by those of CPE. It means that we should deal with the convergence of the vertical velocity of CNS and the absence of the information on the vertical velocity in CPE. Moreover, the pressure index ($\gamma > 4$) in the present work which satisfies the demand of Bresch and Jabin's result [9]($\gamma > \frac{3}{2}(\frac{4}{3} + \frac{\sqrt{10}}{3}) \simeq 3.5$), it improves our previous work [29]($\gamma > 6$). This is the first work to use the relative entropy inequality for proving the hydrostatic approximation at the compressible case. For the introduction of the versatile relative entropy inequality, see [28]. Last but not least, let us mention that the corner-stone analysis of our results is based on the relative energy inequality which was invented by Dafermos, see [16]. Then Germain [31] introduced it into compressible Navier-Stokes equations. It is Feireisl and his co-authors [22, 23, 25] that generalized the relative energy inequality for solving various compressible fluid model problems.

The paper is organized as follows. In Section 2, we recall some useful inequalities. We introduce the definition of weak solutions, strong solution, relative energy and state the main theorem in Section 3. Section 4 is devoted to proof of the convergence.

2 Preliminaries

In this section, we first introduce some basic inequalities needed in the later proof. The first inequality is the so called the generalized Poincaré inequality.

Lemma 2.1. *Let $2 \leq p \leq 6$, and $\rho \geq 0$ such that $0 < \int_\Omega \rho dx \leq M$ and $\int_\Omega \rho^\gamma dx \leq E_0$ for some ($\gamma > 1$) then*

$$\|f\|_{L^p(\Omega)} \leq C\|\nabla f\|_{L^2(\Omega)} + \|\rho^{\frac{1}{2}} f\|_{L^2(\Omega)},$$

where C depends on M and E_0.

The details of proof can be seen at Feireisl's monograph [21]. The following is the famous Gagliardo-Nirenberg inequality.

Lemma 2.2. *For a function $u : \Omega \to \mathbb{R}$ defined on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $\forall 1 \leq q, r \leq \infty$, and a natural number m. Suppose that a real number θ and a natural* *number j are such that*

$$\frac{1}{p}=\frac{j}{n}+(\frac{1}{r}-\frac{m}{n})\theta+\frac{1-\theta}{q},$$

and

$$\frac{j}{m}\leq\theta\leq 1,$$

then there exists constant C independent of u such that

$$\|D^j u\|_{L^p(\Omega)}\leq C\|D^m u\|^{\theta}_{L^r(\Omega)}\|u\|^{1-\theta}_{L^q(\Omega)}.$$

3 Main result

Before showing our main result, we give the definition of a weak solution for CNS and a strong solution for CPE. Recently, Bresch and Jabin [9] consider different compactness method from Lions or Feireisl which can be applied to anisotropical stress tensor. They obtain the global existence of weak solutions for non-monotone pressure. Let us recall their definitions here.

3.1 Dissipative weak solutions of CNS

Definition 3.1. *We say that $[\rho_\epsilon,\mathbf{u}_\epsilon]$ $\mathbf{u}_\epsilon=(\mathbf{v}_\epsilon,w_\epsilon)$ is a finite energy weak solution to the system of* (1.3)*, supplemented with initial data (1.4) if $\rho_\epsilon=\rho_\epsilon(x,t)$ and*

$$\begin{aligned}&\mathbf{u}_\epsilon\in L^2(0,T;H^1(\Omega)),\ \rho|\mathbf{u}_\epsilon|^2\in L^\infty(0,T;L^1(\Omega)),\\&\rho_\epsilon\in L^\infty(0,T;L^\gamma(\Omega))\cap C([0,T],L^1(\Omega)),\end{aligned}\tag{3.1}$$

• *the continuity equation*

$$[\int_\Omega\rho_\epsilon\psi dxdz]_{t=0}^{t=\tau}=\int_0^\tau\int_\Omega\rho_\epsilon\partial_t\psi+\rho_\epsilon\mathbf{v}_\epsilon\cdot\nabla_x\psi+\rho_\epsilon w_\epsilon\partial_z\psi dxdzdt,\tag{3.2}$$

holds for all $\psi\in C_c^\infty([0,T)\times\Omega)$;
• *the momentum equation*

$$\begin{aligned}&[\int_\Omega\rho_\epsilon\mathbf{v}_\epsilon\varphi_{\mathbf{H}}dxdz]_{t=0}^{t=\tau}-\int_0^\tau\int_\Omega\rho_\epsilon\mathbf{v}_\epsilon\partial_t\varphi_{\mathbf{H}}dxdzdt-\int_0^\tau\int_\Omega\rho_\epsilon\mathbf{u}_\epsilon\mathbf{v}_\epsilon\cdot\nabla\varphi_{\mathbf{H}}dxdzdt\\&\quad+\int_0^\tau\int_\Omega\nabla\mathbf{v}_\epsilon\cdot\nabla\varphi_{\mathbf{H}}dxdzdt-\int_0^\tau\int_\Omega p(\rho_\epsilon)\mathrm{div}_x\varphi_{\mathbf{H}}dxdzdt=0,\end{aligned}\tag{3.3}$$

and

$$\epsilon^2[\int_\Omega\rho_\epsilon w_\epsilon\varphi_3 dxdz]_{t=0}^{t=\tau}-\epsilon^2\int_0^\tau\int_\Omega\rho_\epsilon w_\epsilon\partial_t\varphi_3 dxdzdt-\epsilon^2\int_0^\tau\int_\Omega\rho_\epsilon\mathbf{u}_\epsilon w_\epsilon\cdot\nabla\varphi_3 dxdzdt$$

$$+\epsilon^2\int_0^\tau\int_\Omega \nabla w_\epsilon\cdot\nabla\varphi_3 dxdzdt-\int_0^\tau\int_\Omega p(\rho_\epsilon)\partial_z\varphi_3 dxdzdt=0, \quad (3.4)$$

holds for all $\varphi_{\mathbf{H}},\varphi_3\in C_c^\infty([0,T)\times\Omega)$. *Combining* (3.3) − (3.4), *we obtain*

$$\begin{aligned}
&[\int_\Omega \rho_\epsilon \mathbf{v}_\epsilon\varphi_{\mathbf{H}} dxdz+\epsilon^2\int_\Omega \rho_\epsilon w_\epsilon\varphi_3 dxdz]_{t=0}^{t=\tau}\\
&\quad-\int_0^\tau\int_\Omega \rho_\epsilon\mathbf{v}_\epsilon\partial_t\varphi_{\mathbf{H}} dxdzdt-\epsilon^2\int_0^\tau\int_\Omega \rho_\epsilon w_\epsilon\partial_t\varphi_3 dxdzdt\\
&\quad-\int_0^\tau\int_\Omega \rho_\epsilon\mathbf{v}_\epsilon\otimes\mathbf{v}_\epsilon:\nabla_x\varphi_{\mathbf{H}} dxdzdt-\int_0^\tau\int_\Omega \rho_\epsilon\mathbf{v}_\epsilon w_\epsilon\cdot\partial_z\varphi_H dxdzdt\\
&\quad-\epsilon^2\int_0^\tau\int_\Omega \rho_\epsilon\mathbf{v}_\epsilon w_\epsilon\cdot\nabla_x\varphi_3 dxdzdt-\epsilon^2\int_0^\tau\int_\Omega \rho_\epsilon w_\epsilon^2\partial_z\varphi_3 dxdzdt\\
&\quad+\int_0^\tau\int_\Omega \nabla\mathbf{v}_\epsilon:\nabla\varphi_{\mathbf{H}} dxdzdt+\epsilon^2\int_0^\tau\int_\Omega \nabla w_\epsilon\cdot\nabla\varphi_3 dxdzdt-\int_0^\tau\int_\Omega p(\rho_\epsilon)\mathrm{div}\varphi dxdzdt=0,
\end{aligned} \quad (3.5)$$

where $\varphi=(\varphi_{\mathbf{H}},\varphi_3)\in C_c^\infty([0,T)\times\Omega)$ *and* $div\varphi=div_x\varphi_{\mathbf{H}}+\partial_z\varphi_3$,

- *the energy inequality*

$$[\int_\Omega \frac{1}{2}\rho_\epsilon|\mathbf{v}_\epsilon|^2+\frac{\epsilon^2}{2}\rho_\epsilon|w_\epsilon|^2+P(\rho_\epsilon))dxdz]|_{t=0}^{t=\tau}+\int_0^\tau\int_\Omega(|\nabla\mathbf{v}_\epsilon|^2+\epsilon^2|\nabla w_\epsilon|^2)dxdzdt\le 0, \quad (3.6)$$

holds for a.a $\tau\in(0,T)$, *where* $P(\rho)=\rho\int_1^\rho\frac{p(z)}{z^2}dz$.

