Indeterminacy and Stability in a Modified Romer Model

Sergey Slobodyan*
CERGE-EI, Charles University
Politických vězňů 7, Praha 1, 111 21

Abstract

This paper considers the well known Romer model of endogenous technological change and its extension where different intermediate capital goods are complementary, introduced in (Benhabib, Perli, and Xie 1994). They have shown that this modification allows indeterminate steady state for relatively mild degrees of the complementarity. The authors were able to derive analytically sufficient conditions for the indeterminacy and to find specific parameter values producing the indeterminate steady state.

For the modified Romer model of (Benhabib, Perli, and Xie 1994), I derive necessary and sufficient conditions for the steady state to be interior and strictly positive. I show that Hopf bifurcation to the absolutely stable steady state is impossible and the steady state is determinate if the model parameter values belong to a certain set. Considering a simplified version of the model, I calculate necessary conditions for a Hopf bifurcation in one special case and show that it is impossible in another. Using numerical algorithm for multigoal optimization, I obtain several sets of parameter values leading to the loss of stability of the indeterminate steady state through Hopf bifurcation.

Abstrakt

Tato práce vychází z Romerova modelu endogenní technologické změny a jeho modifikace (Benhabib, Perli a Xie, 1994), ve které jsou různé kapitálové statky komplementy. Tato modifikace umožňuje nedeterminovaný stacionární stav v případe relativně malé míry komplementarity mezi těmito statky. Autoři analyticky odvodili postačující podmínky pro nedeterminovanost a našli hodnotu parametru, která vede k nedeterminovanému stacionárnímu stavu.

Pro tento model tato práce odvozuje nutné a postačující podmínky, aby stacionární stav byl interiorní a striktně positivní. Zároveň ukazuje, že Hopfova bifurkace v absolutním stacionárním stavu není možná pro jisté hodnoty parametru modelu a stacionární stav je tak determinovaný. Pro jednoduchou verzi modelu je vypočítaná nutná podmínka Hopfovy bifurkace v jednom specifickém případě a pak je ukázáno, že není možná ani v jiném případě. Numerickými simulacemi se vypočítá několik hodnot parametrů, které vedou k nestabilitě nedeterminovaného stacionárního stavu prostřednictvím Hopfovy bifurkace.

JEL Classification: E32, O41

Keywords: indeterminacy, stability, Romer model

**Sergey.Slobodyan@cerge.cuni.cz. (+420 2) 24 005 211.* *This paper is based on the first essay of my Ph.D. thesis completed at Washington University in St. Louis, USA, and was partially written during a research visit to the Federal Reserve Bank in St. Louis. Financial support from both institutions is gratefully acknowledged.*

1 Introduction

It has been known for almost two decades that in dynamical general equilibrium models equilibria could be indeterminate. The usual definition of a local indeterminacy of some equilibrium dynamics (for example, steady state or limit cycle) is the existence of a non-stationary continuum of perfect foresight eqilibria around the steady state (limit cycle) asymptotically converging to it (Shigoka 1994). Global indeterminacy is defined as the existence of different perfect foresight trajectories asymptotically converging to *different* equilibrium dynamics (steady state or a limit cycle). By the very definition, both local and global indeterminacy imply non–uniqueness of the equilibrium. This feature was considered bad modelling from the 1950's through to the 1970's, but currently is being increasingly used to explain business cycles, monetary transmission mechanism, and divergence in the economic performance of different countries (Benhabib and Farmer 1999).

I study indeterminacy in the continuous time economic growth model with rational expectations. Such models are usually described by a system of ordinary differential equations (ODE). Determinate (locally unique) equilibrium means that the number of constraints, imposed on the perfect foresight dynamics (a trajectory in the state space), is just enough to pinpoint a single trajectory converging to the steady state. The constraints are derived by requiring that the trajectory evolves only along the converging (stable) directions in the state space. The trajectory should be orthogonal to the explosive directions. In more technical terms, the constraints are derived by limiting the trajectory to the stable manifold of a particular steady state; this provides a very simple test for the indeterminacy. One has to compare the number of free (control) variables in the model to the number of explosive, or unstable, directions in the neighborhood of the steady state. If those numbers are equal, generically there is a unique choice of control variables that puts the system onto the stable manifold. This situation is referred to as local determinacy. If the number of constraints (unstable directions) is higher than the number of controls, it is, in general, impossible to satisfy the constraints. Therefore, trajectories will initially diverge from the steady state. This case does not have an established name, but terms "explosive dynamics" or "explosive steady state" are sometimes used[1]. When the number of unstable directions is less than the number of controls, a continuum of values for the controls that put the system onto the stable manifold exists. This is local indeterminacy. Study of local determinacy, indeterminacy, or explosive behavior is thus equivalent to studying the local stability of the steady state of the system of ODEs.

This paper considers the well known Romer model of endogenous technological change (Romer 1990). The original Romer paper did not address the question of the uniqueness of the equilibrium trajectory. However, the question has been studied in several other papers. In (Arnold 2000) the model was simplified by removing unskilled labor from the production function. It was shown that if the model has an interior steady state, then this steady state is locally determinate. Necessary and sufficient conditions for non–existence of complex roots implying oscillatory convergence to the steady state were derived. The

[1] After initial divergence the trajectory can converge to a limit cycle and remain bounded. The limit cycle can be determinate or indeterminate. Alternatively, the trajectory can diverge to infinity. Only in the latter case can we speak about "explosive" behavior. However, proving the existence and (in)determinacy of the limit cycle is usually hard, and possibility of its existence is very often mentioned in passing without further elaboration. For further discussion of the economic interpretation of limit cycles refer to (Kind 1999) and (Benhabib and Miyao 1981).

author of the paper mentions in passing that "unskilled labor in final goods production ... proves to be inessential in (Romer 1990)".

(Benhabib, Perli, and Xie 1994, from now on BPX) generalize the original model to allow for complementarity between different intermediary capital goods. They are able to prove that strong enough complementarities imply the possibility of the indeterminate steady state, but they did not succeed in deriving the necessary and sufficient conditions for the steady state to be indeterminate because of the complexity of expressions involved; only numeric results were obtained.

This paper proceeds by following BPX in describing the most general model with exernalities and unskilled labor in Section 2. A 3–dimensional system of ordinary differential equations that is slightly simpler to analyze is derived. It is confirmed that the sufficient conditions for the steady state to be indeterminate are the same as in BPX. Some further restrictions on parameter values necessary to obtain an interior and positive steady state solution are presented. In Section 3, I verify the claim presented in (Arnold 2000) on the insignificance of the unskilled labor (L) in the original Romer model without complementarities. Specifically, I show that the inclusion of L does not change the main conclusion of that paper. If parameter values are such that the steady state if interior, then it is determinate. This finding leads to an attempt to look at the more complex BPX model from the same point of view, that is, excluding the unskilled labor. This allows some simplification of the steady state Jacobian, and I am able to derive analytical results on the nonexistence of the Hopf bifurcations in several special cases in Section 4. In particular, it is shown that no Hopf bifurcation leading from a determinate steady state to a completely stable one exists. In Section 5, I attempt to find "reasonable" parameter values leading to the absolutely unstable steady state through the Hopf bifurcation. As described in the footnote 1, such a bifurcation can lead to the appearance of the stable limit cycle or absolutely explosive behavior.

