SPECTRUM OF THE LAPLACIAN IN NARROW TUBULAR
NEIGHBOURHOODS OF HYPERSURFACES WITH COMBINED
DIRICHLET AND NEUMANN BOUNDARY CONDITIONS

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Abstract. We consider the Laplacian in a domain squeezed between two parallel hypersurfaces in Euclidean spaces of any dimension, subject to Dirichlet boundary conditions on one of the hypersurfaces and Neumann boundary conditions on the other. We derive two-term asymptotics for eigenvalues in the limit when the distance between the hypersurfaces tends to zero. The asymptotics are uniform and local in the sense that the coefficients depend only on the extremal points where the ratio of the area of the Neumann boundary to the Dirichlet one is locally the biggest.

Keywords: Laplacian in tubes; Dirichlet boundary condition; Neumann boundary condition; eigenvalue asymptotics; dimension reduction; quantum waveguides; mean curvature

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1. Introduction

Let Σ be a connected orientable C^2 hypersurface (compact or non-compact) in $\mathbb{R}^d$, with $d \geqslant 2$, equipped with the Riemannian metric g induced by the embedding. The orientation is specified by a globally defined unit normal vector field $n\colon \Sigma \to \mathbb{S}^{d-1}$. Given a small positive parameter ε, we consider the tubular neighbourhood

$$(1.1) \qquad \Omega_\varepsilon := \{x + \varepsilon t n(x) \in \mathbb{R}^d;\ (x,t) \in \Sigma \times (0,1)\}.$$

We always assume that the map $(x,t) \mapsto x + \varepsilon t n(x)$ is injective on $\overline{\Sigma} \times [0,1]$; in particular, we require that the principal curvatures of Σ, $\kappa_1, \ldots, \kappa_{d-1}$, are bounded

functions. Let $-\Delta_{DN}^{\Omega_\varepsilon}$ be the Laplacian on Ω_ε, subject to Dirichlet and Neumann boundary conditions on Σ and $\Sigma_\varepsilon := \Sigma + \varepsilon n(\Sigma)$, respectively. If the boundary $\partial\Sigma$ is not empty, we impose Dirichlet boundary conditions on the remaining part of $\partial\Omega_\varepsilon$. We arrange the eigenvalues below the essential spectrum of $-\Delta_{DN}^{\Omega_\varepsilon}$ in an increasing order and repeat them according to multiplicity, $\lambda_1(\varepsilon) \leqslant \lambda_2(\varepsilon) \leqslant \lambda_3(\varepsilon) \leqslant \ldots$, with the convention that all eigenvalues are included if the essential spectrum is empty. In fact, we make the sequence always infinite by defining $\lambda_n := \inf \sigma_{\mathrm{ess}}(-\Delta_{DN}^{\Omega_\varepsilon})$ for all $n > N$, if the number of eigenvalues below the essential spectrum is a finite (possibly zero) natural number N.

The objective of this paper is to show that the d-dimensional differential operator $-\Delta_{DN}^{\Omega_\varepsilon}$ can be approximated in the limit as $\varepsilon \to 0$ by the $(d-1)$-dimensional Schrödinger-type operator

$$(1.2) \qquad H_\varepsilon := -\Delta_g + \frac{\kappa}{\varepsilon} \quad \text{on } L^2(\Sigma).$$

Here $-\Delta_g$ denotes the Laplace-Beltrami operator of Σ, subject to Dirichlet boundary conditions if $\partial\Sigma$ is not empty, and $\kappa := \kappa_1 + \ldots + \kappa_{d-1}$ is a $d-1$ multiple of the mean curvature of Σ. Note that the sign of κ depends on the choice of orientation n, that is, on the direction in which the parallel surface Σ_ε is constructed with respect to Σ, cf. Figure 1. We arrange the eigenvalues below the essential spectrum of the operator H_ε using the same conventions as above, $\mu_1(\varepsilon) \leqslant \mu_2(\varepsilon) \leqslant \mu_3(\varepsilon) \leqslant \ldots$.