3.2 Strong solution of CPE

We say that $(r,\mathbf{U})$, $\mathbf{U}=(\mathbf{V},W)$ is a strong solution to the CPE system (1.5) in $(0,T)\times\Omega$, if

$$\begin{aligned}
&r^{\frac{1}{2}}\in L^\infty(0,T;H^2(\Omega)),\ \partial_t r^{\frac{1}{2}}\in L^\infty(0,T;H^1(\Omega)),\ r>0 \text{ for all } (t,x),\\
&\mathbf{V}\in L^\infty(0,T;H^3(\Omega))\cap L^2(0,T;H^4(\Omega)),\ \partial_t\mathbf{V}\in L^2(0,T;H^2(\Omega)),
\end{aligned}$$

with initial data $r_0^{\frac{1}{2}}\in H^2(\Omega)$, $r_0>0$ and $\mathbf{V}_0\in H^3(\Omega)$. Liu and Titi [43] has proved the local existence of strong solution to CPE system (1.5).

Remark 3.1. *As the density is independent of z, we can obtain the following information of vertical velocity for the weak solution of CNS :*

$$\rho w(x,z,t)=-\mathrm{div_x}(\rho\widetilde{\mathbf{v}})+\mathrm{zdiv_x}(\rho\overline{\mathbf{v}}),\ \textit{in the sense of}\ \mathrm{H}^{-1}(\Omega), \quad (3.7)$$

where

$$\widetilde{\mathbf{v}}(x,z,t)=\int_0^z v(x,s,t)ds,\ \overline{v}(x,t)=\int_0^1 v(x,z,t)dz.$$

Similarly, we can obtain the same equation for the strong solution of CPE in the classical sense. There is no information about w, so we need to derive its information. We should emphasize that (3.7) is the key step to obtain the existence of weak solution for CPE in [44, 52], which is inspired by incompressible case.

3.3 Relative entropy inequality

Motivated by [22, 23], for any finite energy weak solution $(\rho, \mathbf{u})$, where $\mathbf{u} = (\mathbf{v}, w)$, to the CNS system, we introduce the relative energy functional

$$\begin{aligned}
\mathcal{E}(\rho, \mathbf{u}|r, \mathbf{U}) &= \int_\Omega [\frac{1}{2}\rho|\mathbf{v} - \mathbf{V}|^2 + \frac{\epsilon^2}{2}\rho|w - W|^2 + P(\rho) - P'(r)(\rho - r) - P(r)]dxdz \\
&= \int_\Omega (\frac{1}{2}\rho|\mathbf{v}|^2 + \frac{\epsilon^2}{2}\rho|w|^2 + P(\rho))dxdz - \int_\Omega (\rho\mathbf{v}\cdot\mathbf{V} + \epsilon^2\rho w W)dxdz \\
&\quad + \int_\Omega [\rho\frac{|\mathbf{V}|^2}{2} + \frac{\epsilon^2}{2}\rho|W|^2 - P'(r)]dxdz + \int_\Omega p(r)dxdz \\
&= \sum_{i=1}^{4} I_i, \qquad (3.8)
\end{aligned}$$

where $r > 0$, $\mathbf{U} = (\mathbf{V}, W)$ are smooth "test" functions, r, $\mathbf{U}$ compactly supported in Ω.

Lemma 3.1. *Let $(\rho, \mathbf{v}, w)$ be a dissipative weak solution introduced in Definition 3.1. Then $(\rho, \mathbf{v}, w)$ satisfy the relative entropy inequality*

$$\begin{aligned}
&\mathcal{E}(\rho, \mathbf{u}|r, \mathbf{U})|_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \big(\nabla\mathbf{v}\cdot(\nabla\mathbf{v} - \nabla\mathbf{V}) + \epsilon^2|\nabla w|^2\big)dxdzdt \\
&\le \int_0^\tau \int_\Omega \rho(\partial_t\mathbf{V} + \mathbf{v}\nabla_x\mathbf{V} + w\partial_z\mathbf{V})(\mathbf{V} - \mathbf{v})dxdt \\
&\quad + \epsilon^2 \int_0^\tau \int_\Omega \rho(\partial_t W + \mathbf{v}\nabla_x W + w\partial_z W)(W - w)dxdzdt + \epsilon^2 \int_0^\tau \int_\Omega \nabla w \cdot \nabla W dxdzdt \\
&\quad - \int_0^\tau \int_\Omega P''(r)((\rho - r)\partial_t r + \rho\mathbf{v}\nabla_x r)dxdzdt - \int_0^\tau \int_\Omega p(r)\mathrm{div}_x\mathbf{V}dxdzdt. \qquad (3.9)
\end{aligned}$$

Proof: From the weak formulation and energy inequality (3.3)-(3.6), we deduce

$$I_1|_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega (|\nabla\mathbf{v}|^2 + \epsilon^2|\nabla w|^2)dxdzdt \le 0, \qquad (3.10)$$

$$\begin{aligned}
I_2|_{t=0}^{t=\tau} = &- \int_0^\tau \int_\Omega \rho\mathbf{v}\partial_t\mathbf{V} + \rho\mathbf{v}\otimes\mathbf{v} : \nabla_x\mathbf{v} + \rho\mathbf{v}w\cdot\partial_z\mathbf{V}dxdzdt \\
&+ \int_0^\tau \int_\Omega \epsilon^2\rho w\partial_t W + \epsilon^2\rho w(\mathbf{v}\cdot\nabla_x)W + \epsilon^2\rho w^2\partial_z W + p(\rho)\mathrm{div}_x\mathbf{V}dxdzdt \\
&+ \int_0^\tau \int_\Omega \nabla\mathbf{v} : \nabla\mathbf{V} + \epsilon^2\nabla w\cdot\nabla W dxdzdt, \qquad (3.11)
\end{aligned}$$

$$
\begin{aligned}
I_3|_{t=0}^{t=\tau} = & \int_0^\tau \int_\Omega \rho\partial_t \frac{|\mathbf{V}|^2}{2} + \rho\mathbf{v}\cdot\nabla_x \frac{|\mathbf{V}|^2}{2} + \rho w \partial_z \frac{|\mathbf{V}|^2}{2} dxdzdt \\
& + \epsilon^2 \int_0^\tau \int_\Omega \rho\partial_t \frac{|W|^2}{2} + \rho\mathbf{v}\cdot\nabla_x \frac{|W|^2}{2} + \rho w \partial_z \frac{|W|^2}{2} dxdzdt \\
& - \int_0^\tau \int_\Omega \rho\partial_t P'(r) + \rho\mathbf{v}\cdot\nabla_x P'(r) + \rho w \partial_z P'(r) dxdzdt \\
= & \int_0^\tau \int_\Omega \rho\mathbf{V}\partial_t\mathbf{V} + \rho\mathbf{v}(\mathbf{V}\cdot\nabla_x)\mathbf{V} + \rho w \mathbf{V}\partial_z\mathbf{V} dxdzdt \\
& + \epsilon^2 \int_0^\tau \int_\Omega (\rho W\partial_t W + \rho W\mathbf{v}\cdot\nabla_x W + \rho w W \partial_z W) dxdzdt \\
& - \int_0^\tau \int_\Omega \rho P''(r)\partial_t r + P''(r)\rho\mathbf{v}\cdot\nabla_x r dxdzdt, \qquad (3.12)
\end{aligned}
$$

$$
I_4|_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \partial_t p(r) dxdzdt. \qquad (3.13)
$$

Summing (3.10)-(3.13) together, we obtain Lemma 3.1.

