On the long–time behavior of compressible fluid flows: Ergodic hypothesis and beyond

Eduard Feireisl

based on joint work with F. Fanelli (Lyon I), M. Hofmanová (TU Bielefeld)

Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague
Technische Universität Berlin

Thematic Einstein Semester, WIAS Berlin, October 26 – 30, 2020

EINSTEIN Foundation.de

Einstein Stiftung Berlin
Einstein Foundation Berlin

GAČR
CZECH SCIENCE FOUNDATION

Motto

Energetically closed systems

Clausius:

The energy of the world is constant; its entropy tends to a maximum

Energetically open systems – ergodic hypothesis

Time averages along trajectories of the flow converge, for large enough times, to an ensemble average given by a certain probability measure.

Dynamical system

$$\mathbf{U}(t,\cdot) : [0,\infty) \times X \to X$$

- **Closed system:** $\mathbf{U}(t, X_0) \to \mathbf{U}_\infty$ equilibrium solution as $t \to \infty$
- **Open system:** $\displaystyle \frac{1}{T}\int_0^T F(\mathbf{U}(t,X_0))\mathrm{d}t \to \int_X F(X)\ \mathrm{d}\mu,\ \ T \to \infty$

μ a.s. in X_0

Principal problems

Low regularity of global in time solutions

Global in time solutions necessary. For many problems in fluid dynamics – Navier–Stokes or Euler system – only weak solutions available

Lack of uniqueness

Solutions do not, or at least are not known to, depend uniquely on the initial data. Spaces of trajectories: Sell, Nečas, Temam and others

Propagation of oscillations

Realistic systems are partly hyperbolic: propagation of oscillations "from the past", singularities

Abstract setting

Space of entire trajectories

$$\mathcal{T} = C_{\rm loc}(R; X), \ t \in (-\infty, \infty)$$

ω**–limit set**

$$\omega[\mathbf{U}(\cdot, X_0)] \subset \mathcal{T}$$

$$\omega[\mathbf{U}(\cdot, X_0)] = \left\{ \mathbf{V} \in \mathcal{T} \ \middle| \ \mathbf{U}(\cdot + t_n, X_0) \to \mathbf{V} \text{ in } \mathcal{T} \right\}$$

Stationary (statistical) solution

$\mathbf{V} : R \to X$ stationary process (law is time shift invariant)

$\mathbf{V} \in \omega[\mathbf{U}(\cdot, X_0)]$ a.s.

$\mathbf{V}$ solves the associated evolutionary equation a.s.

Strong and weak ergodic hypothesis

Krylov – Bogolyubov construction

$$T \mapsto \frac{1}{T}\int_0^T \delta_{\mathbf{U}(t,X_0)}\mathrm{d}t \text{ – a family of probability measures on } \mathcal{T}$$

$$\text{tightness in } \mathcal{T} \ \Rightarrow \ T_n \mapsto \frac{1}{T_n}\int_0^{T_n} \delta_{\mathbf{U}(t,X_0)}\mathrm{d}t \to \mu \in \mathcal{P}[\mathcal{T}]$$

$[\mathcal{T}, \mu]$ stationary statistical solution

$$\textbf{Ergodic hypothesis} \ \Leftrightarrow \ \mu \text{ is unique} \ \Rightarrow \ T \mapsto \frac{1}{T}\int_0^T \delta_{\mathbf{U}(t,X_0)}\mathrm{d}t \to \mu$$

unique $\approx$ unique on $\omega[\mathbf{U}(\cdot, X_0)]$

Weak ergodic hypothesis

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \delta_{\mathbf{U}(t,X_0)}\mathrm{d}t = \mu \text{ exists in the narrow sense in } \mathcal{P}[\mathcal{T}]$$

$[\mathcal{T}, \mu]$ stationary statistical solution

Barotropic Navier–Stokes system

Field equations

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \varrho \mathbf{g}$$

Constitutive equations

- barotropic (isentropic) pressure–density EOS $p = p(\varrho)$ $(p = a\varrho^\gamma)$
- Newton's rheological law

$$\mathbb{S} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0, \ \eta \geq 0$$

- Gravitational external force

$$\mathbf{g} = \nabla_x F, \ F = F(x)$$

Energy

$$E(\varrho, \mathbf{m}) \equiv \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - \varrho F, \ P'(\varrho)\varrho - P(\varrho) = p(\varrho), \ \mathbf{m} = \varrho \mathbf{u}$$