2 Romer's Model with Complementarities, Interior Solution

In this Section, the description of the model and derivation of the system of ODEs describing it follow BPX unless otherwise noted. The economy consists of 3 sectors; research, intermediate, and final. Research and final sectors are competitive while the intermediate sector consists of monopolists holding infinitely living patents on "designs" of different intermediate goods. The inputs in the economy are unskilled and skilled labor L and H, capital K and knowledge A, L and H are fixed and are supplied inelastically. η units of foregone consumption is needed to produce a unit of an intermediate good. Assuming there is a continuum of "designs", total capital in the economy is given by $K = \int_0^A x(i)di$, where A is the level of knowledge currently available. Final good production technology is given by

$$Y = H_Y^\alpha L^\beta (\int_0^A x(i)^{\frac{\gamma}{\xi}} di)^\xi$$

where $\gamma = 1 - \alpha - \beta$ and ξ is the degree of complementarity between different intermediate capital goods, $\xi \geq 1$. In a symmetric equilibrium, all kinds of intermediate capital are produced in the same amount, and $x(i) = x$. Total capital accumulates without depreciation,

$$\dot{K} = Y - C = \eta^{-\gamma} K^{\gamma} A^{\xi-\gamma} H_Y^{\alpha} L^{\beta} - C. \quad (1)$$

The firms in the final sector are perfect competitors. Therefore, they take price for intermediate capital goods as given and calculate the desired level of demand by maximizing profit,

$$\max_x \left\{ H_Y^{\alpha} L^{\beta} (\int_0^A x(i)^{\frac{\gamma}{\xi}} di)^{\xi} - \int_0^A p(i)x(i)di \right\}, \quad (2)$$

with the solution given by

$$p(j) = \gamma H_Y^{\alpha} L^{\beta} (\int_0^A x(i)^{\frac{\gamma}{\xi}} di)^{\xi-1} x(j)^{\frac{\gamma}{\xi}-1}.$$

The solution of this maximization problem, $x(p)$, is taken as given by monopolies producing intermediate capital goods. They are using only foregone consumption as an input to their production function, transforming η units of it into one unit of the intermediate capital. The optimization problem for the firm producing j^{th} capital good is

$$\max_x \left\{ \gamma H_Y^{\alpha} L^{\beta} (\int_0^A x(i)^{\frac{\gamma}{\xi}} di)^{\xi-1} x(j)^{\frac{\gamma}{\xi}-1} - r\eta x(j) \right\},$$

where r is the interest rate (it is assumed that intermediate sector firms rent their capital). It is possible to express the interest rate through other variables in the model as

$$r = \frac{\gamma^2 \eta^{-\gamma}}{\xi} K^{\gamma-1} A^{\xi-\gamma} H_Y^{\alpha} L^{\beta}. \quad (3)$$

Comparing this with the expression for $p(j)$, one arrives at the expression of the intermediate firm's profits,

$$\pi = p(j)x(j) - rx(j) = \frac{\eta(\xi-\gamma)}{\gamma} rx(j). \quad (4)$$

The research sector is competitive and uses skilled capital and total stock of knowledge as inputs. The production function in this sector is given by

$$\dot{A} = \delta H_A A = \delta(H - H_Y)A, \tag{5}$$

with H_A denoting skilled labor employed in the research. Firms in the research sector produce "designs" of new intermediate capital goods, receive infinitely living patents on them, and sell them to the intermediate sector's monopolists. Perfect competition in the research sector implies that the price for a new "design" is exactly equal to the present value of profits derived from it,

$$P_A(t) = \int_0^\infty \pi(\tau)\exp(-\int_t^\tau r(s)ds)d\tau. \tag{6}$$

Differentiating (6) with respect to time, one gets

$$\dot{P_A} = rP_A - \pi. \tag{7}$$

Noting that the wages of skilled labor in research and the final sector should be the same, one obtains

$$P_A = \frac{\alpha\eta^{-\gamma}}{\delta}K^\gamma A^{\xi-\gamma-1}H_Y^{\alpha-1}L^\beta. \tag{8}$$

The model is closed by introducing representative infinitely living households maximizing their lifetime utility

$$\max_C \int_0^\infty \frac{C^{1-\sigma}}{1-\sigma}\exp(-\rho t)dt$$

subject to budget constraint,

$$\int_0^\infty (C - w_H H - w_L L - rK)\exp(-\int_0^t r(s)ds)dt = 0.$$

As is well known, the solution to this problem is given by

$$\frac{\dot{C}}{C} = \frac{r-\rho}{\sigma} \tag{9}$$

plus appropriate transversality condition.

Substituting the expression for r into (1) and introducing the new variable $q = \frac{C}{K}$ gives me

$$\frac{\dot{q}}{q} = \frac{r-\rho}{\sigma} - \frac{\xi}{\gamma^2}r + q \tag{10}$$

$$\frac{\dot{K}}{K} = \frac{\xi}{\gamma^2}r - q. \tag{11}$$

Noting that $K = \eta A x$ in a symmetric equilibrium, substituting this expression and (8) into (4), one gets

$$\frac{\dot{P}_A}{P_A} = r - \frac{\delta}{\Lambda}H_Y, \tag{12}$$

where Λ is given by $\frac{\alpha\xi}{\gamma(\xi-\gamma)}$. Taking logs in (8), differentiating with respect to time, and comparing the result with (7), I get the following equation:

$$\frac{\dot{r}}{r} + \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{H}_Y}{H_Y} = r - \frac{\delta}{\Lambda}H_Y.$$

Finally, taking logs in (3), differentiating with respect to time, and substituting (10), I arrive at the following system of equations:

$$\begin{aligned}
\frac{\dot{r}}{r} &= r - \frac{\delta}{\Lambda}H_Y - (\frac{\xi}{\gamma^2}r - q) + \frac{\dot{A}}{A} + \frac{\dot{H}_Y}{H_Y} \\
\frac{\dot{r}}{r} &= \alpha\frac{\dot{H}_Y}{H_Y} + (\xi-\gamma)\frac{\dot{A}}{A} - (1-\gamma)(\frac{\xi}{\gamma^2}r - q) \\
\frac{\dot{A}}{A} &= \delta(H - H_Y) \\
\frac{\dot{q}}{q} &= \frac{r-\rho}{\sigma} - \frac{\xi}{\gamma^2}r + q.
\end{aligned}$$

Solving the first two equations of the above system for $\frac{\dot{r}}{r}$ and $\frac{\dot{H}_Y}{H_Y}$, I obtain the final system of differential equations,

$$
\begin{aligned}
\frac{\dot r}{r} &= \frac{1}{1-\alpha}\left\{(\xi-1+\beta)\delta(H-h)-\beta(\tfrac{\xi}{\gamma^2}r-q)-\alpha(r-\tfrac{\delta}{\Lambda}h)\right\}\\
\frac{\dot h}{h} &= \frac{1}{1-\alpha}\left\{(\xi-1-\gamma)\delta(H-h)+\gamma(\tfrac{\xi}{\gamma^2}r-q)-(r-\tfrac{\delta}{\Lambda}h)\right\}\\
\frac{\dot q}{q} &= \frac{r-\rho}{\sigma}-\frac{\xi}{\gamma^2}r+q\\
\frac{\dot A}{A} &= \delta(H-h).
\end{aligned}
\tag{13}
$$