Based on the relative entropy inequality, we can obtain the following lemma from [22]

Lemma 3.2. *Let $0 < a < b < \infty$. Then there exists $c = c(a,b) > 0$ such that for all $\rho \in [0,\infty)$ and $r \in [a,b]$ there holds*

$$
P(\rho) - P'(r)(\rho - r) - P(r) \geq \begin{cases} C|\rho - r|^2, & \text{when } \frac{r}{2} < \rho < r, \\ C(1+\rho^\gamma), & \text{otherwise,} \end{cases}
$$

where $C = C(a,b)$.

Moreover, from [22], we learn that

$$
\begin{aligned}
& \mathcal{E}(\rho,\mathbf{u}|r,\mathbf{U})(t) \in L^\infty(0,T), \ \int_\Omega \chi_{\rho\geq r}\rho^\gamma dxdz \leq C\mathcal{E}(\rho,\mathbf{u}|r,\mathbf{U})(t), \\
& \int_\Omega \chi_{\rho\leq\frac{r}{2}} 1 dxdz \leq C\mathcal{E}(\rho,\mathbf{u}|r,\mathbf{U})(t), \ \int_\Omega \chi_{\frac{r}{2}<\rho<r}(\rho - r)^2 dxdz \leq C\mathcal{E}(\rho,\mathbf{u}|r,\mathbf{U})(t). \qquad (3.14)
\end{aligned}
$$

For a rigorous proof of Lemma 3.2 and (3.14), the reader is referred to [22].

3.4 Main result

Now, we are ready to state our main result.

Theorem 3.1. *Let $\gamma > 4$, $T_{max} > 0$ be the life time of strong solution to CPE system (1.5) corresponding to initial data $[r_0, \mathbf{V}_0]$. Let $(\rho_\epsilon, \mathbf{u}_\epsilon)$, $\mathbf{u}_\epsilon = (\mathbf{v}_\epsilon, w_\epsilon)$ be a sequence of dissipative weak solutions to the CNS system (1.3) from the initial data $(\rho_{0,\epsilon}, \mathbf{u}_{0,\epsilon})$. Suppose that*

$$
\mathcal{E}(\rho_{0,\epsilon}, \mathbf{u}_{0,\epsilon}|r_0, \mathbf{U}_0) \to 0,
$$

where $\mathbf{U}_0 = (\mathbf{V}_0, W_0)$*, then*

$$ess \sup_{t\in(0,T_{max})} \mathcal{E}(\rho_\epsilon, \mathbf{u}_\epsilon|r, \mathbf{U}) \to 0,$$

where $\mathbf{U} = (\mathbf{V}, W)$ *and the couple* $(r, \mathbf{U})$ *satisfy the CPE system (1.5) on the time interval* $[0, T_{max})$*.*

Remark 3.2. *Recently, Bresch and Burtea [10] proved existence of weak solutions for the anisotropic compressible Stokes system.*

Remark 3.3. *It is important to point out that our convergence holds on a fixed time interval due to the local existence of CPE. Some results [26, 38, 48] concerning the incompressible PE model were shown the global convergence based on the global existence in time under assumptions on the smallness of initial data.*

Section 4 is devoted to the proof of the above theorem.

4 Convergence

In this section, we will prove the Theorem 3.1. First, we will explain our idea of the proof in the following.

4.1 Main idea of Proof

The proof of Theorem 3.1 depends on the relative energy inequality by considering the strong solution $(r, \mathbf{U})$, where $\mathbf{U} = (\mathbf{V}, W)$, as test function in the relative energy inequality (3.8). Firstly, let us recall the relative energy inequality

$$\begin{aligned}
&\mathcal{E}(\rho_\epsilon, \mathbf{u}_\epsilon|r, \mathbf{U})|_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \big(\nabla \mathbf{v}_\epsilon \cdot (\nabla \mathbf{v}_\epsilon - \nabla \mathbf{V}) + \epsilon^2 \nabla w_\epsilon (\nabla w_\epsilon - \nabla W)\big) dxdzdt \\
&\quad \leq \int_0^\tau \int_\Omega \rho_\epsilon (\partial_t \mathbf{V} + \mathbf{v}_\epsilon \nabla_x \mathbf{V} + w_\epsilon \partial_z \mathbf{V})(\mathbf{V} - \mathbf{v}_\epsilon) dxdt \\
&\quad + \epsilon^2 \int_0^\tau \int_\Omega \rho_\epsilon (\partial_t W + \mathbf{v}_\epsilon \nabla_x W + w_\epsilon \partial_z W)(W - w_\epsilon) dxdzdt + \epsilon^2 \int_0^\tau \int_\Omega \nabla w_\epsilon \cdot \nabla W dxdzdt \\
&\quad - \int_0^\tau \int_\Omega P''(r)((\rho_\epsilon - r)\partial_t r + \rho_\epsilon \mathbf{v}_\epsilon \nabla_x r) dxdzdt - \int_0^\tau \int_\Omega p(r) \mathrm{div}_x \mathbf{V} dxdzdt. \qquad (4.1)
\end{aligned}$$

The goal now is to find the an estimate of the left hand side of (4.1) in the following form

$$\mathcal{E}(\rho_\epsilon, \mathbf{u}_\epsilon|r, \mathbf{U})(t) + C \int_0^t \|\nabla \mathbf{v}_\epsilon - \nabla \mathbf{V}\|_{L^2}^2 dt + \epsilon^2 \int_0^t \|\nabla \mathbf{w}_\epsilon\|_{L^2}^2 dt$$

and of the right hand side in the form

$$C(\delta)\int_0^t h(t)\mathcal{E}(\rho_\epsilon,\mathbf{u}_\epsilon|r,\mathbf{U})dt+\delta\int_0^t \|\nabla\mathbf{v}_\epsilon-\nabla\mathbf{V}\|^2_{W^{1,2}}dt+o(\epsilon^2),$$

with any $\delta>0$, where C is independent of δ and ϵ, $h\in L^1(0,T)$ and $o(\epsilon^2)\to 0$ when $\epsilon\to 0$.

If we establish the above bounds, we can deduce

$$\mathcal{E}(\rho_\epsilon,\mathbf{u}_\epsilon|r,\mathbf{U})(\tau)\le C\int_0^\tau h(t)\mathcal{E}(\rho_\epsilon,\mathbf{u}_\epsilon|r,\mathbf{U})(t)dt+o(\epsilon^2),$$

that implies our result by using the Gronwall inequality. In the rest of this section, we will perform this programme.

4.2 Step 1

We write

$$\int_\Omega \rho_\epsilon\mathbf{v}_\epsilon(\mathbf{V}-\mathbf{v}_\epsilon)\cdot\nabla_x\mathbf{V}dxdz=$$
$$\int_\Omega \rho_\epsilon(\mathbf{v}_\epsilon-\mathbf{V})(\mathbf{V}-\mathbf{v}_\epsilon)\cdot\nabla_x\mathbf{V}dxdz+\int_\Omega \rho_\epsilon\mathbf{V}(\mathbf{V}-\mathbf{v}_\epsilon)\cdot\nabla_x\mathbf{V}dxdz.$$

As $[r,\mathbf{V},W]$ is a strong solution, it is obvious to obtain that

$$\int_\Omega \rho_\epsilon(\mathbf{v}_\epsilon-\mathbf{V})(\mathbf{V}-\mathbf{v}_\epsilon)\cdot\nabla_x\mathbf{V}dxdz\le C\mathcal{E}(\rho_\epsilon,\mathbf{u}_\epsilon|r,\mathbf{U}). \tag{4.2}$$

Moreover, the momentum equation reads as

$$(r\mathbf{V})_t+\mathrm{div}_x(r\mathbf{V}\otimes\mathbf{V})+\partial_z(r\mathbf{V}W)+\nabla_x p(r)=\Delta\mathbf{V}=\Delta_x\mathbf{V}+\partial_{zz}\mathbf{V},$$

which implies that

$$\mathbf{V}_t+\mathbf{V}\cdot\nabla_x\mathbf{V}+W\partial_z\mathbf{V}=-\frac{1}{r}\nabla_x p(r)+\frac{1}{r}\Delta_x\mathbf{V}+\frac{1}{r}\partial_{zz}\mathbf{V}.$$