Energetically insulated system

Conservative boundary conditions

$$\Omega \subset R^d \text{ bounded (sufficiently regular) domain}$$

- **impermeability** $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$
- **no–slip** $[\mathbf{u}]_{\rm tan}|_{\partial\Omega} = 0$

Long–time behavior – Clausius scenario

- Total mass conserved

$$\int_\Omega \varrho(t, \cdot) \ {\rm d}x = M_0$$

- Total energy – Lyapunov function

$$\frac{\rm d}{{\rm d}t} \int_\Omega E(\varrho, \mathbf{m}) \ {\rm d}x + \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ {\rm d}x = (\leq) 0, \ \int_\Omega E(\varrho, \mathbf{m}) \ {\rm d}x \searrow \mathcal{E}_\infty$$

- Stationary solution

$$\mathbf{m}_\infty = 0, \ \nabla_x p(\varrho_\infty) = \varrho_\infty \nabla_x F, \ \int_\Omega \varrho_\infty \ {\rm d}x = M_0, \int_\Omega E(\varrho_\infty, 0) \ {\rm d}x = \mathcal{E}_\infty$$

Energetically open system

In/out flow boundary conditions

$$\mathbf{u} = \mathbf{u}_b \text{ on } \partial\Omega$$

$$\Gamma_{\rm in} = \left\{ x \in \partial\Omega \ \Big| \ \mathbf{u}_b(x) \cdot \mathbf{n}(x) < 0 \right\}, \ \Gamma_{\rm out} = \left\{ x \in \partial\Omega \ \Big| \ \mathbf{u}_b(x) \cdot \mathbf{n}(x) \geq 0 \right\}$$

Density (pressure) on the inflow boundary

$$\varrho = \varrho_b \text{ on } \Gamma_{\rm in}$$

Energy balance

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{2} \varrho |\mathbf{u} - \mathbf{u}_b|^2 + P(\varrho) \ \mathrm{d}x + \int_\Omega \mathbb{S} : \nabla_x \mathbf{u} \ \mathrm{d}x\mathrm{d}t \\
&\quad + \int_{\Gamma_{\rm in}} P(\varrho_b) \mathbf{u}_b \cdot \mathbf{n} \ \mathrm{d}S_x + \int_{\Gamma_{\rm out}} P(\varrho) \mathbf{u}_b \cdot \mathbf{n} \ \mathrm{d}S_x \\
&\quad = (\leq) - \int_\Omega \left[\varrho \mathbf{u} \otimes \mathbf{u} + p(\varrho) \mathbb{I} \right] : \nabla_x \mathbf{u}_b \ \mathrm{d}x + \frac{1}{2} \int_\Omega \varrho \mathbf{u} \cdot \nabla_x |\mathbf{u}_b|^2 \ \mathrm{d}x\mathrm{d}t \\
&\quad + \int_\Omega \mathbb{S} : \nabla_x \mathbf{u}_b \ \mathrm{d}x\mathrm{d}t + \int_\Omega \varrho \nabla_x F \cdot (\mathbf{u} - \mathbf{u}_b) \ \mathrm{d}x
\end{aligned}$$

Global bounded trajectories

Global in time weak solutions

$\mathbf{U} = [\varrho, \mathbf{m} = \varrho \mathbf{u}]$ – weak solution of the Navier–Stokes system satisfying energy inequality and defined for $t > T_0$

Bounded energy

$$\limsup_{t \to \infty} \int_\Omega E(\varrho, \mathbf{m}) \ \mathrm{d}x \leq \mathcal{E}_\infty$$

Available results

- **Existence:** T. Chang, B. J. Jin, and A. Novotný, *SIAM J. Math. Anal.*, **51**(2):1238–1278, 2019
  H. J. Choe, A. Novotný, and M. Yang *J. Differential Equations*, **266**(6):3066–3099, 2019
- **Globally bounded solutions:** F. Fanelli, E. F., and M. Hofmanová **arxiv preprint No. 2006.02278**, 2020
  J. Březina, E. F., and A. Novotný, Communications in PDE's 2020

ω – limit sets

$$p \approx a\varrho^{\gamma},\ \gamma > \frac{d}{2}$$

Trajectory space

$$X = \left\{ \varrho, \mathbf{m} \ \Big| \ \varrho(t,\cdot) \in L^{\gamma}(\Omega),\ \mathbf{m}(t,\cdot) \in L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d) \hookrightarrow W^{-k,2} \right\}$$

$$\mathcal{T} = C_{\rm loc}(R; L^{\gamma} \times W^{-k,2})$$

Fundamental result on compactness [Fanelli, EF, Hofmanová, 2020]