In this paper we establish the following spectral asymptotics:

Theorem 1.1. *For all $n \geqslant 1$,*

$$(1.3) \qquad \lambda_n(\varepsilon) = \left(\frac{\pi}{2\varepsilon}\right)^2 + \mu_n(\varepsilon) + \mathcal{O}(1) \quad \text{as } \varepsilon \to 0.$$

This asymptotic expansion was proved previously by the author for $d = 2$ in [7]. Moreover, some form of norm-resolvent convergence of $-\Delta_{DN}^{\Omega_\varepsilon}$ to H_ε was established and the result (1.3) for $d = 3$ was announced there. In the present paper we extend the validity of formula (1.3) to any dimension and provide some details of the variational proof which were missing in [7].

Using known results about the strong-coupling/semiclassical asymptotics of eigenvalues of the Schrödinger-type operator (1.2), one has, for all $n \geqslant 1$,

$$(1.4) \qquad \mu_n(\varepsilon) = \frac{\inf \kappa}{\varepsilon} + o(\varepsilon^{-1}) \quad \text{as } \varepsilon \to 0.$$

This result seems to be well known; we refer to [4], Appendix A, for a proof in a general Euclidean case, which extends to the present situation.

Combining (1.3) with (1.4), we see that the two leading terms in the ε-expansion of $\lambda_n(\varepsilon)$ are independent of n. Furthermore, the geometry of Ω_ε is seen in these terms only *locally*, through the minimal value of the mean curvature of Σ. In view of the leading role of the mean curvature κ in the surface element of Σ_ε, cf. (2.2), we see that the minimal values of the mean curvature on Σ correspond to points for which, roughly, the Neumann boundary has "locally the largest area" with respect to the opposite Dirichlet one; see also Figure 1. The results (1.3)–(1.4) are thus consistent with the physical intuition that "Dirichlet conditions raise energies and Neumann conditions lower energies".
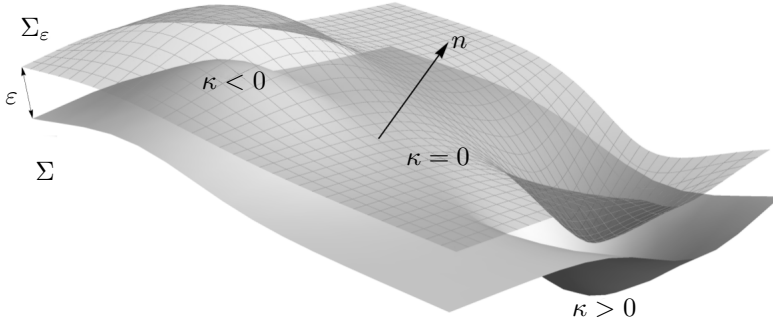

Figure 1. The geometry of the tubular neighbourhood Ω_ε for $d = 3$.

The particular form of the thin-width expansions (1.3) has important physical consequences for spectral properties of quantum waveguides as explained in [7]. Let us also mention that the local character resembles situations of Dirichlet tubes of variable radius [6], [5], [1], [2], [9].

The case of Neumann or Dirichlet tubes of uniform radius differs from the present situation in many respects. Let us denote by $\{\lambda_n^N(\varepsilon)\}_{n=1}^\infty$ and $\{\lambda_n^D(\varepsilon)\}_{n=1}^\infty$ the set of eigenvalues below the essential spectrum of the Neumann and Dirichlet Laplacian on $L^2(\Omega_\varepsilon)$, respectively, with the same conventions as used above for $\{\lambda_n(\varepsilon)\}_{n=1}^\infty$. The case of the Neumann Laplacian is trivial in the sense that its spectrum is known to converge to the spectrum of the underlying manifold Σ, cf. [10]. More precisely,

$$(1.5) \qquad \lambda_n^N(\varepsilon) = 0 + \mu_n^N + o(1) \quad \text{as} \ \ \varepsilon \to 0,$$