So we rewrite the preceding two items on the right side of (4.1) as

$$\begin{aligned}\int_\Omega \rho_\epsilon[\partial_t\mathbf{V}+\mathbf{V}\nabla_x\mathbf{V}+W\partial_z\mathbf{V}+(\mathbf{v}_\epsilon-\mathbf{V})\nabla_x\mathbf{V}+(w_\epsilon-W)\partial_z\mathbf{V}](\mathbf{V}-\mathbf{v}_\epsilon)dxdz\\ =\int_\Omega \frac{\rho_\epsilon}{r}(\mathbf{V}-\mathbf{v}_\epsilon)(\Delta_x\mathbf{V}+\partial_{zz}\mathbf{V}-\nabla_x p(r))dxdz\\ +\int_\Omega \rho_\epsilon(w_\epsilon-W)(\mathbf{V}-\mathbf{v}_\epsilon)\cdot\partial_z\mathbf{V}dxdz-\int_\Omega \rho_\epsilon(\mathbf{V}-\mathbf{v}_\epsilon)^2\nabla_x\mathbf{V},\end{aligned}$$

and

$$\epsilon^2\int_0^\tau\int_\Omega \rho_\epsilon(\partial_t W+\mathbf{v}_\epsilon\nabla_x W+w_\epsilon\partial_z W)(W-w_\epsilon)dxdzdt$$

$$\leq \mathcal{E}(\rho_\epsilon, \mathbf{u}_\epsilon|r, \mathbf{U}) + \epsilon^4 \int_0^\tau \int_\Omega \rho_\epsilon(\partial_t W + \mathbf{v}_\epsilon \nabla_x W + w_\epsilon \partial_z W)^2 dxdzdt$$
$$= \mathcal{E}(\rho_\epsilon, \mathbf{u}_\epsilon|r, \mathbf{U}) + \epsilon^4 \int_0^\tau \int_\Omega \rho_\epsilon(\partial_t W + \mathbf{V} \nabla_x W + W \partial_z W)^2 dxdzdt$$
$$+ \epsilon^4 \int_0^\tau \int_\Omega \rho_\epsilon((\mathbf{v}_\epsilon - \mathbf{V}) \nabla_x W + (w_\epsilon - W) \partial_z W)^2 dxdzdt. \tag{4.3}$$

Noticing Lemma 3.2, we have

$$\int_\Omega \rho_\epsilon(\partial_t W + \mathbf{V}\nabla_x W + W\partial_z W)^2 dxdzdt$$
$$= \int_\Omega \chi_{\rho_\epsilon < \frac{r}{2}} \rho_\epsilon(\partial_t W + \mathbf{V}\nabla_x W + W\partial_z W)^2 dxdz$$
$$+ \int_\Omega \chi_{\frac{r}{2} \leq \rho_\epsilon \leq r} \rho_\epsilon(\partial_t W + \mathbf{V}\nabla_x W + W\partial_z W)^2 dxdz$$
$$+ \int_\Omega \chi_{\rho_\epsilon > r} \rho_\epsilon(\partial_t W + \mathbf{V}\nabla_x W + W\partial_z W)^2 dxdzdt$$
$$\leq \int_\Omega \chi_{\rho_\epsilon < \frac{r}{2}} r(\partial_t W + \mathbf{V}\nabla_x W + W\partial_z W)^2 dxdz$$
$$+ \int_\Omega \chi_{\rho_\epsilon > r} \rho_\epsilon(\partial_t W + \mathbf{V}\nabla_x W + W\partial_z W)^2 dxdzdt$$
$$+ C\int_\Omega \chi_{\frac{r}{2} \leq \rho_\epsilon \leq r} (\rho_\epsilon - r)(\partial_t W + \mathbf{V}\nabla_x W + W\partial_z W)^2 dxdz$$
$$\leq C\mathcal{E}(\rho_\epsilon, \mathbf{u}_\epsilon|r, \mathbf{U}) + C\int_\Omega \chi_{\frac{r}{2} \leq \rho_\epsilon \leq r} (\rho_\epsilon - r)^2 dxdz + C\int_\Omega \chi_{\rho_\epsilon > r} \rho_\epsilon^\gamma dxdz + C$$
$$\leq C\mathcal{E}(\rho_\epsilon, \mathbf{u}_\epsilon|r, \mathbf{U}) + C. \tag{4.4}$$

Putting (4.4) into (4.3) yields

$$\epsilon^2 \int_0^\tau \int_\Omega \rho_\epsilon(\partial_t W + \mathbf{v}_\epsilon \nabla_x W + w_\epsilon \partial_z W)(W - w_\epsilon) dxdzdt \leq C\mathcal{E}(\rho_\epsilon, \mathbf{u}_\epsilon|r, \mathbf{U}) + o(\epsilon^2).$$

Moreover, a simple application of Cauchy inequality leads to the following

$$\epsilon^2 \int_0^\tau \int_\Omega \nabla w_\epsilon \cdot \nabla W dxdzdt \leq \frac{\epsilon^2}{2} \int_0^\tau \int_\Omega |\nabla w_\epsilon|^2 dxdzdt + o(\epsilon^2).$$

Thus, we obtain that

$$\mathcal{E}(\rho_\epsilon, \mathbf{u}_\epsilon|r, \mathbf{U})|_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \big(\nabla \mathbf{v}_\epsilon \cdot (\nabla \mathbf{v}_\epsilon - \nabla \mathbf{V}) + \frac{\epsilon^2}{2}|\nabla w_\epsilon|^2\big) dxdzdt$$
$$\leq C\int_0^\tau \mathcal{E}(\rho_\epsilon, \mathbf{u}_\epsilon|r, \mathbf{U})dt - \int_0^\tau \int_\Omega P''(r)((\rho_\epsilon - r)\partial_t r + \rho_\epsilon \mathbf{v}_\epsilon \nabla_x r) dxdzdt$$
$$+ \int_0^\tau \int_\Omega \frac{\rho_\epsilon}{r}(\mathbf{V} - \mathbf{v}_\epsilon)(\Delta_x \mathbf{V} + \partial_{zz}\mathbf{V}) dxdz - \int_0^\tau \int_\Omega \frac{\rho_\epsilon}{r}(\mathbf{V} - \mathbf{v}_\epsilon)\nabla_x p(r) dxdz$$
$$+ \int_0^\tau \int_\Omega \rho_\epsilon(w_\epsilon - W)(\mathbf{V} - \mathbf{v}_\epsilon) \cdot \partial_z \mathbf{V} dxdzdt - \int_0^\tau \int_\Omega p(\rho_\epsilon)\mathrm{div}_x \mathbf{V} dxdzdt + o(\epsilon^2).$$

4.3 Step 2

The major challenges of the analysis is to estimate the complicated nonlinear term $\int_\Omega \rho_\epsilon(w_\epsilon - W)(\mathbf{V}-\mathbf{v}_\epsilon)\cdot\partial_z\mathbf{V}dxdz$, we rewrite it as

$$\begin{aligned}&\int_\Omega \rho_\epsilon(w_\epsilon - W)(\mathbf{V}-\mathbf{v}_\epsilon)\cdot\partial_z\mathbf{V}dxdz\\&\qquad=\int_\Omega \rho_\epsilon w_\epsilon(\mathbf{V}-\mathbf{v}_\epsilon)\cdot\partial_z\mathbf{V}dxdz-\int_\Omega \rho_\epsilon W(\mathbf{V}-\mathbf{v}_\epsilon)\cdot\partial_z\mathbf{V}dxdz.\end{aligned}\tag{4.5}$$