The ω–limit set $\omega[\varrho, \mathbf{m}]$ of each global in time trajectory with globally bounded energy is:

- $\boxed{non - empty}$
- $\boxed{compact}$ in $\mathcal{T}$
- time shift invariant
- consists of entire (defined for all $t \in R$) weak solutions of the Navier–Stokes system

Propagation of oscillations

Equation of continuity

$$\partial_t \varrho + \mathbf{u} \cdot \nabla_x \varrho = -\varrho \mathrm{div}_x \mathbf{u}$$

Renormalized equation of continuity

$$\partial_t b(\varrho) + \mathrm{div}_x (b(\varrho)\mathbf{u}) + \Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u} = 0$$

Weak convergence

$$b(\varrho_n) \to \overline{b(\varrho)} \text{ weakly in } L^1$$

$$\partial_t \left[\overline{b(\varrho)} - b(\varrho) \right] + \mathrm{div}_x \left(\overline{b(\varrho)\mathbf{u}} - b(\varrho)\mathbf{u} \right)$$

$$= \Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u} - \overline{\Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u}}$$

$$\left[\overline{b(\varrho)} - b(\varrho) \right] (0, \cdot) = 0 \text{ is needed!}$$

Vanishing oscillation defect

Compactness of densities

$$\varrho_n \equiv \varrho(\cdot + T_n) \to \varrho \text{ in } C_{\mathrm{weak,loc}}(R; L^\gamma(\Omega))$$

$$\varrho_n \log(\varrho_n) \to \overline{\varrho \log(\varrho)} \geq \varrho \log(\varrho)$$

$$D(t) \equiv \int_\Omega \overline{\varrho \log(\varrho)} - \varrho \log(\varrho) \ \mathrm{d}x \geq 0$$

Problem: Unlike in the *existence* proof, there is no information on oscillations of "initial data"!

Crucial differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} D + \Psi(D) \leq 0, \ 0 \leq D \leq \overline{D}, \ t \in R$$

$$\Psi \in C(R), \ \Psi(0) = 0, \ \Psi(Z)Z > 0 \text{ for } Z \neq 0$$

$$\Rightarrow$$

$$D \equiv 0$$

Statistical stationary solutions

Application of Krylov – Bogolyubov method

$$\frac{1}{T_n}\int_0^{T_n} \delta_{\varrho(t,\cdot),\mathbf{m}(t,\cdot)}\ \mathrm{d}t \to \mu \in \mathcal{P}[\mathcal{T}] \text{ narrowly}$$

$[\mathcal{T},\mu]$ (canonical representation) – statististical stationary solution

$\mu(t)|_X$ (marginal) independent of $t \in R$

Application of Birkhoff – Khinchin ergodic theorem

$$\frac{1}{T}\int_0^{T} F(\varrho(t,\cdot),\mathbf{m}(t,\cdot))\mathrm{d}t \to \overline{F} \text{ as } T \to \infty$$

F bounded Borel measurable on X

for $\mu-$a.a. $(\varrho,\mathbf{m}) \in \omega$

Related results for incompressible Navier–Stokes system with conservative boundary conditions

F.Flandoli and D. Gatarek, F.Flandoli and M.Romito (stochastic forcing), P. Constantin and I. Procaccia, C. Foiaș, O. Manley, R. Rosa, and R. Temam, M. Vishik and A. Fursikov etc (deterministic forcing)

Back to ergodic hypothesis – conclusion

Ergodicity

$$\mu \text{ ergodic } \Leftrightarrow \mathcal{B} \subset \omega[\varrho, \mathbf{m}] \text{ shift invariant } \Rightarrow \mu[\mathcal{B}] = 1 \text{ or } \mu[\mathcal{B}] = 0$$

$$\mu \in \text{conv}\Big\{ \text{ergodic measures on } \omega[\varrho, \mathbf{m}] \Big\}$$

State of the art for compressible Navier–Stokes system

- Each bounded energy global trajectory generates a stationary statistical solution – a shift invariant measure μ – sitting on its $\omega-$limit set $\omega[\varrho, \mathbf{m}]$
- The weak ergodic hypothesis (the existence of limits of ergodic averages for any Borel measurable F) holds on $\omega[\varrho, \mathbf{m}]$ $\mu-$a.s.
- The (strong) ergodic hypothesis definitely holds for enegetically isolated systems and a class of potential forces F, where all solutions tend to equilibrium