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Abstract

We consider steady compressible Navier–Stokes–Fourier system in a bounded two-dimensional domain with the pressure law $p(\varrho, \vartheta) \sim \varrho\vartheta + \varrho \ln^\alpha(1+\varrho)$. For the heat flux $\mathbf{q} \sim -(1+\vartheta^m)\nabla\vartheta$ we show the existence of a weak solution provided $\alpha > \max\{1, 1/m\}$, $m > 0$. This improves the recent result from [11].

1 Introduction, main result

We consider the following system of partial differential equations

$$\operatorname{div}(\varrho\mathbf{u}) = 0, \tag{1.1}$$

$$\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}\mathbb{S} + \nabla p = \varrho\mathbf{f}, \tag{1.2}$$

$$\operatorname{div}(\varrho E\mathbf{u}) = \varrho\mathbf{f}\cdot\mathbf{u} - \operatorname{div}(p\mathbf{u}) + \operatorname{div}(\mathbb{S}\mathbf{u}) - \operatorname{div}\mathbf{q}. \tag{1.3}$$

It is a well-known model for steady flow of a compressible heat conducting fluid. Here, ϱ is the density of the fluid, $\mathbf{u}$ is the velocity field, $\mathbb{S}$ the viscous part of the stress tensor, p the pressure, $\mathbf{f}$ the external force, E the specific total energy and $\mathbf{q}$ the heat flux. We consider system (1.1)–(1.3) in a bounded domain $\Omega \subset \mathbb{R}^2$. At the boundary $\partial\Omega$ we assume the boundary conditions

$$\mathbf{u} = \mathbf{0}, \tag{1.4}$$

$$-\mathbf{q}\cdot\mathbf{n} + L(\vartheta - \Theta_0) = 0, \tag{1.5}$$

with $\mathbf{n}$ the outer normal to $\partial\Omega$, $L = const > 0$ and $\Theta_0 = \Theta_0(x) > 0$, both given. Furthermore, the total mass of the fluid

$$\int_\Omega \varrho \, \mathrm{d}x = M > 0 \tag{1.6}$$

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is also given.

We have to specify the constitutive relations for the quantities $\mathbb{S}$, p, E and $\mathbf{q}$. The fluid is assumed to be newtonian, i.e. we have

$$\mathbb{S} = \mathbb{S}(\vartheta, \mathbf{u}) = \mu(\vartheta)\Big[\nabla\mathbf{u} + (\nabla\mathbf{u})^T - \operatorname{div}\mathbf{u}\mathbb{I}\Big] + \xi(\vartheta)\operatorname{div}\mathbf{u}\mathbb{I}, \tag{1.7}$$

with viscosity coefficients $\mu(\vartheta)$ and $\xi(\vartheta)$. In our paper, we consider the viscosity coefficients to be given globally Lipschitz functions of the temperature ϑ such that

$$c_1(1+\vartheta) \le \mu(\vartheta) \le c_2(1+\vartheta), \qquad 0 \le \xi(\vartheta) \le c_2(1+\vartheta). \tag{1.8}$$

The heat flux satisfies the Fourier law

$$\mathbf{q} = \mathbf{q}(\vartheta, \nabla\vartheta) = -\kappa(\vartheta)\nabla\vartheta, \tag{1.9}$$

with $\kappa(\cdot) \in C([0,\infty))$ such that for a certain $m > 0$

$$c_3(1+\vartheta^m) \le \kappa(\vartheta) \le c_4(1+\vartheta^m). \tag{1.10}$$

The specific total energy E has the form

$$E = E(\varrho, \vartheta, \mathbf{u}) = \frac{1}{2}|\mathbf{u}|^2 + e(\varrho, \vartheta), \tag{1.11}$$

where e stands for the specific internal energy; we will specify this quantity below.