A similar heuristic argument from [22, 37] shows that the second term on the right side of (4.5) will be split into three parts

$$\begin{aligned}&\int_\Omega \rho_\epsilon W(\mathbf{V}-\mathbf{v}_\epsilon)\cdot\partial_z\mathbf{U}dxdz\\&=\int_\Omega \chi_{\rho_\epsilon\le\frac{r}{2}}\rho_\epsilon W(\mathbf{V}-\mathbf{v}_\epsilon)\cdot\partial_z\mathbf{V}dxdz+\int_\Omega \chi_{\frac{r}{2}<\rho_\epsilon<r}\rho_\epsilon W(\mathbf{V}-\mathbf{v}_\epsilon)\cdot\partial_z\mathbf{V}dxdz\\&\quad+\int_\Omega \chi_{\rho_\epsilon\ge r}\rho_\epsilon W(\mathbf{V}-\mathbf{v}_\epsilon)\cdot\partial_z\mathbf{V}dxdz\\&\le\|\chi_{\rho_\epsilon\le\frac{r}{2}}1\|_{L^2(\Omega)}\|r\|_{L^\infty}\|W\partial_z\mathbf{V}\|_{L^3}\|\mathbf{V}-\mathbf{v}_\epsilon\|_{L^6(\Omega)}+\int_\Omega \chi_{\rho_\epsilon\ge r}\rho_\epsilon^{\frac{\gamma}{2}}W\partial_z\mathbf{V}\cdot(\mathbf{V}-\mathbf{v}_\epsilon)dxdz\\&\quad+C\|\chi_{\frac{r}{2}<\rho_\epsilon<r}(\rho_\epsilon-r)\|_{L^2(\Omega)}\|W\partial_z\mathbf{V}\|_{L^3}\|\mathbf{V}-\mathbf{v}_\epsilon\|_{L^6(\Omega)}\\&\le C\int_\Omega \chi_{\rho_\epsilon\le\frac{r}{2}}1dxdz+C\int_\Omega \chi_{\frac{r}{2}<\rho_\epsilon<r}(\rho_\epsilon-r)^2dxdz\\&\quad+C\int_\Omega \chi_{\rho_\epsilon\ge r}\rho_\epsilon^\gamma dxdz+\delta\|\mathbf{V}-\mathbf{v}_\epsilon\|_{L^6(\Omega)}^2\\&\le C\mathcal{E}(\rho_\epsilon,\mathbf{u}_\epsilon|r,\mathbf{U})+\delta\|\nabla_x\mathbf{V}-\nabla_x\mathbf{v}_\epsilon\|_{L^2(\Omega)}^2+\delta\|\partial_z\mathbf{V}-\partial_z\mathbf{v}_\epsilon\|_{L^2(\Omega)}^2,\end{aligned}\tag{4.6}$$

where in the last inequality, we have used Lemma 2.1.

We now turn to analyze the first term on the right hand of (4.5), which is the crucial and difficult part in our proof. Taking (3.7) into it, we have

$$\begin{aligned}&\int_\Omega \rho_\epsilon w_\epsilon(\mathbf{V}-\mathbf{v}_\epsilon)\cdot\partial_z\mathbf{V}dxdz\\&\qquad=\int_\Omega[-\mathrm{div}_x(\rho_\epsilon\widetilde{\mathbf{v}}_\epsilon)+z\mathrm{div}_x(\rho_\epsilon\overline{\mathbf{v}}_\epsilon)]\partial_z\mathbf{V}\cdot(\mathbf{V}-\mathbf{v}_\epsilon)dxdz\\&\qquad=\int_\Omega(\rho_\epsilon\widetilde{\mathbf{v}}_\epsilon-z\rho_\epsilon\overline{\mathbf{v}}_\epsilon)\partial_z\nabla_x\mathbf{V}\cdot(\mathbf{V}-\mathbf{v}_\epsilon)dxdz\\&\qquad\quad+\int_\Omega(\rho_\epsilon\widetilde{\mathbf{v}}_\epsilon-z\rho_\epsilon\overline{\mathbf{v}}_\epsilon)\partial_z\mathbf{V}\cdot(\nabla_x\mathbf{V}-\nabla_x\mathbf{v}_\epsilon)dxdz.\end{aligned}\tag{4.7}$$

In the following, we will estimate the terms on the right hand side of (4.7). We need only consider the most complicated terms, the remaining terms can be completed by the similar method. Firstly, we deal with $\int_\Omega \rho_\epsilon \widetilde{\mathbf{v}}_\epsilon \partial_z \nabla_x \mathbf{V}\cdot(\mathbf{V}-\mathbf{v}_\epsilon)dxdz$ in the following,

$$
\begin{aligned}
&\int_\Omega \rho_\epsilon \widetilde{\mathbf{v}}_\epsilon \partial_z \nabla_x \mathbf{V}\cdot(\mathbf{V}-\mathbf{v}_\epsilon)dxdz\\
&\qquad = \int_\Omega \rho_\epsilon (\widetilde{\mathbf{v}}_\epsilon - \widetilde{\mathbf{V}})\partial_z \nabla_x \mathbf{U}\cdot(\mathbf{V}-\mathbf{v}_\epsilon)dxdz + \int_\Omega \rho_\epsilon \widetilde{\mathbf{V}}\partial_z \nabla_x \mathbf{V}\cdot(\mathbf{V}-\mathbf{v}_\epsilon)dxdz\\
&\qquad = J_1 + J_2,
\end{aligned}
$$

where $\widetilde{\mathbf{V}} = \int_0^z \mathbf{V}(x,s,t)ds$.

Similar to the above analysis, we decompose the term J_2 into three parts

$$
\begin{aligned}
J_2 &= \int_\Omega \rho_\epsilon \widetilde{\mathbf{V}}\partial_z \nabla_x \mathbf{V}\cdot(\mathbf{V}-\mathbf{v}_\epsilon)dxdz\\
&= \int_\Omega \chi_{\rho_\epsilon \le \frac{r}{2}} \rho_\epsilon \widetilde{\mathbf{V}}\partial_z \nabla_x \mathbf{V}\cdot(\mathbf{V}-\mathbf{v}_\epsilon)dxdz + \int_\Omega \chi_{\frac{r}{2}<\rho_\epsilon<r} \rho_\epsilon \widetilde{\mathbf{V}}\partial_z \nabla_x \mathbf{V}\cdot(\mathbf{V}-\mathbf{v}_\epsilon)dxdz\\
&\quad + \int_\Omega \chi_{\rho_\epsilon \ge r} \rho_\epsilon \widetilde{\mathbf{V}}\partial_z \nabla_x \mathbf{V}\cdot(\mathbf{V}-\mathbf{v}_\epsilon)dxdz\\
&\le \|\chi_{\rho_\epsilon\le\frac{r}{2}}1\|_{L^2(\Omega)}\|r\|_{L^\infty}\|\widetilde{\mathbf{V}}\partial_z\nabla_x\mathbf{V}\|_{L^3}\|\mathbf{V}-\mathbf{v}_\epsilon\|_{L^6(\Omega)}\\
&\quad + \|\chi_{\rho_\epsilon\ge r}\rho_\epsilon^{\frac{\gamma}{2}}\|_{L^2(\Omega)}\|\widetilde{\mathbf{V}}\partial_z\nabla_x\mathbf{V}\|_{L^3(\Omega)}\|\mathbf{V}-\mathbf{v}_\epsilon\|_{L^6(\Omega)}\\
&\quad + C\|\chi_{\frac{r}{2}<\rho_\epsilon<r}(\rho_\epsilon - r)\|_{L^2(\Omega)}\|\widetilde{\mathbf{V}}\partial_z\nabla_x\mathbf{V}\|_{L^3(\Omega)}\|\mathbf{V}-\mathbf{v}_\epsilon\|_{L^6(\Omega)}\\
&\le C\mathcal{E}(\rho_\epsilon,\mathbf{u}_\epsilon|r,\mathbf{U})(t) + \delta\|\nabla_x\mathbf{V}-\nabla_x\mathbf{v}_\epsilon\|^2_{L^2(\Omega)} + \delta\|\partial_z\mathbf{V}-\partial_z\mathbf{v}_\epsilon\|^2_{L^2(\Omega)}.
\end{aligned}
$$