For simplicity, H_Y is denoted h in the above system and the following discussion. It is immediately obvious that A does not enter differential equations for r, h, or q. The evolution of A will determine the levels of the capital and consumption, but will not influence determination of the growth rates in the economy or stability of the Balanced Growth Path (BGP). Therefore, I could safely drop it from consideration and concentrate on the first 3 equations in (13). This system is equivalent to the system of (14), (15), and (16) in (Benhabib, Perli, and Xie 1994) when one substitutes y in the latter system with its expression as a function of r and h. The system (13) derived above is easier to analyze because only second degree polynomials are present on the right hand side. The unique non-zero solution of (13) is given by a triple (h^*, r^*, q^*), where

$$h^* = \frac{\Lambda}{\delta}\frac{\delta H[\sigma(\xi-\gamma)-(\xi-1)]+\rho(1-\gamma)}{\Lambda[\sigma(\xi-\gamma)-(\xi-1)]+(1-\gamma)}, \tag{14a}$$

$$r^* = \frac{1}{1-\frac{1}{\sigma}}\left[\frac{\delta}{\Lambda}h^*-\delta(H-h^*)-\frac{\rho}{\sigma}\right], \qquad \sigma \neq 1 \tag{14b}$$

$$r^* = \frac{1}{1-\gamma}\left[(\xi-1-\gamma)\delta(H-h^*)+\frac{\delta}{\Lambda}h^*-\gamma\rho\right], \quad \sigma = 1 \tag{14c}$$

$$q^* = \left(\frac{\xi}{\gamma^2}-\frac{1}{\sigma}\right)r^*+\frac{\rho}{\sigma}. \tag{14d}$$

There are several necessary conditions that need to be satisfied. First, the variables (r, h, q) are by construction positive, therefore the steady state values should be positive. Second, h^* should be less than the total amount of the skilled labor, H. Third, the household's utility should be finite along the BGP. And fourth, transversality condition should hold at the steady state.

Consider first constraints $h^* > 0$ and $H - h^* > 0$. Two inequalities below should be satisfied at the same time:

$$\begin{aligned}
\frac{\Lambda}{\delta}\frac{\delta H\left[\sigma(\xi-\gamma)-(\xi-1)\right]+\rho(1-\gamma)}{\Lambda[\sigma(\xi-\gamma)-(\xi-1)]+(1-\gamma)} &> 0, \\
\frac{(\delta H-\Lambda\rho)\,(1-\gamma)}{\Lambda[\sigma(\xi-\gamma)-(\xi-1)]+(1-\gamma)} &> 0.
\end{aligned}$$

Λ is a positive number. If the term in square parentheses is positive, then $\delta H - \Lambda\rho > 0$ satisfies both inequalities. Denote $\Psi = [\sigma(\xi-\gamma)-(\xi-1)]$. Suppose that Ψ is negative. In this case, one of the two cases must be true:

$$\begin{array}{ccc}
\delta H\Psi + \rho(1-\gamma) > 0, & & \delta H\Psi + \rho(1-\gamma) < 0, \\
\Lambda\Psi + (1-\gamma) > 0, & \text{or} & \Lambda\Psi + (1-\gamma) < 0, \\
\rho < \frac{\delta}{\Lambda}H. & & \rho > \frac{\delta}{\Lambda}H.
\end{array} \tag{15}$$

Assume $\delta H\Psi + \rho(1-\gamma) > 0$ and $\rho < \frac{\delta}{\Lambda}H$. Then the following chain of inequalities hold:

$$0 < \delta H\Psi + \rho(1-\gamma) < \rho\Lambda\Psi + \rho(1-\gamma) = \rho\left[\Lambda\Psi + (1-\gamma)\right], \tag{16}$$

and $\Lambda\Psi + (1-\gamma) > 0$ is satisfied automatically. Similarly, if $\delta H\Psi + \rho(1-\gamma) < 0$ and $\rho > \frac{\delta}{\Lambda}H$, then

$$0 > \delta H\Psi + \rho(1-\gamma) > \rho\Lambda\Psi + \rho(1-\gamma) = \rho\left[\Lambda\Psi + (1-\gamma)\right], \tag{17}$$

and $\Lambda\Psi + (1-\gamma) < 0$ is also satisfied. Combining the results for positive and negative Ψ, I obtain the following Claim:

Claim 1 *Restrictions $h^* > 0$ and $H - h^* > 0$ are satisfied if and only if model parameters belong to one of the following two sets:*

$$\begin{aligned}
\Theta_1 &= \left\{\delta H\left[\sigma(\xi-\gamma)-(\xi-1)\right]+\rho(1-\gamma) < 0 \text{ and } \rho > \frac{\delta}{\Lambda}H\right\}, \\
\Theta_2 &= \left\{\delta H\left[\sigma(\xi-\gamma)-(\xi-1)\right]+\rho(1-\gamma) > 0 \text{ and } \rho < \frac{\delta}{\Lambda}H\right\}.
\end{aligned}$$

Obviously, sets in Claim 1 are equivalent to those derived in the (Benhabib, Perli, and Xie 1994). Note that for $\sigma = 1$, $\Psi = 1-\gamma > 0$. Ψ is increasing in σ, and only the case Θ_2 is possible if $\sigma \geq 1$. Similarly, if $\xi = 1$ then $\Psi = \sigma(1-\gamma) > 0$ and Ψ is decreasing in ξ for $\sigma < 1$. Therefore, the case Θ_1, which requires $\Psi < 0$, is possible only for low σ combined with high ξ.

Claim 2 *Case Θ_1 is realized only for sufficiently low $\sigma < 1$ and sufficiently high $\xi > 1$.*

Consider now r^*. The situation will be different for $\sigma \neq 1$ and $\sigma = 1$. Assume $\sigma \neq 1$. Rewrite expression for h^* as $\delta(H-h^*)\frac{[\sigma(\xi-\gamma)-(\xi-1)]}{1-\gamma} + \rho - \frac{\delta}{\Lambda}h^* = 0$. Substitute $\frac{\delta}{\Lambda}h^*$ into r^* and rewrite (14b) as

$$
\begin{aligned}
r^* &= \frac{1}{1-\frac{1}{\sigma}}\left[\rho\left(1-\frac{1}{\sigma}\right) + \delta(H-h^*)\left(\frac{\Psi}{1-\gamma}-1\right)\right] = \\
&= \rho + \delta(H-h^*)\frac{(\xi-\gamma)(\sigma-1)}{\left(1-\frac{1}{\sigma}\right)(1-\gamma)} = r^* = \rho + \frac{(\delta H - \Lambda\rho)(\xi-\gamma)\sigma}{\Lambda\Psi + (1-\gamma)}.
\end{aligned}
$$

It is now obvious that if I have case Θ_1 or Θ_2, the second term is positive, and therefore r^* is positive.