where $\{\mu_n^N\}_{n=1}^\infty$ is the set of eigenvalues below the essential spectrum (with the aforementioned conventions) of the Laplace-Beltrami operator $-\Delta_g$ on $L^2(\Sigma)$, subject to Neumann boundary conditions on $\partial\Sigma$. In order to consistently compare (1.5) with (1.3) (and (1.6) below), we included into (1.5) the vanishing lowest Neumann eigenvalue of the transverse interval $(0, \varepsilon)$ and will refer to μ_n^N as the "second term"

in the expansion of $\lambda_n^N(\varepsilon)$. In the Dirichlet case, we have [8]

$$(1.6) \qquad \lambda_n^D(\varepsilon) = \left(\frac{\pi}{\varepsilon}\right)^2 + \mu_n^D + \mathcal{O}(1) \quad \text{as} \ \varepsilon \to 0,$$

where $\{\mu_n^D\}_{n=1}^\infty$ is the set of eigenvalues below the essential spectrum (again with the aforementioned conventions) of the Schrödinger-type operator $-\Delta_g + V_{\mathrm{eff}}$ on $L^2(\Sigma)$, subject to Dirichlet boundary conditions on $\partial\Sigma$. Here V_{eff} is a purely geometric, ε-independent potential, expressed solely in terms of the principal curvatures,

$$(1.7) \qquad V_{\mathrm{eff}} := -\frac{\kappa_1^2 + \ldots + \kappa_{d-1}^2}{2} + \frac{(\kappa_1 + \ldots + \kappa_{d-1})^2}{4}.$$

Summing up, contrary to Theorem 1.1, in the purely Neumann or Dirichlet case the second term in the asymptotic expansion of eigenvalues is independent of ε and determined by the *global* geometry of Σ.

In addition to this introductory part, the paper consists of Section 2, in which we collect some auxiliary material, and Section 3 devoted to the proof of Theorem 1.1.

2. Preliminaries

We refer to [8] for a necessary geometric background of tubes about hypersurfaces. Using the Fermi "coordinates" (x, t) that appear in (1.1), Ω_ε can be identified with the Riemannian manifold $\Sigma \times (0, 1)$ equipped with the metric G of the block-diagonal structure $G = G_{\mu\nu}\, dx^\mu\, dx^\nu + \varepsilon^2\, dt^2$. Here the range of Greek indices is assumed to be $1, \ldots, d-1$ and the Einstein summation convention is employed. We shall not need the explicit formulae for the coefficients $G_{\mu\nu}$, just the bounds:

$$(2.1) \qquad (1 - C\varepsilon)(g_{\mu\nu}) \leqslant (G_{\mu\nu}) \leqslant (1 + C\varepsilon)(g_{\mu\nu}).$$

(Of course, we implicitly assume that ε is so small that $1 - C\varepsilon$ is positive.) Here and in the sequel, we adopt the convention that C, c and the constants involved in the "big $\mathcal{O}$" notation possibly depend on the supremum norm of the principal curvatures $\kappa_1, \ldots, \kappa_{d-1}$ and may vary from line to line. On the other hand, we shall need the formula for the determinant $|G| = \varepsilon^2 |g| h_\varepsilon^2$, where

$$(2.2) \qquad h_\varepsilon(\cdot, t) := \prod_{\mu=1}^{d-1} (1 - \varepsilon\kappa_\mu t) = 1 - \varepsilon\kappa t + \mathcal{O}(\varepsilon^2).$$

The volume element of $(\Sigma \times (0, 1), G)$ is thus given by $d\Omega_\varepsilon = \varepsilon h_\varepsilon\, d\Sigma \wedge dt$, where $d\Sigma = |g|^{1/2}\, dx^1 \wedge \ldots \wedge dx^{d-1}$ is the surface element of (Σ, g).