On the other hand, by virtue of Cauchy inequality, it follows that

$$
\begin{aligned}
J_1 &= \int_\Omega \rho_\epsilon (\widetilde{\mathbf{v}}_\epsilon - \widetilde{\mathbf{V}})\partial_z \nabla_x \mathbf{V}\cdot(\mathbf{V}-\mathbf{v}_\epsilon)dxdz\\
&\le \|\partial_z\nabla_x\mathbf{V}\|_{L^\infty}\int_\Omega \rho_\epsilon|\widetilde{\mathbf{v}}_\epsilon - \widetilde{\mathbf{V}}|^2dxdz + \int_\Omega \rho_\epsilon|\mathbf{V}-\mathbf{v}_\epsilon|^2dxdz\\
&\le C\int_\Omega \rho_\epsilon|\int_0^z(\mathbf{u}_\epsilon(s)-\mathbf{U}(s))ds|^2dxdz + \mathcal{E}(\rho_\epsilon,\mathbf{u}_\epsilon|r,\mathbf{U})\\
&\le C\int_\Omega \rho_\epsilon\big(\int_0^1|\mathbf{V}-\mathbf{v}_\epsilon|^2ds\big)dxdz + \mathcal{E}(\rho_\epsilon,\mathbf{u}_\epsilon|r,\mathbf{U})\\
&\le C\int_0^1\int_\Omega \rho_\epsilon|\mathbf{V}-\mathbf{v}_\epsilon|^2dxdzds + \mathcal{E}(\rho_\epsilon,\mathbf{u}_\epsilon|r,\mathbf{U})\\
&\le C\int_\Omega \rho_\epsilon|\mathbf{V}-\mathbf{v}_\epsilon|^2dxdz + \mathcal{E}(\rho_\epsilon,\mathbf{u}_\epsilon|r,\mathbf{U})\\
&\le C\mathcal{E}(\rho_\epsilon,\mathbf{u}_\epsilon|r,\mathbf{U}).
\end{aligned}
\tag{4.8}
$$

Secondly, we will investigate another complicated nonlinear term $\int_\Omega \rho_\epsilon \widetilde{\mathbf{v}}_\epsilon \partial_z \mathbf{V}\cdot(\nabla_x\mathbf{V}-\nabla_x\mathbf{v}_\epsilon)dxdz$. It is straightforward to show that

$$
\int_\Omega \rho_\epsilon \widetilde{\mathbf{v}}_\epsilon \partial_z \mathbf{V}\cdot(\nabla_x\mathbf{V}-\nabla_x\mathbf{v}_\epsilon)dxdz
$$

$$
= \int_\Omega \chi_{\rho_\epsilon<r}\rho_\epsilon\widetilde{\mathbf{v}}_\epsilon\partial_z\mathbf{V}\cdot(\nabla_x\mathbf{V}-\nabla_x\mathbf{v}_\epsilon)dxdz + \int_\Omega \chi_{\rho_\epsilon\geq r}\rho_\epsilon\widetilde{\mathbf{v}}_\epsilon\partial_z\mathbf{V}\cdot(\nabla_x\mathbf{V}-\nabla_x\mathbf{v}_\epsilon)dxdz, \tag{4.9}
$$

where the first term on the right side of (4.9) is split into two parts as

$$
\begin{aligned}
&\int_\Omega \chi_{\rho_\epsilon<r}\rho_\epsilon\widetilde{\mathbf{v}}_\epsilon\partial_z\mathbf{V}\cdot(\nabla_x\mathbf{V}-\nabla_x\mathbf{v}_\epsilon)dxdz\\
&\quad= \int_\Omega \chi_{\rho_\epsilon<r}\rho_\epsilon(\widetilde{\mathbf{v}}_\epsilon-\widetilde{\mathbf{V}})\partial_z\mathbf{V}\cdot(\nabla_x\mathbf{V}-\nabla_x\mathbf{v}_\epsilon)dxdz\\
&\qquad+ \int_\Omega \chi_{\rho_\epsilon<r}\rho_\epsilon\widetilde{\mathbf{V}}\partial_z\mathbf{V}\cdot(\nabla_x\mathbf{V}-\nabla_x\mathbf{v}_\epsilon)dxdz\\
&\quad= \int_\Omega \chi_{\rho_\epsilon<r}\rho_\epsilon(\widetilde{\mathbf{v}}_\epsilon-\widetilde{\mathbf{V}})\partial_z\mathbf{V}\cdot(\nabla_x\mathbf{V}-\nabla_x\mathbf{v}_\epsilon)dxdz\\
&\qquad+ \int_\Omega \chi_{\frac{r}{2}<\rho_\epsilon<r}\rho_\epsilon\widetilde{\mathbf{V}}\partial_z\mathbf{V}\cdot(\nabla_x\mathbf{V}-\nabla_x\mathbf{v}_\epsilon)dxdz\\
&\qquad+ \int_\Omega \chi_{\rho_\epsilon\leq\frac{r}{2}}\rho_\epsilon\widetilde{\mathbf{V}}\partial_z\mathbf{V}\cdot(\nabla_x\mathbf{V}-\nabla_x\mathbf{v}_\epsilon)dxdz\\
&\quad\leq \|\chi_{\rho_\epsilon<r}\rho_\epsilon^{\frac{1}{2}}\|_{L^\infty(\Omega)}\|\sqrt{\rho_\epsilon}(\widetilde{\mathbf{v}}_\epsilon-\widetilde{\mathbf{V}})\|_{L^2(\Omega)}\|\partial_z\mathbf{V}\|_{L^\infty(\Omega)}\|\nabla_x\mathbf{V}-\nabla_x\mathbf{v}_\epsilon\|_{L^2(\Omega)}\\
&\qquad+ \|\chi_{\frac{r}{2}<\rho_\epsilon<r}\rho_\epsilon\|_{L^2(\Omega)}\|\widetilde{\mathbf{V}}\partial_z\mathbf{V}\|_{L^\infty(\Omega)}\|\nabla_x\mathbf{V}-\nabla_x\mathbf{v}_\epsilon\|_{L^2(\Omega)}\\
&\qquad+ \|\chi_{\rho_\epsilon\leq\frac{r}{2}}1\|_{L^2(\Omega)}\|r\|_{L^\infty(\Omega)}\|\widetilde{\mathbf{V}}\partial_z\mathbf{V}\|_{L^\infty(\Omega)}\|\nabla_x\mathbf{V}-\nabla_x\mathbf{v}_\epsilon\|_{L^2(\Omega)}\\
&\quad\leq C\mathcal{E}(\rho_\epsilon,\mathbf{u}_\epsilon|r,\mathbf{U})(t)+\delta\|\nabla_x\mathbf{V}-\nabla_x\mathbf{v}_\epsilon\|^2_{L^2(\Omega)}.
\end{aligned}
$$

The decomposition of remainder of (4.9) is identical to the above as:

$$
\begin{aligned}
&\int_\Omega \chi_{\rho_\epsilon\geq r}\rho_\epsilon\widetilde{\mathbf{v}}_\epsilon\partial_z\mathbf{V}\cdot(\nabla_x\mathbf{V}-\nabla_x\mathbf{v}_\epsilon)dxdz\\
&= \int_\Omega \chi_{\rho_\epsilon\geq r}\rho_\epsilon(\widetilde{\mathbf{v}}_\epsilon-\widetilde{\mathbf{V}})\partial_z\mathbf{V}\cdot(\nabla_x\mathbf{V}-\nabla_x\mathbf{v}_\epsilon)dxdz + \int_\Omega \chi_{\rho_\epsilon\geq r}\rho_\epsilon\widetilde{\mathbf{V}}\partial_z\mathbf{V}\cdot(\nabla_x\mathbf{V}-\nabla_x\mathbf{v}_\epsilon)dxdz\\
&= K_1+K_2,
\end{aligned}\tag{4.10}
$$

where

$$
\begin{aligned}
K_2 &\leq \int_\Omega \chi_{\rho_\epsilon\geq r}\rho_\epsilon^{\frac{\gamma}{2}}\widetilde{\mathbf{V}}\partial_z\mathbf{V}\cdot(\nabla_x\mathbf{V}-\nabla_x\mathbf{v}_\epsilon)dxdz\\
&\leq \|\chi_{\rho_\epsilon\geq r}\rho_\epsilon^{\frac{\gamma}{2}}\|_{L^2(\Omega)}\|\widetilde{\mathbf{V}}\partial_z\mathbf{V}\|_{L^\infty(\Omega)}\|\nabla_x\mathbf{V}-\nabla_x\mathbf{v}_\epsilon\|_{L^2(\Omega)}\\
&\leq C\|\chi_{\rho_\epsilon\geq r}\rho_\epsilon^{\frac{\gamma}{2}}\|^2_{L^2(\Omega)}+\delta\|\nabla_x\mathbf{V}-\nabla_x\mathbf{v}_\epsilon\|^2_{L^2(\Omega)}\\
&\leq C\mathcal{E}(\rho_\epsilon,\mathbf{u}_\epsilon|r,\mathbf{U})(t)+\delta\|\nabla_x\mathbf{V}-\nabla_x\mathbf{v}_\epsilon\|^2_{L^2(\Omega)}.
\end{aligned}\tag{4.11}
$$