Assume $\sigma = 1$. Direct computation in this case gives

$$
\begin{aligned}
r^* &= \frac{1}{1-\gamma}\left[(\xi-1-\gamma)\delta(H-h^*) + \frac{\delta}{\Lambda}h^* - \gamma\rho\right] = \\
&= \frac{1}{1-\gamma}\left[(\xi-1-\gamma)\frac{\delta H - \Lambda\rho}{1+\Lambda} + \frac{\delta H + \rho}{1+\Lambda} - \gamma\rho\right] = \\
&= \frac{(\xi-\gamma)\delta H + \rho(1-\gamma-\Lambda(\xi-1))}{(1-\gamma)(1+\Lambda)}.
\end{aligned}
$$

By Claim 2 only case Θ_2 is possible for $\sigma = 1$, and $\rho < \frac{\delta}{\Lambda}H$. Therefore, substituting δH in the numerator, one gets

$$
\begin{aligned}
r^* &= \frac{(\xi-\gamma)\delta H + \rho(1-\gamma-\Lambda(\xi-1))}{(1-\gamma)(1+\Lambda)} > \\
&> \frac{\rho(\Lambda(\xi-\gamma) + 1 - \gamma - \Lambda(\xi-1))}{(1-\gamma)(1+\Lambda)} = \frac{\rho(1-\gamma)(1+\Lambda)}{(1-\gamma)(1+\Lambda)} = \rho > 0.
\end{aligned}
$$

and therefore $r^* > \rho > 0$.

Claim 3 *If the model parameters belong to* $\{\Theta_1, \Theta_2\}$ *then* r^* *is positive.*

My next step will be to ensure that the household utility remains finite. The utility is given by $U = \int_0^\infty \frac{C^{1-\sigma}}{1-\sigma} \exp(-\rho t)dt$. Along the Balanced Growth Path, r, h, and q remain constant. Therefore, knowledge grows with the rate given by $\delta(H-h^*)$. Interest rate r is proportional to $K^{\gamma-1}A^{\xi-\gamma}$. If r is constant then the capital growth rate is given by $\delta(H-h^*)\frac{\xi-\gamma}{1-\gamma}$. $q = \frac{C}{K}$ is also constant along BGP and so C grows with the same rate as K, $\delta(H-h^*)\frac{\xi-\gamma}{1-\gamma}$. To ensure convergence of the utility integral, the following condition should hold:

$$(1-\sigma)\,\delta(H-h^*)\frac{\xi-\gamma}{1-\gamma} - \rho < 0. \tag{18}$$

It is immediately obvious that when the model parameters are in $\{\Theta_1, \Theta_2\}$ and $\sigma \geq 1$, this inequality is trivially satisfied.

After substitution of (14a) into the above expression and simplifying, I obtain

$$\frac{(1-\sigma)\,(\xi-\gamma)\delta H - \rho(1-\gamma)(1+\Lambda)}{\Lambda\Psi + (1-\gamma)} < 0. \tag{19}$$

This expression provides an additional constraint on the model parameters when $\sigma < 1$. Finiteness of the utility integral also implies that the transversality condition at the steady state holds, $\lim_{t\to\infty}[\lambda K \exp(-\rho t) = C^{-\sigma}K\exp(-\rho t)] = 0$. Both C and K grow at the same rate, and this condition is equivalent to $(1-\sigma)\,\delta(H-h^*)\frac{\xi-\gamma}{1-\gamma} - \rho < 0$ which is exactly the necessary condition for the utility integral to converge.

Finally, I have to check the positivity of $q^* = \left(\frac{\xi}{\gamma^2} - \frac{1}{\sigma}\right) r^* + \frac{\rho}{\sigma}$. If $\sigma \geq 1$, $\frac{\xi}{\gamma^2} - \frac{1}{\sigma} > 0$, r^* was shown above to be positive, q^* is then positive. In case $\sigma < 1$, rewrite r^* as $r^* = \frac{1}{1-\frac{1}{\sigma}}\left[\rho\left(1-\frac{1}{\sigma}\right) + \delta(H-h^*)\left(\frac{\Psi}{1-\gamma} - 1\right)\right]$. Then q^* is given by

$$q^* = \frac{\left(\frac{\xi}{\gamma^2} - \frac{1}{\sigma}\right)}{1 - \frac{1}{\sigma}} \left[\rho\left(1 - \frac{1}{\sigma}\right) + \delta(H - h^*)\left(\frac{\Psi}{1-\gamma} - 1\right)\right] + \frac{\rho}{\sigma},$$

$$q^* = \frac{\xi}{\gamma^2}\rho - \frac{\left(\frac{\xi}{\gamma^2} - \frac{1}{\sigma}\right)}{1 - \frac{1}{\sigma}} (1-\sigma)\,\delta(H - h^*)\frac{\xi - \gamma}{1 - \gamma}.$$

For $\sigma < 1$, (19) guarantees that $(1-\sigma)\,\delta(H-h^*)\frac{\xi-\gamma}{1-\gamma} - \rho < 0$. Therefore,

$$q^* > (1-\sigma)\,\delta(H-h^*)\frac{\xi-\gamma}{1-\gamma}\left\{\frac{\xi}{\gamma^2} - \frac{\left(\frac{\xi}{\gamma^2} - \frac{1}{\sigma}\right)}{1-\frac{1}{\sigma}}\right\} =$$

$$(1-\sigma)\,\delta(H-h^*)\frac{\xi-\gamma}{1-\gamma}\left\{\frac{1}{\sigma}\frac{1-\frac{\xi}{\gamma^2}}{1-\frac{1}{\sigma}}\right\}.$$

The term in the figure parentheses is always positive because $\frac{\xi}{\gamma^2} > 1$, $\frac{1}{\sigma} > 1$, and so q^* is also positive.

Claim 4 *If the model parameters belong to one of the following two sets:*

$$\Theta_1 = \left\{\begin{array}{c} \delta H\left[\sigma(\xi-\gamma) - (\xi-1)\right] + \rho(1-\gamma) < 0 \\ \rho > \frac{\delta}{\Lambda}H \\ (1-\sigma)\,(\xi-\gamma)\delta H - \rho(1-\gamma)(1+\Lambda) > 0 \end{array}\right\},$$

$$\Theta_2 = \left\{\begin{array}{c} \delta H\left[\sigma(\xi-\gamma) - (\xi-1)\right] + \rho(1-\gamma) > 0 \\ \rho < \frac{\delta}{\Lambda}H \\ (1-\sigma)\,(\xi-\gamma)\delta H - \rho(1-\gamma)(1+\Lambda) < 0 \end{array}\right\},$$

then the system (13) has an interior BGP solution along which household's utility integral converges and the transversality condition holds.

As was stated above, the system (13) is equivalent to the one derived in Section 2 of BPX. Therefore, indeterminacy is still possible in my system only for parameter values belonging to the set Θ_1.

3 Influence of Unskilled Labor, no Complementarities

Stability analysis of a simplified version of the original Romer model (no complementarities, $\xi = 1$) was performed in (Arnold 2000) by assuming that unskilled labor, L, does not enter the production function. This assumption meant a significant simplification of the steady state Jacobian. Arnold was then able to show that if the positive steady state of the model is interior, then it is determinate with two positive and one negative eigenvalues of the steady state Jacobian. In this Section, I undertake to verify that inclusion of the unskilled labor into the original Romer model does not change this result.