188

Using the above geometric preliminaries, the Hilbert space $L^2(\Omega_\varepsilon)$ can be identified with $\mathcal{H}_\varepsilon := L^2(\Sigma \times (0,1), \varepsilon h_\varepsilon \, \mathrm{d}\Sigma \wedge \mathrm{d}t)$. The Laplacian $-\Delta_{DN}^{\Omega_\varepsilon}$ can be in turn identified with the self-adjoint operator on $\mathcal{H}_\varepsilon$ associated with the quadratic form

$$Q_\varepsilon[\psi] := \langle \partial_{x^\mu} \psi, G^{\mu\nu} \partial_{x^\nu} \psi \rangle_{\mathcal{H}_\varepsilon} + \varepsilon^{-2} \|\partial_t \psi\|^2_{\mathcal{H}_\varepsilon},$$

$$\psi \in \mathfrak{D}(Q_\varepsilon) := \{\psi \in W^{1,2}(\Sigma \times (0,1)); \ \psi = 0 \quad \text{on } \partial(\Sigma \times (0,1)) \setminus (\Sigma \times \{1\})\}.$$

Here the boundary values of ψ are understood in the sense of traces. Similarly, the operator H_ε is associated with the form

$$q_\varepsilon[\varphi] := \langle \partial_{x^\mu} \varphi, g^{\mu\nu} \partial_{x^\nu} \varphi \rangle_{L^2(\Sigma)} + \varepsilon^{-1} \langle \varphi, \kappa \varphi \rangle_{L^2(\Sigma)},$$

$$\varphi \in \mathfrak{D}(q_\varepsilon) := W_0^{1,2}(\Sigma).$$

The spectral numbers $\{\lambda(\varepsilon)\}_{n=1}^\infty$ as defined above can be fully characterized by the Rayleigh-Ritz variational formula [3], Section 4.5,

$$(2.3) \qquad \lambda_n(\varepsilon) = \inf_{\mathcal{L}_n} \sup_{\psi \in \mathcal{L}_n} \frac{Q_\varepsilon[\psi]}{\|\psi\|^2_{\mathcal{H}_\varepsilon}},$$

where the infimum is taken over all n-dimensional subspaces $\mathcal{L}_n \subset \mathfrak{D}(Q_\varepsilon)$. An analogous formula holds for the spectral numbers $\{\mu(\varepsilon)\}_{n=1}^\infty$ of H_ε. It follows from (2.3) that the presence of the multiplicative factor ε in the weight of $\mathcal{H}_\varepsilon$ has no effect on the spectrum of $-\Delta_{DN}^{\Omega_\varepsilon}$.

Our strategy to prove Theorem 1.1 will be to show that the forms Q_ε and q_ε are close to each other in a sense as $\varepsilon \to 0$. Since the forms act on different Hilbert spaces, this requires a suitable identification of $\mathcal{H}_\varepsilon$ with $L^2(\Sigma)$. First, notice that it follows from (2.2) that $\mathcal{H}_\varepsilon$ (up to the irrelevant factor ε) approaches the ε-independent Hilbert space $\mathfrak{H} := L^2(\Sigma \times (0,1), \mathrm{d}\Sigma \wedge \mathrm{d}t)$. For this Hilbert space, we use the orthogonal-sum decomposition

$$(2.4) \qquad \mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_1^\perp,$$

where the subspace $\mathfrak{H}_1$ consists of functions ψ_1 such that

$$(2.5) \qquad \psi_1(x,t) = \varphi(x)\chi_1(t) \quad \text{with } \varphi \in L^2(\Sigma), \quad \chi_1(t) := \sqrt{2}\sin(\pi t/2).$$