We consider the pressure law of the form

$$p = p(\varrho, \vartheta) = \varrho\vartheta + \frac{\varrho^2}{\varrho+1}\ln^\alpha(1+\varrho) \tag{1.12}$$

with $\alpha > 0$. In agreement with the second law of thermodynamics, there exists specific entropy, a function of ϱ and ϑ, given up to an additive constant by the Gibbs relation

$$\frac{1}{\vartheta}\Big(De(\varrho,\vartheta) + p(\varrho,\vartheta)D\Big(\frac{1}{\varrho}\Big)\Big) = Ds(\varrho,\vartheta). \tag{1.13}$$

The specific entropy, due to (1.2) and (1.3), fulfills

$$\operatorname{div}(\varrho s\mathbf{u}) + \operatorname{div}\Big(\frac{\mathbf{q}}{\vartheta}\Big) = \frac{\mathbb{S}:\nabla\mathbf{u}}{\vartheta} - \frac{\mathbf{q}\cdot\nabla\vartheta}{\vartheta^2} \tag{1.14}$$

provided all quantities are smooth enough. As a matter of fact, working with only weak solutions, we will replace (1.14) by inequality. Provided all quantities are smooth, the Gibbs relation immediately implies the Maxwell relation

$$\frac{\partial e(\varrho,\vartheta)}{\partial\varrho} = \frac{1}{\varrho^2}\Big(p(\varrho,\vartheta) - \vartheta\frac{\partial p(\varrho,\vartheta)}{\partial\vartheta}\Big) \tag{1.15}$$

which gives the specific internal energy up to an unknown function of temperature. Assuming this function linear, we get

$$e = e(\varrho,\vartheta) = \frac{\ln^{\alpha+1}(1+\varrho)}{\alpha+1} + c_v\vartheta, \qquad c_v = const > 0, \tag{1.16}$$

and

$$s(\varrho,\vartheta) = \ln \frac{\vartheta^{c_v}}{\varrho} + s_0. \tag{1.17}$$

Indeed, we could also treat more general pressure law with asymptotics as in (1.12) and (1.16), but to avoid additional technicalities we will omit it.

As already mentioned, our aim is to construct a weak solution to our problem

Definition 1 *The triple $(\varrho, \mathbf{u}, \vartheta)$ is called a weak solution to system (1.1)–(1.17), if $\varrho \geq 0$ a.e. in Ω, $\varrho \in L^r(\Omega;\mathbb{R})$, $r > 1$, $\int_\Omega \varrho \,\mathrm{d}x = M$, $\boldsymbol{u} \in W_0^{1,2}(\Omega;\mathbb{R}^2)$, $\vartheta > 0$ a.e. in Ω, $\vartheta \in W^{1,r}(\Omega;\mathbb{R})$ $\forall r < 2$, and*

$$\int_\Omega \varrho\mathbf{u}\cdot\nabla\psi \,\mathrm{d}x = 0 \qquad \forall \psi \in C^\infty(\overline{\Omega};\mathbb{R}), \tag{1.18}$$

$$\int_\Omega \big(-\varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\varphi} - p(\varrho,\vartheta)\,\mathrm{div}\,\boldsymbol{\varphi} + \mathbb{S}(\vartheta,\mathbf{u}) : \nabla\boldsymbol{\varphi}\big)\,\mathrm{d}x = \int_\Omega \varrho\mathbf{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x \quad \forall \boldsymbol{\varphi} \in C_0^\infty(\Omega;\mathbb{R}^2), \tag{1.19}$$

$$\begin{aligned} \int_\Omega -\Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\Big)\mathbf{u}\cdot\nabla\psi\,\mathrm{d}x &= \int_\Omega \big(\varrho\mathbf{f}\cdot\mathbf{u}\psi + p(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi\big)\,\mathrm{d}x \\ - \int_\Omega \big((\mathbb{S}(\vartheta,\mathbf{u})\mathbf{u}) \cdot\nabla\psi + \kappa(\vartheta)\nabla\vartheta\cdot\nabla\psi\big)\,\mathrm{d}x &- \int_{\partial\Omega} L(\vartheta-\Theta_0)\psi\,\mathrm{d}\sigma \quad \forall \psi \in C^\infty(\overline{\Omega};\mathbb{R}). \end{aligned} \tag{1.20}$$