Rewrite (13) as

$$
\begin{aligned}
\frac{\dot{r}}{r} &= \frac{1}{1-\alpha}\left\{\beta\delta(H-h) - \beta(\frac{1}{\gamma^2}r - q) - \alpha(r - \frac{\delta}{\Lambda}h)\right\} \\
\frac{\dot{h}}{h} &= \frac{1}{1-\alpha}\left\{-\gamma\delta(H-h) + \gamma(\frac{1}{\gamma^2}r - q) - (r - \frac{\delta}{\Lambda}h)\right\} \\
\frac{\dot{q}}{q} &= \frac{r-\rho}{\sigma} - \frac{1}{\gamma^2}r + q
\end{aligned}
\tag{20}
$$

by substituting $\xi = 1$. It is easy to verify that the steady state is given by

$$h^* = \frac{\Lambda}{\delta}\frac{\delta H + \frac{\rho}{\sigma}}{\Lambda + \frac{1}{\sigma}}, \tag{21a}$$

$$r^* = \frac{\delta}{\Lambda}h^*, \tag{21b}$$

$$q^* = \left(\frac{1}{\gamma^2} - \frac{1}{\sigma}\right)r^* + \frac{\rho}{\sigma} = \frac{\delta H(\frac{1}{\gamma^2} - \frac{1}{\sigma}) + \frac{\rho}{\sigma}(\frac{1}{\gamma^2} + \Lambda)}{\Lambda + \frac{1}{\sigma}}. \tag{21c}$$

It is immediately obvious that h^* and r^* are always positive. $H - h^*$ is given by $\frac{1}{\sigma}\frac{\delta H - \rho\Lambda}{\Lambda + \frac{1}{\sigma}}$ and so $\rho < \frac{\delta}{\Lambda}H$ guarantees an interior solution. In the absence of externalities the growth rates of knowledge, consumption, and capital are the same. Therefore, to ensure finiteness of the utility integral, one needs

$$0 > (1-\sigma)\,\delta(H-h^*) - \rho = \frac{(1-\sigma)\,\delta H - \rho(1+\Lambda)}{\Lambda + \frac{1}{\sigma}}. \tag{22}$$

Finally, note that $q^* > 0$ for $\sigma \geq 1$ because $\frac{1}{\gamma^2} - \frac{1}{\sigma}$ is positive. For $\sigma < 1$, the following chain of inequalities proves that q^* is positive:

$$\begin{aligned} q^* &= \frac{\delta H(\frac{1}{\gamma^2} - \frac{1}{\sigma}) + \frac{\rho}{\sigma}(\frac{1}{\gamma^2} + \Lambda)}{\Lambda + \frac{1}{\sigma}} > \frac{\delta H(\frac{1}{\gamma^2} - \frac{1}{\sigma}) + \frac{1}{\sigma}(\frac{1}{\gamma^2} + \Lambda)\frac{1-\sigma}{1+\Lambda}}{\Lambda + \frac{1}{\sigma}} = \\ &= \frac{\delta H(\frac{1}{\gamma^2} - 1)(1+\sigma\Lambda)}{(\Lambda + \frac{1}{\sigma})\sigma(1+\Lambda)} > 0. \end{aligned}$$

Therefore, the following Claim is true:

Claim 5 *If the model parameters belong to the following set,*

$$\Theta = \left\{ \begin{array}{c} \rho < \frac{\delta}{\Lambda} H \\ (1-\sigma)\,\delta H - \rho(1+\Lambda) < 0 \end{array} \right\},$$

then the system (20) has an interior BGP solution along which household's utility integral converges and the transversality condition holds.

It is interesting to note that for $\sigma < 1$, only $\rho \in [\underline{\rho}, \overline{\rho}]$ where $\overline{\rho} > \underline{\rho} > 0$, ensure an interior solution. Economies with highly impatient agents do not allocate skilled labor to the research sector; economies with very high patience grow too fast, and the utility integral diverges.

Determinacy, indeterminacy, or explosiveness of the steady state is determined by the number of unstable directions near the steady state and the number of the free, or control, variables. After excluding the fourth equation for A from consideration, only one variable remains predetermined — total capital K. Two other variables, C and h, are free. After changing the variables to (r, h, q), the structure with one predetermined and two free variables is preserved. Therefore, indeterminacy requires 1 positive (unstable) direction. 2

unstable directions mean that the steady state is a saddle, and 3 positives imply explosive behavior. As is well known, the number of stable and unstable directions near the steady state is obtained by calculating the Jacobian of the system linearized around the steady state and then evaluating the number of stable and unstable eigenvalues. In addition, for 3–dimensional systems of ODE, the number of positive eigenvalues for the matrix A is given by the number of sign changes in the following sequence:

$$\left(-1, \qquad trace(A), \qquad -BA + \frac{Det(A)}{trace(A)}, \qquad Det(A)\right), \tag{23}$$

where BA is the sum of the second order minors. See, for example, (Gantmacher 1960) for proof.

The Jacobian of the linearized system (20) is given by

$$J^* = \begin{bmatrix} -r^* \frac{\alpha + \frac{1-\alpha-\gamma}{\gamma^2}}{1-\alpha} & \delta r^* \frac{\frac{\alpha}{\Lambda} - (1-\alpha-\gamma)}{1-\alpha} & r^* \frac{1-\alpha-\gamma}{1-\alpha} \\ h^* \frac{\frac{1}{\gamma} - 1}{1-\alpha} & \delta h^* \frac{\frac{1}{\Lambda} + \gamma}{1-\alpha} & \frac{\gamma}{1-\alpha} h^* \\ q^* (\frac{1}{\sigma} - \frac{1}{\gamma^2}) & 0 & q^* \end{bmatrix}. \tag{24}$$

Calculating the determinant of J^*, one obtains

$$Det(J^*) = -\frac{1}{1-\alpha} r^* \delta h^* q^* (1-\gamma)(1 + \frac{1}{\Lambda\sigma}) < 0. \tag{25}$$

For the trace of J^*, long calculations produce

$$trace(J^*) = \delta H \frac{\gamma\Lambda + \frac{1}{\gamma} + (1-\alpha)(1-\frac{1}{\sigma})}{(\Lambda + \frac{1}{\sigma})(1-\alpha)} + \frac{\rho}{\sigma} \frac{\gamma\Lambda + \frac{1}{\gamma} + (1-\alpha)(1+\Lambda)}{(\Lambda + \frac{1}{\sigma})(1-\alpha)}. \tag{26}$$

Partially substituting (22) in the above expression, one gets

$$trace(J^*) > \frac{(\delta H + \frac{\rho}{\sigma})(\gamma\Lambda + \frac{1}{\gamma})}{(\Lambda + \frac{1}{\sigma})(1-\alpha)} > 0. \tag{27}$$

The following Claim is, therefore, true:

Claim 6 *For parameter values in* Θ, *the Jacobian of the linearized around the positive steady state system (20) satisfies* $trace(J^*) > 0 > Det(J^*)$.

This is exactly the conclusion achieved in (Arnold 2000) under the assumption that $\beta = 0$. Therefore, the main result of that paper holds: for parameter values ensuring that there exists an interior solution with finite utility integral, the solution is saddle path stable, because the sequence (23) becomes $(-, +, ?, -)$ and the only possibility is 2 changes of sign meaning 2 positive eigenvalues. As noted above, 2 positive eigenvalues mean saddle path stability in the model.

Note that assuming $\beta = 0$ simplifies the problem. For example, the Jacobian (24) has zero in the first row, third column. It might be, therefore, advisable to assume $\beta = 0$ in the complicated problem of Sections 1 and 2 and to study its stability with the hope that the results are not very sensitive to the particular value of β. This is undertaken in the next Section.