Notice that χ_1 is the first eigenfunction of the Laplacian on $L^2((0,1))$, subject to the Dirichlet and Neumann boundary condition at 0 and 1, respectively. This operator has eigenvalues $\{(n\pi/2)^2\}_{n=1}^\infty$, where the lowest one is of course related to the leading term in (1.3). Since χ_1 is normalized, we clearly have $\|\psi_1\|_{\mathfrak{H}} = \|\varphi\|_{L^2(\Sigma)}$. Given any $\psi \in \mathfrak{H}$, we have the decomposition

$$(2.6) \qquad \psi = \psi_1 + \psi_\perp \quad \text{with } \psi_1 \in \mathfrak{H}_1, \ \psi_\perp \in \mathfrak{H}_1^\perp,$$

where ψ_1 has the form (2.5) with $\varphi(x) := \int_0^1 \psi(x,t)\chi_1(t)\,\mathrm{d}t$. Note that $\psi_1, \psi_\perp \in \mathfrak{D}(Q_\varepsilon)$ if $\psi \in \mathfrak{D}(Q_\varepsilon)$. The inclusion $\psi_\perp \in \mathfrak{H}_1^\perp$ means that

$$(2.7) \qquad \int_0^1 \psi_\perp(x,t)\chi_1(x)\,\mathrm{d}t = 0 \quad \text{for a.e. } x \in \Sigma.$$

If in addition $\psi_\perp \in \mathfrak{D}(Q_\varepsilon)$, then one can differentiate the last identity to get

$$(2.8) \qquad \int_0^1 \partial_{x^\mu}\psi_\perp(x,t)\chi_1(t)\,\mathrm{d}t = 0 \quad \text{for a.e. } x \in \Sigma.$$

Since $\mathcal{H}_\varepsilon$ and $\mathfrak{H}$ can be identified as vector spaces for any fixed $\varepsilon > 0$, the decomposition (2.4) can be equally used for each function $\psi \in \mathcal{H}_\varepsilon$. In view of the isomorphism $L^2(\Sigma) \ni \varphi \mapsto \psi_1 \in \mathfrak{H}_1$, we may think of H_ε as acting on $\mathfrak{H}_1$ as well.

3. Proof of Theorem 1.1

Expansion (1.3) will follow as a consequence of upper and lower bounds to $\lambda_n(\varepsilon)$ that have the same leading order terms in their asymptotics. It is convenient to define the shifted form $\widetilde{Q}_\varepsilon := Q_\varepsilon - \pi^2/(2\varepsilon)^2$ and focus on the first nontrivial term $\mu_n(\varepsilon)$ in (1.3). Let us decompose any $\psi \in \mathfrak{D}(Q_\varepsilon)$ according to (2.6). A straightforward calculation employing integration by parts yields

$$(3.1) \quad \|\partial_t\psi\|_{\mathcal{H}_\varepsilon}^2 - \left(\frac{\pi}{2}\right)^2\|\psi\|_{\mathcal{H}_\varepsilon}^2 = \|\partial_t\psi_\perp\|_{\mathcal{H}_\varepsilon}^2 - \left(\frac{\pi}{2}\right)^2\|\psi_1\|_{\mathcal{H}_\varepsilon}^2 - 2\varepsilon\Re\int\overline{\varphi}\chi_1'\psi_\perp\partial_t h_\varepsilon$$
$$+ \frac{\varepsilon}{2}\int|\varphi|^2\chi_1^2\partial_t^2 h_\varepsilon - \varepsilon\int_\Sigma|\varphi|^2\partial_t h_\varepsilon\Big|_{t=1}.$$

Here and in the sequel, $\int$ and $\int_\Sigma$ abbreviate the integrals over $\Sigma \times (0,1)$ and Σ with the integration measures $\mathrm{d}\Sigma \wedge \mathrm{d}t$ and $\mathrm{d}\Sigma$, respectively, and we do not write the variables on which the integrated functions depend. Using (2.2) and recalling that χ_1 is normalized, we easily verify

$$(3.2) \qquad \left|\frac{1}{\varepsilon^2}\int|\varphi|^2\chi_1^2\partial_t^2 h_\varepsilon\right| \leqslant C\int_\Sigma|\varphi|^2,$$