We will also work with the renormalized solutions to the continuity equation

Definition 2 *Let $\boldsymbol{u} \in W^{1,2}_{loc}(\mathbb{R}^2;\mathbb{R}^2)$ and $\varrho \in L^q_{loc}(\mathbb{R}^2;\mathbb{R})$, $q > 1$, solve*

$$\mathrm{div}(\varrho\boldsymbol{u}) = 0 \textit{ in } \mathcal{D}'(\mathbb{R}^2).$$

Then the pair $(\varrho, \mathbf{u})$ is called a renormalized solution to the continuity equation, if

$$\mathrm{div}(b(\varrho)\mathbf{u}) + \big(\varrho b'(\varrho) - b(\varrho)\big)\,\mathrm{div}\,\boldsymbol{u} = 0 \textit{ in } \mathcal{D}'(\mathbb{R}^2)$$

for all $b \in C^1([0,\infty)) \cap W^{1,\infty}(0,\infty)$ with $zb'(z) \in L^\infty(0,\infty)$.

The main result of this paper is

Theorem 1 *Let $\Omega \in C^2$ be a bounded domain in $\mathbb{R}^2$, $\mathbf{f} \in L^\infty(\Omega;\mathbb{R}^2)$, $\Theta_0 \geq K_0 > 0$ a.e. at $\partial\Omega$, $\Theta_0 \in L^1(\partial\Omega)$, $L > 0$. Let $\alpha > \max\{1, \frac{1}{m}\}$. Then there exists a weak solution to (1.1)–(1.17) in the sense of Definition 1. Moreover, $(\varrho, \mathbf{u})$, extended by zero outside of Ω, is a renormalized solution to the continuity equation in the sense of Definition 2.*

Remark 1.1 Note that our solution also fulfills the entropy inequality, see beginning of Section 4.

This note improves the recent result from [11], where the authors showed the existence of a weak solution assuming $\alpha > 1$ and $\alpha \geq \frac{2}{m}$. The main ingredients of the proof, the strong convergence of density, is much simpler in [11] as the a priori estimates there allow to get immediately that the limit density and velocity fulfill the renormalized continuity equation which is not the case here. Moreover, note that in [11] the authors got the a priori estimates assuming $\alpha > 1$ and $\alpha \geq \frac{2}{m}$, while here, we get them for $\alpha > 1$ only. However, the bound $\alpha > \frac{1}{m}$ is needed in order to get the strong convergence of density. In [11], also the case $\gamma > 1$ (with $p(\varrho, \vartheta) = \varrho\vartheta + \varrho^\gamma$) was treated. Let us also mention the paper [13], where the two dimensional case was studied for constant viscosity and Navier boundary condition for the velocity. Assuming $L = L(\vartheta) \sim (1+\vartheta)^l$, the existence of a weak solution is shown there for $\gamma > 2$ and $m = l+1 > \frac{\gamma-1}{\gamma-2}$; however, this solution is more regular in the sense that the density is bounded and $\nabla \mathbf{u}$, $\nabla\vartheta$ belong to all $L^q(\Omega)$, $q < \infty$.

In three space dimensions, the existence of a weak solution (in some cases only of a variational solution) has been recently established in [9], [10] for $\gamma > \frac{3+\sqrt{41}}{8}$ and $m > \max\{\frac{2}{3}, \frac{2}{3(\gamma-1)}, \frac{2}{9}\frac{\gamma(4\gamma-1)}{4\gamma^2-3\gamma-2}\}$ for viscosity coefficients satisfying (1.8). The case of constant viscosity in three space dimension was considered in [7] and [8], where the existence of a solution was established for $\gamma > \frac{7}{3}$ and $m = l+1 > \frac{3\gamma-1}{3\gamma-7}$; if $\gamma > 3$ and the velocity fulfills the Navier boundary conditions, the solution has the same regularity as in [13]. Note also that paper [7] is actually the first paper giving existence of a weak solution for arbitrary large data for the steady compressible Navier–Stokes–Fourier system.