4 Model with Complementarities, no Unskilled Labor

If I assume $\beta = 0$, then $\gamma = 1 - \alpha$. Linearizing (13) near the positive steady state and substituting away α, I get the following steady state Jacobian:

$$J^* = \begin{bmatrix} -\frac{1-\gamma}{\gamma} r^* & \frac{\delta}{\gamma} r^* (\frac{1-\gamma}{\Lambda} - (\xi - 1)) & 0 \\ \frac{1}{\gamma} h^* (\frac{\xi}{\gamma} - 1) & \frac{\delta}{\gamma} h^* (\frac{1}{\Lambda} - (\xi - 1 - \gamma)) & -h^* \\ q^* (\frac{1}{\sigma} - \frac{\xi}{\gamma^2}) & 0 & q^* \end{bmatrix}, \tag{28}$$

where (r^*, h^*, q^*) are given by (14a)–(14d). It will be impossible to obtain detailed stability boundaries in this case. My task here will be, therefore, very limited. I will ask if the Hopf bifurcation — passing of the imaginary axis by two complex conjugate eigenvalues with non–zero speed — is possible in 3 special cases. The importance of possibility (or impossibility) of the Hopf bifurcation is obvious from the following consideration. Suppose one starts from a saddle path stable steady state (2 positive and 1 negative eigenvalue).

A Hopf bifurcation then means that all the eigenvalues become negative and the steady state is absolutely stable. This means that the steady state becomes indeterminate. Any choice of controls will select a trajectory converging to the steady state. An absolutely stable steady state in three dimensions might also imply the presence of chaotic behavior. Suppose now that the steady state was initially indeterminate (1 positive, 2 negative eigenvalues). A Hopf bifurcation then creates an absolutely unstable steady state. As explained in footnote 1, this might imply either truly explosive behavior, when all the trajectories leave the neighborhood of the steady state and possibly diverge to infinity, or the existence of a stable limit cycle. In the latter case, the Balanced Growth Path becomes the Balanced Growth Business Cycle, which is indeterminate.

Recall that the number of positive eigenvalues is determined by the number of sign changes in the following sequence:

$$\left(-1, \qquad trace(A), \qquad -BA + \frac{Det(A)}{trace(A)}, \qquad Det(A)\right).$$

For a generic system, one does not expect that $trace(A)$ and $Det(A)$ or $trace(A)$ and BA change the sign simultaneously. Suppose that $trace(A)$ passes through zero from negative to positive values. Let the sign of $Det(A)$ be positive (negative). Then $-BA + \frac{Det(A)}{trace(A)}$ changes sign by diverging to minus (plus) infinity and appearing again at plus (minus) infinity, changing sign together with $trace(A)$. Possible changes in the above sequence are then $(-,-,-,+)$ to $(-,+,+,+)$ or $(-,-,+,-)$ to $(-,+,-,-)$. In both cases the number of positive eigenvalues does not change. If the sign of $Det(A)$ changes, at most 1 positive eigenvalue appears or disappears. If, on the other hand, $-BA+\frac{Det(A)}{trace(A)}$ changes sign with the signs of $Det(A)$ and $trace(A)$ fixed, it is possible to have a Hopf bifurcation, as in $(-,+,+,+)$ to $(-,+,-,+)$ (1 to 3, indeterminacy to absolutely unstable steady state) or $(-,-,+,-)$ to $(-,-,-,-)$ (2 to 0, determinate steady state to an absolute stable one). Therefore, the curve in the parameter space along which $-BA + \frac{Det(A)}{trace(A)} = 0$ is of utmost interest.

Alternatively, a very simple method could be used to rule out Hopf bifurcations. If I can show that for some parameter values all the eigenvalues are real, then the Hopf bifurcation cannot happen.

I consider three special cases: $\frac{\xi}{\gamma^2} = \frac{1}{\sigma}$, $\xi = 1 + \frac{1-\gamma}{\Lambda}$, and the model parameters belong to Θ_2.

4.1 Case $\frac{\xi}{\gamma^2} = \frac{1}{\sigma}$

In this case the steady state Jacobian simplifies even more,

$$J^* = \begin{bmatrix} -\frac{1-\gamma}{\gamma} r^* & \frac{\delta}{\gamma} r^* (\frac{1-\gamma}{\Lambda} - (\xi - 1)) & 0 \\ \frac{1}{\gamma} h^* (\frac{\xi}{\gamma} - 1) & \frac{\delta}{\gamma} h^* (\frac{1}{\Lambda} - (\xi - 1 - \gamma)) & -h^* \\ 0 & 0 & q^* \end{bmatrix} \tag{29}$$

One of the eigenvalues is given by $q^* = \frac{\rho}{\sigma} > 0$. Two remaining eigenvalues are the eigenvalues of the 2-dimensional matrix,

$$J_1^* = \begin{bmatrix} -\frac{1-\gamma}{\gamma} r^* & \frac{\delta}{\gamma} r^* (\frac{1-\gamma}{\Lambda} - (\xi - 1)) \\ \frac{1}{\gamma} h^* (\frac{\xi}{\gamma} - 1) & \frac{\delta}{\gamma} h^* (\frac{1}{\Lambda} - (\xi - 1 - \gamma)) \end{bmatrix}. \tag{30}$$

The Hopf bifurcation then requires $Det(J_1^*) > 0$, $trace(J_1^*) = 0$. Determinant is proportional to $-\left[-\xi^2 + \xi(1 + \gamma + \gamma^2) - \gamma^2(1 + \gamma)\right]$ which is positive for large ξ. The expression for $trace(J_1^*)$ can be written as $f(\rho, \delta H, \xi, \gamma)$ that cannot be signed using restrictions analogous to those derived in Claim 4. Function f is cubic in ξ and γ. However, a relatively easy necessary condition can be derived here. Note that if $trace(J_1^*) = 0$, then $J_1^*(2,2) > 0$ and $\frac{1}{\Lambda} - (\xi - 1 - \gamma) > 0$. Simplifying the last expression, it is easy to see that the following two constraints should hold at the Hopf bifurcation point:

$$\xi^2 - \xi(1+\gamma+\gamma^2) + \gamma^2(1+\gamma) > 0, \tag{31a}$$

$$\xi^2 - \xi(\frac{1+\gamma-\gamma^2}{1-\gamma}) - \frac{\gamma^2}{1-\gamma} < 0. \tag{31b}$$

The first quadratic equation has 2 roots, $\xi_1 = \gamma^2$ and $\xi_2 = 1+\gamma$. ξ is assumed to be greater or equal to one, and (31b) is then satisfied for $\xi > 1+\gamma$. The second equation has 2 real roots,

$$\widetilde{\xi}_{1,2} = \frac{1+\gamma-\gamma^2 \pm \sqrt{(1+\gamma-\gamma^2)^2 - 4\gamma^2(1+\gamma)}}{2(1-\gamma)}. \tag{32}$$

Obvious calculations demonstrate that

$$\widetilde{\xi}_1 = \frac{1+\gamma-\gamma^2 + \sqrt{(1+\gamma-\gamma^2)^2 - 4\gamma^2(1+\gamma)}}{2(1-\gamma)} > 1+\gamma,$$

$$\widetilde{\xi}_2 = \frac{1+\gamma-\gamma^2 - \sqrt{(1+\gamma-\gamma^2)^2 - 4\gamma^2(1+\gamma)}}{2(1-\gamma)} < 1+\gamma.$$

Summarizing the above results I can state the following:

Claim 7 *The necessary condition for the existence of the Hopf bifurcation in case* $\frac{\xi}{\gamma^2} = \frac{1}{\sigma}$ *is* $\xi \in [1+\gamma, \widetilde{\xi}_1]$.