It remains to estimate K_1. Due to Hölder inequality, it follows that

$$
K_1 \leq \|\chi_{\rho_\epsilon\geq r}\rho_\epsilon\|_{L^4(\Omega)}\|\chi_{\rho_\epsilon\geq r}(\widetilde{\mathbf{v}}_\epsilon-\widetilde{\mathbf{V}})\|_{L^4(\Omega)}\|\partial_z\mathbf{V}\|_{L^\infty(\Omega)}\|\nabla_x\mathbf{V}-\nabla_x\mathbf{v}_\epsilon\|_{L^2(\Omega)}
$$

$$
\begin{aligned}
&\le C\|\chi_{\rho_\epsilon\ge r}\rho_\epsilon\|_{L^4(\Omega)}^2\|\chi_{\rho_\epsilon\ge r}(\widetilde{\mathbf{v}}_\epsilon-\widetilde{\mathbf{V}})\|_{L^4(\Omega)}^2+\delta\|\nabla_x\mathbf{V}-\nabla_x\mathbf{v}_\epsilon\|_{L^2(\Omega)}^2\\
&\le C\|\chi_{\rho_\epsilon\ge r}\rho_\epsilon\|_{L^4(\Omega)}^2\|\chi_{\rho_\epsilon\ge r}(\widetilde{\mathbf{v}}_\epsilon-\widetilde{\mathbf{V}})\|_{L^3(\Omega)}\|\chi_{\rho_\epsilon\ge r}(\nabla\widetilde{\mathbf{v}}_\epsilon-\nabla\widetilde{\mathbf{V}})\|_{L^2(\Omega)}+\delta\|\nabla_x\mathbf{v}_\epsilon-\nabla_x\mathbf{V}\|_{L^2(\Omega)}^2\\
&\le C\|\chi_{\rho_\epsilon\ge r}\rho_\epsilon\|_{L^4(\Omega)}^4\|\chi_{\rho_\epsilon\ge r}(\widetilde{\mathbf{v}}_\epsilon-\widetilde{\mathbf{V}})\|_{L^3(\Omega)}^2+\delta\|\nabla_x\widetilde{\mathbf{v}}_\epsilon-\nabla_x\widetilde{\mathbf{V}}\|_{L^2(\Omega)}^2\\
&\quad+\delta\|\partial_z\widetilde{\mathbf{v}}_\epsilon-\partial_z\widetilde{\mathbf{V}}\|_{L^2(\Omega)}^2+\delta\|\nabla_x\mathbf{v}_\epsilon-\nabla_x\mathbf{V}\|_{L^2(\Omega)}^2\\
&\le \|\chi_{\rho_\epsilon\ge r}\rho_\epsilon\|_{L^4(\Omega)}^4\|\chi_{\rho_\epsilon\ge r}(\widetilde{\mathbf{v}}_\epsilon-\widetilde{\mathbf{V}})\|_{L^2(\Omega)}\|\chi_{\rho_\epsilon\ge r}(\widetilde{\mathbf{v}}_\epsilon-\widetilde{\mathbf{V}})\|_{H^1(\Omega)}+\delta\|\nabla_x\widetilde{\mathbf{v}}_\epsilon-\nabla_x\widetilde{\mathbf{V}}\|_{L^2(\Omega)}^2\\
&\quad+\delta\|\partial_z\widetilde{\mathbf{v}}_\epsilon-\partial_z\widetilde{\mathbf{V}}\|_{L^2(\Omega)}^2+\delta\|\nabla_x\mathbf{v}_\epsilon-\nabla_x\mathbf{V}\|_{L^2(\Omega)}^2\\
&\le \|\chi_{\rho_\epsilon\ge r}\rho_\epsilon\|_{L^4(\Omega)}^8\|\chi_{\rho_\epsilon\ge r}(\widetilde{\mathbf{v}}_\epsilon-\widetilde{\mathbf{V}})\|_{L^2(\Omega)}^2+\delta\|\chi_{\rho_\epsilon\ge r}(\widetilde{\mathbf{v}}_\epsilon-\widetilde{\mathbf{V}})\|_{L^2(\Omega)}^2+\delta\|\nabla_x\widetilde{\mathbf{v}}_\epsilon-\nabla_x\widetilde{\mathbf{V}}\|_{L^2(\Omega)}^2\\
&\quad+\delta\|\partial_z\widetilde{\mathbf{v}}_\epsilon-\partial_z\widetilde{\mathbf{V}}\|_{L^2(\Omega)}^2+\delta\|\nabla_x\mathbf{v}_\epsilon-\nabla_x\mathbf{V}\|_{L^2(\Omega)}^2
\end{aligned}
$$

where we have used the Lemma 2.2

$$
\|f\|_{L^4}\le\|\nabla f\|_{L^2}^{\frac12}\|f\|_{L^3}^{\frac12}\text{ and }\|f\|_{L^3}\le\|f\|_{L^2}^{\frac12}\|f\|_{H^1}^{\frac12}.
$$

Recalling (3.14) and (4.8), we have

$$
\|\chi_{\rho_\epsilon\ge r}\rho_\epsilon\|_{L^4(\Omega)}^8=\Big(\int_{\rho_\epsilon\ge r}\rho_\epsilon^4dxdz\Big)^2\le C\Big(\int_\Omega\rho_\epsilon^\gamma dxdz\Big)^{\frac8\gamma}\le\mathcal{E}(\rho_\epsilon,\mathbf{u}_\epsilon|r,\mathbf{U})^{\frac8\gamma}(t),
$$

and

$$
\begin{aligned}
\|\chi_{\rho_\epsilon\ge r}(\widetilde{\mathbf{v}}_\epsilon-\widetilde{\mathbf{V}})\|_{L^2(\Omega)}^2&=\int_{\rho_\epsilon\ge r}|\widetilde{\mathbf{v}}_\epsilon-\widetilde{\mathbf{V}}|^2dxdz=\int_{\rho_\epsilon\ge r}\frac{1}{\rho_\epsilon}\rho_\epsilon|\widetilde{\mathbf{v}}_\epsilon-\widetilde{\mathbf{V}}|^2dxdz\\
&\le\frac{1}{\|r\|_{\infty(\Omega)}}\mathcal{E}(\rho_\epsilon,\mathbf{u}_\epsilon|r,\mathbf{U})(t).
\end{aligned}
$$

An argument similar to the one used in (4.8) yields

$$
\|\nabla_x\widetilde{\mathbf{v}}_\epsilon-\nabla_x\widetilde{\mathbf{V}}\|_{L^2(\Omega)}^2\le\|\nabla_x\mathbf{v}_\epsilon-\nabla_x\mathbf{V}\|_{L^2(\Omega)}^2,\ \ \|\partial_z\widetilde{\mathbf{v}}_\epsilon-\partial_z\widetilde{\mathbf{V}}\|_{L^2(\Omega)}^2\le\|\partial_z\mathbf{v}_\epsilon-\partial_z\mathbf{V}\|_{L^2(\Omega)}^2.
$$

Combining the above estimates, we arrive at the conclusion that

$$
\int_0^\tau K_1dt\le C\int_0^\tau h(t)\mathcal{E}(\rho_\epsilon,\mathbf{u}_\epsilon|r,\mathbf{U})(t)dt+\delta\int_0^\tau\|\nabla_x\mathbf{v}_\epsilon-\nabla_x\mathbf{V}\|_{L^2(\Omega)}^2+\|\partial_z\mathbf{v}_\epsilon-\partial_z\mathbf{V}\|_{L^2(\Omega)}^2dt,
$$

where $h(t)\in L^1(0,T)$.