$$\left|-\frac{1}{\varepsilon^2}\int_\Sigma|\varphi|^2\partial_t h_\varepsilon|_{t=1} - \varepsilon^{-1}\langle\varphi, \kappa\varphi\rangle_{L^2(\Sigma)}\right| \leqslant C\int_\Sigma|\varphi|^2,$$

which reveals the source of the potential term of (1.2). At the same time, using (2.1),

$$(3.3) \qquad \pm\varepsilon^{-1}\langle\partial_{x^\mu}\psi, G^{\mu\nu}\partial_{x^\nu}\psi\rangle_{\mathcal{H}_\varepsilon} \leqslant \pm(1 \pm C\varepsilon)\langle\partial_{x^\mu}\psi, g^{\mu\nu}\partial_{x^\nu}\psi\rangle_\mathfrak{H},$$
$$\pm\varepsilon^{-1}\|\psi\|_{\mathcal{H}_\varepsilon}^2 \leqslant \pm(1 \pm C\varepsilon)\|\psi\|_\mathfrak{H}^2.$$

Here, by the normalization of χ_1 and (2.7)–(2.8),

$$(3.4) \qquad \langle \partial_{x^\mu} \psi, g^{\mu\nu} \partial_{x^\nu} \psi \rangle_{\mathfrak{H}} = \langle \partial_{x^\mu} \varphi, g^{\mu\nu} \partial_{x^\nu} \varphi \rangle_{L^2(\Sigma)} + \langle \partial_{x^\mu} \psi_\perp, g^{\mu\nu} \partial_{x^\nu} \psi_\perp \rangle_{\mathfrak{H}},$$
$$\|\psi\|_{\mathfrak{H}}^2 = \|\varphi\|_{L^2(\Sigma)}^2 + \|\psi_\perp\|_{\mathfrak{H}}^2.$$

3.1. Upper bound. Let us restrict the subspaces $\mathcal{L}_n$ in the formula (2.3) to the decoupled functions (2.5), where $\varphi \in \mathfrak{D}(q_\varepsilon)$. Using (3.1)–(3.4) with $\psi_\perp = 0$, we get the upper bound

$$(3.5) \qquad \frac{\widetilde{Q}_\varepsilon[\psi_1]}{\|\psi_1\|_{\mathcal{H}_\varepsilon}^2} \leqslant \frac{(1 + C\varepsilon)q_\varepsilon[\varphi] + C\|\varphi\|_{L^2(\Sigma)}^2}{(1 - C\varepsilon)\|\varphi\|_{L^2(\Sigma)}^2},$$

which yields

$$(3.6) \qquad \lambda_n(\varepsilon) - \left(\frac{\pi}{2\varepsilon}\right)^2 \leqslant \frac{1 + C\varepsilon}{1 - C\varepsilon}\mu_n(\varepsilon) + \frac{C}{1 - C\varepsilon}.$$

Observing that, for each $n \geqslant 1$,

$$(3.7) \qquad -\|\kappa\|_\infty \leqslant \varepsilon\nu_n - \|\kappa\|_\infty \leqslant \varepsilon\mu_n(\varepsilon) \leqslant \varepsilon\nu_n + \|\kappa\|_\infty,$$

where ν_n are the "eigenvalues" of $-\Delta_g$ as defined by (2.3), we conclude from (3.6) the desired asymptotic upper bound

$$(3.8) \qquad \lambda_n(\varepsilon) \leqslant \left(\frac{\pi}{2\varepsilon}\right)^2 + \mu_n(\varepsilon) + \mathcal{O}(1) \quad \text{as} \ \ \varepsilon \to 0.$$

It is worth noticing that the constant C in (3.6) does not depend on n; a possible dependence of the constants appearing in $\mathcal{O}(1)$ enters through the upper bound of (3.7) only.