4.2 Case $\xi = 1 + \frac{1-\gamma}{\Lambda}$

When $\xi = 1 + \frac{1-\gamma}{\Lambda}$, the steady state Jacobian becomes

$$J^* = \begin{bmatrix} -\frac{1-\gamma}{\gamma} r^* & 0 & 0 \\ \frac{1}{\gamma} h^* (\frac{\xi}{\gamma} - 1) & \frac{\delta}{\gamma} h^* (\frac{1}{\Lambda} - (\xi - 1 - \gamma)) & -h^* \\ q^* (\frac{1}{\sigma} - \frac{\xi}{\gamma^2}) & 0 & q^* \end{bmatrix}. \tag{33}$$

The three eigenvalues of J^* are given by $\lambda_{1,2,3} = (-\frac{1-\gamma}{\gamma} r^*, \frac{\delta}{\gamma} h^* (\frac{1}{\Lambda} - (\xi - 1 - \gamma)), q^*)$. All three are real, and Hopf bifurcation is impossible here. Assuming that the positive solution is interior, $\lambda_1 < 0$, $\lambda_3 > 0$, $\lambda_2 = \frac{\delta}{\gamma} h^* (\frac{1}{\Lambda} - (\xi - 1 - \gamma)) =$

$\frac{\delta}{\gamma}h^*\gamma(1+\frac{1}{\Lambda}) > 0$. Therefore, in this case there are 2 positive eigenvalues and 1 negative, and the positive solution is determinate.

Claim 8 *If* $\xi = 1 + \frac{1-\gamma}{\Lambda}$, *then the positive interior solution is determinate. No Hopf bifurcations are possible.*

Note that $\xi = 1 + \frac{1-\gamma}{\Lambda} = 1 + \gamma(1 - \frac{\gamma}{\xi}) < 1 + \gamma$.

The two cases just considered point to the following conclusion: low ξ excludes Hopf bifurcations. Numerical simulations conducted in the following Section support this conjecture.

4.3 Case Θ_2

Determinant of the steady state Jacobian was calculated in (Benhabib, Perli, and Xie 1994) and remains the same in the current model. Rewriting it using my notation, I get

$$Det(J^*) = -\frac{r^* q^* \delta h^*}{\sigma\Lambda}(\Lambda\Psi + 1 - \gamma). \tag{34}$$

Expression $(\Lambda\Psi + 1 - \gamma)$ is positive (negative) if the model parameters are in Θ_2 (Θ_1) and therefore $Det(J^*) < 0$ for Θ_2. To show that Hopf bifurcation is impossible in this case it suffices to demonstrate that $trace(J^*) > 0$ as the sequence of signs becomes $(-, +, ?, -)$ and only 2 sign changes are possible. Express r^* and q^* through $\delta(H - h^*)$ and ρ only as described above. After very long and tedious chain of calculations that are omitted here, the expression for $trace(J^*)$ becomes

$$\begin{aligned} trace(J^*) = \rho\frac{\xi+\gamma+\Lambda}{\gamma} + \delta(H-h^*)\frac{\sigma}{\gamma(1-\gamma)}\times \\ \times\left[\Psi(\xi+\Lambda) + (\xi-\gamma)(\gamma + \frac{1}{\sigma}(\frac{\xi-1}{\gamma} - \gamma)\right]. \end{aligned} \tag{35}$$

I evaluate this expression separately for $\sigma \geq 1$ and $\sigma < 1$[2].

Note that the first term is always positive no matter what σ is. Consider first the case $\sigma \geq 1$. For $\sigma = 1$ the expression in square parentheses reduces to

$$(\xi + \Lambda)(1-\gamma) + (\xi - \gamma)\frac{\xi - 1}{\gamma} > 0. \tag{36}$$

$\Psi = \sigma(\xi - \gamma) - (\xi - 1)$ is increasing in σ and positive for $\sigma = 1$. The term $\frac{\xi-1}{\gamma} - \gamma$ is the only one that can be negative. It is multiplied by $\frac{\xi-\gamma}{\sigma}$ which is positive and decreasing in σ. Therefore, the whole square bracket term is either always positive if $\frac{\xi-1}{\gamma} - \gamma > 0$ or increasing in σ and positive when $\sigma = 1$ if $\frac{\xi-1}{\gamma} - \gamma < 0$. In both cases it is positive for $\sigma \geq 1$. The whole $trace(J^*)$ is thus positive.

Consider now $\sigma < 1$. In this case (18) is not moot and $\rho > (1-\sigma)\,\delta(H - h^*)\frac{\xi-\gamma}{1-\gamma}$. Substitute this inequality into (35) and after simplification obtain

$$trace(J^*) > \frac{\delta(H - h^*)}{\gamma(1-\gamma)}\left\{(\xi + \Lambda)\left[(1-\sigma)(\xi - \gamma) + \sigma\Psi\right] + \frac{\xi - 1}{\gamma}\right\}. \tag{37}$$

The only part of the last expression that can be negative is given by $\Phi = (1-\sigma)(\xi - \gamma) + \sigma\Psi$. Write it as a quadratic equation in σ,

$$\sigma^2(\xi - \gamma) - \sigma(\xi - \gamma + \xi - 1) + (\xi - \gamma). \tag{38}$$

Note that (38) is positive for $\sigma = 0$ and $\sigma = 1$, and the coefficient at σ^2 is positive. Therefore, there is a minimum at $\sigma_{\min} = \frac{1}{2}(1 + \frac{\xi-1}{\xi-\gamma})$. I want to show that (38) evaluated at $\sigma_{\min}$ is positive and positive everywhere as a result.

If the model parameters belong to Θ_2 then (16) guarantees that $\Lambda\Psi + 1 - \gamma > 0$. Substitute this inequality into Φ to obtain

$$(1-\sigma)(\xi - \gamma) + \sigma\Psi > (\xi - \gamma)\left[1 - \sigma - \frac{\sigma\gamma}{(1-\gamma)\xi}\right]. \tag{39}$$

[2] Note that only Case Θ_2 is possible for $\sigma \geq 1$.

It is easy to show that $1-\sigma-\frac{\sigma\gamma}{(1-\gamma)\xi}>0$ when $\sigma<\widehat{\sigma}=\frac{(1-\gamma)\xi}{\gamma+(1-\gamma)\xi}$. Finally, calculate $\sigma_{\min}-\widehat{\sigma}$ to obtain

$$\sigma_{\min}-\widehat{\sigma}=-2\xi^2(1-\gamma)^2-2\xi\gamma^2-\xi(1+\gamma)(1-\gamma)-(1+\gamma)\gamma<0. \tag{40}$$

I have just proven that given Θ_2, Φ reaches a minimum at a point where it is greater than the positive number $(\xi-\gamma)\left[1-\sigma-\frac{\sigma\gamma}{(1-\gamma)\xi}\right]$. Therefore, it is positive everywhere for $\sigma<1$. But $\Phi=(1-\sigma)(\xi-\gamma)+\sigma\Psi$ was the only term inside the figure parentheses in (37) that could possibly be negative. This means that $trace(J^*)$ is greater than some positive number for $\sigma<1$ and so is positive. Combining the results for $\sigma\geq 1$ and $\sigma<1$ I get that $trace(J^*)>0$ if the model parameters belong to Θ_2. The sequence of signs becomes $(-,+,?,-)$ and only 2 sign changes are possible. Therefore, Case Θ_2 implies that there are no Hopf bifurcations and the interior steady state is determinate (saddle path stable).