The estimate of remainder in (4.7) can be completed by the analogous method. Therefore, we can summarize what we have proved as the following

$$
\begin{aligned}
&\mathcal{E}(\rho_\epsilon,\mathbf{u}_\epsilon|r,\mathbf{U})|_{t=0}^{t=\tau}+\int_0^\tau\int_\Omega\big(\nabla\mathbf{v}_\epsilon\cdot(\nabla\mathbf{v}_\epsilon-\nabla\mathbf{V})+\epsilon^2|\nabla w_\epsilon|^2\big)dxdzdt\\
&\le C\int_0^\tau h(t)\mathcal{E}(\rho_\epsilon,\mathbf{u}_\epsilon|r,\mathbf{U})dt+\delta\int_0^\tau\|\nabla_x\mathbf{v}_\epsilon-\nabla_x\mathbf{V}\|_{L^2(\Omega)}^2+\|\partial_z\mathbf{v}_\epsilon-\partial_z\mathbf{V}\|_{L^2(\Omega)}^2dt\\
&\quad+\int_0^\tau\int_\Omega\frac{\rho_\epsilon}{r}(\mathbf{V}-\mathbf{v}_\epsilon)(\Delta_x\mathbf{V}+\partial_{zz}\mathbf{V})dxdzdt-\int_0^\tau\int_\Omega\frac{\rho_\epsilon}{r}(\mathbf{V}-\mathbf{v}_\epsilon)\nabla_xp(r)dxdzdt
\end{aligned}
$$

$$-\int_0^\tau\int_\Omega P''(r)((\rho_\epsilon-r)\partial_t r+\rho_\epsilon\mathbf{v}_\epsilon\nabla_x r)dxdzdt-\int_0^\tau\int_\Omega p(\rho_\epsilon)\mathrm{div}_x\mathbf{V}dxdzdt+o(\epsilon^2).$$

Then we deduce that

$$\begin{aligned}
&\mathcal{E}(\rho_\epsilon,\mathbf{u}_\epsilon|r,\mathbf{U})|_{t=0}^{t=\tau}+\int_0^\tau\int_\Omega\big((\nabla_x\mathbf{v}_\epsilon-\nabla_x\mathbf{V}):(\nabla_x\mathbf{v}_\epsilon-\nabla_x\mathbf{V})+|\partial_z\mathbf{v}_\epsilon-\partial_z\mathbf{V}_\epsilon|^2+\epsilon^2|\nabla w_\epsilon|^2\big)dxdzdt\\
&\quad\le C\int_0^\tau h(t)\mathcal{E}(\rho_\epsilon,\mathbf{u}_\epsilon|r,\mathbf{U})dt+\delta\int_0^\tau\|\nabla_x\mathbf{v}_\epsilon-\nabla_x\mathbf{V}\|_{L^2(\Omega)}^2+\|\partial_z\mathbf{v}_\epsilon-\partial_z\mathbf{V}\|_{L^2(\Omega)}^2dt\\
&\quad+\int_0^\tau\int_\Omega(\frac{\rho_\epsilon}{r}-1)(\mathbf{V}-\mathbf{v}_\epsilon)(\Delta_x\mathbf{V}+\partial_{zz}\mathbf{V})dxdzdt-\int_0^\tau\int_\Omega\frac{\rho_\epsilon}{r}(\mathbf{V}-\mathbf{v}_\epsilon)\nabla_x p(r)dxdzdt\\
&\quad-\int_0^\tau\int_\Omega P''(r)((\rho_\epsilon-r)\partial_t r+\rho_\epsilon\mathbf{v}_\epsilon\nabla_x r)dxdzdt-\int_0^\tau\int_\Omega p(\rho_\epsilon)\mathrm{div}_x\mathbf{V}dxdzdt+o(\epsilon^2).
\end{aligned}\tag{4.12}$$

4.4 Step 3

We are now in a position to estimate the remaining terms in the relative energy inequality (4.12). It is clear to check that

$$\begin{aligned}
&-\int_0^\tau\int_\Omega\frac{\rho_\epsilon}{r}(\mathbf{V}-\mathbf{v}_\epsilon)\nabla_x p(r)+p(\rho_\epsilon)\mathrm{div}_x\mathbf{V}+P''(r)((\rho_\epsilon-r)\partial_t r+\rho_\epsilon\mathbf{v}_\epsilon\nabla_x r)dxdzdt\\
&\qquad=-\int_0^\tau\int_\Omega(\rho_\epsilon-r)P''(r)\partial_t r+P''(r)\rho_\epsilon\mathbf{v}_\epsilon\cdot\nabla_x r+\rho_\epsilon P''(r)(\mathbf{V}-\mathbf{v}_\epsilon)\cdot\nabla_x r+p(\rho_\epsilon)\mathrm{div}_x\mathbf{V}dxdzdt\\
&\qquad=-\int_0^\tau\int_\Omega(\rho_\epsilon-r)P''(r)\partial_t r+P''(r)\rho_\epsilon\mathbf{V}\cdot\nabla_x r+p(\rho_\epsilon)\mathrm{div}_x\mathbf{V}dxdzdt\\
&\qquad=-\int_0^\tau\int_\Omega\rho_\epsilon P''(r)(\partial_t r+\mathbf{V}\cdot\nabla_x r)-rP''(r)\partial_t r+p(\rho_\epsilon)\mathrm{div}_x\mathbf{V}dxdzdt\\
&\qquad=-\int_0^\tau\int_\Omega\rho_\epsilon P''(r)(-r\mathrm{div}_x\mathbf{V}-r\partial_z W)-rP''(r)\partial_t r+p(\rho_\epsilon)\mathrm{div}_x\mathbf{V}dxdzdt\\
&\qquad=-\int_0^\tau\int_\Omega\mathrm{div}_x\mathbf{V}\big(p(\rho_\epsilon)-p'(r)(\rho_\epsilon-r)-p(r)\big)dxdzdt+\int_0^\tau\int_\Omega p'(r)(\rho_\epsilon-r)\partial_z W dxdzdt,
\end{aligned}\tag{4.13}$$

where we have used the fact that $\partial_t r+\mathrm{div}_x\mathbf{V}r+\mathbf{U}\cdot\nabla_x r+r\partial_z W=0$.

Using the analogous argument as in [46] Section 2.2.5, we can easily carry out the following estimate:

$$|\int_0^\tau\int_\Omega\mathrm{div}_x\mathbf{V}\big(p(\rho_\epsilon)-p'(r)(\rho_\epsilon-r)-p(r)\big)dxdzdt|\le C\int_0^\tau h(t)\mathcal{E}(\rho_\epsilon,\mathbf{u}_\epsilon|r,\mathbf{U})dt.\tag{4.14}$$

According to the periodic boundary condition, it follows that

$$\int_0^\tau\int_\Omega p'(r)(\rho_\epsilon-r)\partial_z W dxdzdt=\int_0^\tau dt\int_{\mathbb{T}^2}(\int_0^1\partial_z W dz)p'(r)(\rho_\epsilon-r)dx=0.\tag{4.15}$$

Furthermore, an argument similar to the one used in [37] Section 6.3 shows that

$$\int_\Omega (\frac{\rho_\epsilon}{r} - 1)(\mathbf{V} - \mathbf{v}_\epsilon)(\Delta_x \mathbf{V} + \partial_{zz}\mathbf{V})dxdz \\ \leq C\mathcal{E}(\rho_\epsilon, \mathbf{u}_\epsilon | r, \mathbf{U}) + \delta\|\nabla_x \mathbf{v}_\epsilon - \nabla_x \mathbf{V}\|_{L^2}^2 + \delta\|\partial_z \mathbf{v}_\epsilon - \partial_z \mathbf{V}\|_{L^2}^2. \tag{4.16}$$

Therefore, putting (4.12) − (4.16) together, we have

$$\mathcal{E}(\rho_\epsilon, \mathbf{u}_\epsilon | r, \mathbf{U})(\tau) \leq C\int_0^\tau h(t)\mathcal{E}(\rho_\epsilon, \mathbf{u}_\epsilon | r, \mathbf{U})(t)dt + o(\epsilon^2). \tag{4.17}$$

Then applying the Gronwall's inequality, we finish the proof of Theorem 3.1.

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