3.2. Lower bound. As usual, lower bounds are more difficult to establish. In our situation, we need to carefully exploit the Hilbert-space decomposition (2.4). Since $\psi_\perp \in \mathfrak{H}_1^\perp$, we have $\int_0^1 |\partial_t \psi_\perp(x,t)|^2 \, \mathrm{d}t \geqslant \pi^2 \int_0^1 |\psi_\perp(x,t)|^2 \, \mathrm{d}t$ for a.e. $x \in \Sigma$. This Poincaré-type estimate extends to $\mathfrak{H}$ by Fubini's theorem. Hence, using (2.1) to estimate h_ε in $\mathrm{d}\Omega_\varepsilon$, we get

$$\|\partial_t \psi_\perp\|_{\mathcal{H}_\varepsilon}^2 - \left(\frac{\pi}{2\varepsilon}\right)^2 \|\psi_\perp\|_{\mathcal{H}_\varepsilon}^2 \geqslant \varepsilon\left[\frac{3\pi^2}{4\varepsilon^2} - C\frac{5\pi^2}{4\varepsilon}\right]\|\psi_\perp\|_{\mathfrak{H}}^2 \geqslant \varepsilon\frac{c}{\varepsilon^2}\|\psi_\perp\|_{\mathfrak{H}}^2,$$

where the second inequality holds with a positive constant c for all sufficiently small ε. Using (2.2) and the Young inequality, the last term on the right hand side of (3.1) can be estimated as follows:

$$\frac{1}{\varepsilon^2}\left|2\Re \int \overline{\varphi}\chi_1' \psi_\perp \partial_t h_\varepsilon\right| \leqslant \frac{C}{\varepsilon}2 \int |\varphi \chi_1' \psi_\perp| \leqslant \frac{C^2}{\delta}\|\varphi \chi_1'\|_{\mathfrak{H}}^2 + \frac{\delta}{\varepsilon^2}\|\psi_\perp\|_{\mathfrak{H}}^2$$

with any positive δ. Here $\|\varphi\chi_1'\|_{\mathfrak{H}} = (\pi/2)\|\varphi\|_{L^2(\Sigma)}$. Choosing δ sufficiently small and using (3.1)–(3.4), we thus get the lower bound

$$(3.9) \qquad \frac{\widetilde{Q}_\varepsilon[\psi]}{\|\psi\|_{\mathcal{H}_\varepsilon}^2} \geqslant \frac{(1-C\varepsilon)q_\varepsilon[\varphi] - C\|\varphi\|_{L^2(\Sigma)}^2 + c\varepsilon^{-2}\|\psi_\perp\|_{\mathfrak{H}}^2}{(1+C\varepsilon)(\|\varphi\|_{L^2(\Sigma)}^2 + \|\psi_\perp\|_{\mathfrak{H}}^2)}.$$

Here the numerator is in fact the quadratic form of an operator direct sum $T_\varepsilon \oplus T_\varepsilon^\perp$ with respect to the decomposition (2.4), where $T_\varepsilon := (1-C\varepsilon)H_\varepsilon - C$ and $T_\varepsilon^\perp := c\varepsilon^{-2}$. In view of (3.7), the spectrum of $T_\varepsilon^\perp$ diverges faster as $\varepsilon \to 0$ than that of T_ε. This enables us to conclude from (3.9) with help of (2.3) that for any $n \geqslant 1$ there exist C, c such that, for all $\varepsilon \leqslant c$, we have

$$(3.10) \qquad \lambda_n(\varepsilon) - \left(\frac{\pi}{2\varepsilon}\right)^2 \geqslant \frac{1-C\varepsilon}{1+C\varepsilon}\mu_n(\varepsilon) - \frac{C}{1+C\varepsilon}.$$

Using (3.7), we conclude from (3.10) the desired asymptotic lower bound

$$(3.11) \qquad \lambda_n(\varepsilon) \geqslant \left(\frac{\pi}{2\varepsilon}\right)^2 + \mu_n(\varepsilon) + \mathcal{O}(1) \quad \text{as} \ \ \varepsilon \to 0.$$

Combining (3.11) with (3.8), we complete the proof of Theorem 1.1. $\qquad\qquad\square$

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