Claim 9 *If the model parameters belong to set Θ_2, then the unique interior steady state is determinate. No Hopf bifurcations are possible.*

5 Numerical Search for the Hopf Bifurcation Boundary

(Benhabib, Perli, and Xie 1994) provided some numerical results for the model described in Section 2. They were able to demonstrate that assuming parameter values implying reasonable magnitudes for the steady state interest rate and shares of the skilled and unskilled labor one can generate both determinate and indeterminate steady state. Indeterminacy required a rather low value of σ, in agreement with Claim 2. In this Section, I use numerical optimization algorithm to search for parameter values putting the model onto the Hopf bifurcation

γ	ξ	$\delta,\%$	$\rho,\%$	σ	$h^*,\%$	$g_A,\%$	$g_C,\%$
0.5588	2.123	3.082	3.292	0.2865	77.42	0.69573	2.4682
0.5899	2.546	2.372	3.094	0.3671	78.21	0.51672	2.4655
0.5837	2.456	2.262	2.419	0.5399	71.23	0.65061	2.9275
0.5716	2.288	3.78	3.923	0.1851	97.27	0.10321	0.41374
0.6171	2.985	1.807	2.682	0.4928	76.10	0.4321	2.6727
0.5257	1.748	3.964	3.334	0.2233	70.79	1.158	2.983
0.6222	3.412	1.581	2.13	0.6822	75.53	0.3869	2.857

Table 1: Points on a Hopf bifurcation boundary. $r^* = 4\%$, $S_K = 25\%$, parameters in Θ_1, $\beta = 0$

boundary. I demonstrate that it is possible to obtain Hopf bifurcation for the "reasonable" parameter values. I also show that the conjecture on insignificance of β holds for Hopf bifurcation boundaries.

The first set of calculations was performed using the model with $\beta = 0$. I fixed the steady state values of the interest rate at 4% and the skilled labor share at 75%, leaving 25% to the capital as in BPX. The parameter values were constrained to belong to the set Θ_1. As stated above, Θ_1 means positive $Det(J^*)$, and the Hopf bifurcation happens if the sign sequence goes from $(-,+,+,+)$ to $(-,+,-,+)$. Therefore, the full problem was to find a vector $\nu = (\gamma, \xi, \delta, \rho, \sigma)$ such that $\nu \in \Theta_1$, $trace(J^*) > 0$, $r^* = 4\%$, capital share is S_K =25%, and $-BJ^* + \frac{Det(J^*)}{trace(J^*)} = 0$. MATLAB 5.3 multigoal optimization utility `fgoalattain` was used to perform the calculations. Some of the resulting parameter vectors plus steady state values of the h^* — share of skilled labor allocated to manufacturing, and steady state rates of growth for knowledge and consumption/capital g_A and g_C are presented below.

Some results here deserve special attention. For example, note that ξ is always greater than $1 + \gamma$, which supports the tentative conclusion reached in the previous Section. I was able to generate Hopf bifurcation for values of σ

γ	ξ	$\delta, \%$	$\rho, \%$	σ	$h^*, \%$	$g_A, \%$	$g_C, \%$
0.5891	2.534	2.111	3.515	0.4139	88.28	0.2474	1.171
0.5622	2.166	2.776	3.964	0.3114	98.86	0.03159	0.1158
0.6203	3.04	1.617	3.256	0.4724	84.71	0.2473	1.576
0.5699	2.266	2.689	4	0.3083	≈100	≈0	≈0
0.4993	3.375	3.305	4.217	0.6001	94.6	0.1783	1.024
0.7105	1.795	1.421	3.162	0.1391	99.51	0.00698	0.02616
0.4896	2.639	1.721	2.017	0.4839	97.17	0.04868	0.2051

Table 2: Hopf bifurcation boundary $r^* = 4\%$, $S_K = 25\%$, parameters in Θ_1, $\beta = \alpha/2$

as high as 0.6822, but the value of ξ was also unrealistically high at 3.412. In general, I can generate Hopf bifurcation for values of σ higher than those presented in Table 1 of BPX, and those values are simply points in Θ_1, not the bifurcation boundary points. However, the bifurcation boundary points generally have lower productivity of skilled labor in research (δ), discount rate (ρ), and rates of growth of the knowledge and capital than those presented in (Benhabib, Perli, and Xie 1994). As expected, an analogous search in the set Θ_2 did not bring any results.

To test the conjecture on insignificance of the unskilled labor for the stability of the model, a similar search was performed assuming $\beta > 0$. In Table 2, I used the constraint $\beta = \frac{\alpha}{2}$. This is the same assumption as the one used in BPX. As in Table 1, the steady state interest rate was restricted to be equal to 4%, and the share of capital to 25%. Points on the bifurcation boundary for the model with $\beta = 0$ were used as initial points in the numerical search in an attempt to facilitate the comparison of the two models.

The comparison of the results shows that the bifurcation boundary in the model with unskilled labor is achieved for higher ratios of the skilled labor in manufacturing and correspondingly lower growth rates. There is no discernible effect on the values of σ, δ tends to be lower, and ρ and ξ higher. Still, the values of the discount rate ρ needed to generate the Hopf bifurcation are probably too low, as are the steady state rates of growth of knowledge and consumption.

Attempts to make β a free parameter, not constrained to the value $\alpha/2$, did not bring any significantly different results. Inclusion of nonzero β implies that the Hopf bifurcation boundary is achieved for steady states with a very high share of the skilled labor in manufacturing and low growth rates.

Summarizing the numerical results presented above I can say that Arnold's statement on the insignificance of the unskilled labor for the stability of the original Romer's model is qualitatively correct in the extended version of the model. Nonzero value of β does not preclude indeterminate steady state or Hopf bifurcation to the absolutely unstable steady state. It does not allow a bifurcation from the determinate to the absolutely stable steady state. The major result of its inclusion is the shift of the bifurcation boundary towards economies allocating less resources to research. If this bifurcation leads to absolutely explosive behavior rather that a stable limit cycle, I can claim that the inclusion of β increases the range of economies that could be described by the extended Romer model.

References

ARNOLD, L. G. (2000): "Stability of the Market Equilibrium in Romer's Model of Endogenous Thechnological Change: A Complete Characterization," *Journal of Macroeconomics*, 22, 69–84.

BENHABIB, J., AND R. E. A. FARMER (1999): "Indeterminacy and Sunspots in Macroeconomics," in *Handbook of Macroeconomics*, ed. by J. Tailor, and M. Woodford. North Holland, Amsterdam.

BENHABIB, J., AND T. MIYAO (1981): "Some New Results on the Dynamics of the Generalized Tobin Model," *International Economic Review*, 22, 589–596.

BENHABIB, J., R. PERLI, AND D. XIE (1994): "Monopolistic Competition, Indeterminacy and Growth," *Ricerche Economiche*, 48, 279–298.

GANTMACHER, F. R. (1960): *The Theory of Matrices*. Chelsea, Nyw York.

KIND, C. (1999): "Remarks on the Economic Interpretation of Hopf Bifurcation," *Economics Letters*, 62, 147–154.

ROMER, P. M. (1990): "Endogenous Technological Change," *Journal of Political Economy*, 98, S71–S103.

SHIGOKA, T. (1994): "A Note on Woodford's Conjecture: Constructing Stationary Sunspot Equilibria in a Continuous Time Model," *Journal of Economic Theory*, 64, 531–540.