In three space dimensions, the existence of a solution for arbitrary large data is established for larger interval for γ than in the barotropic case (i.e. system (1.1)–(1.2) with $\vartheta = const$); in this case, the existence is known only for $\gamma > \frac{4}{3}$ (see [3]). This is not the case in two space dimensions, cf. [4], where the existence is known also for $p(\varrho) = a\varrho$ only. However, the linear relation between pressure and density is fundamental, as weak convergence of the density is enough to pass from the approximative system to the original, hence the proof becomes much simpler. Finally, let us also mention paper [1], where the evolutionary problem for barotropic flow was considered and the existence of a weak solution was shown for $\alpha > 1$.

In the next section, we introduce several useful tools, mostly connected with Orlicz spaces. Then we mention a priori estimates for a certain regularization of our problem and show new a priori estimates of the density sequence. Finally we pass to the limit to get existence for our problem; here the main difficulty (basically the only one) is the strong convergence of the density which will be considered in the last section.

2 Preliminaries

In what follows, we use standard notation for the Lebesgue space $L^p(\Omega)$ endowed with the norm $\|\cdot\|_{p,\Omega}$ and Sobolev spaces $W^{k,p}(\Omega)$ endowed with the norm $\|\cdot\|_{k,p,\Omega}$. If no confusion may arise, we skip the domain Ω in the norm. The vector-valued functions will be printed in boldface, the tensor-valued functions with a special font. We will use notation $\varrho \in L^p(\Omega;\mathbb{R})$, $\mathbf{u} \in L^p(\Omega;\mathbb{R}^2)$, and $\mathbb{S} \in L^p(\Omega;\mathbb{R}^{2\times 2})$. The generic constants are denoted by C and their values may vary even in the same formula or in the same line. We also use summation convention over twice repeated indeces, from 1 to 2; e.g. u_iv_i means $\sum_{i=1}^2 u_iv_i$.

We recall the following version of the Korn inequality (see e.g. [11] for the proof)

Lemma 1 *We have for $\boldsymbol{u} \in W_0^{1,2}(\Omega;\mathbb{R}^2)$, $\vartheta > 0$ and $\mathbb{S}(\vartheta,\mathbf{u})$ satisfying (1.7)–(1.8)*

$$\int_\Omega \frac{\mathbb{S}(\vartheta,\mathbf{u}) : \nabla\mathbf{u}}{\vartheta}\,\mathrm{d}x \geq C\|\mathbf{u}\|_{1,2}^2 \quad \textit{and} \quad \int_\Omega \mathbb{S}(\vartheta,\mathbf{u}) : \nabla\mathbf{u}\,\mathrm{d}x \geq C\|\mathbf{u}\|_{1,2}^2. \tag{2.1}$$

Next we recall basic properties of a certain class of Orlicz spaces. For more details as well as for proofs of results below see e.g. [5] or [6]; see also [11].

Let Φ be the Young function. We denote by $E_\Phi(\Omega)$ the set of all measurable functions u such that

$$\int_\Omega \Phi(|u(x)|)\,\mathrm{d}x < +\infty,$$

and by $L_\Phi(\Omega)$ the set of all measurable functions u such that the Luxemburg norm

$$\|u\|_\Phi = \inf\Big\{k > 0; \int_\Omega \Phi\Big(\frac{1}{k}|u(x)|\Big)dx \leq 1\Big\} < +\infty.$$

We say that Φ satisfies the Δ_2-condition if there exist $k > 0$ and $c \geq 0$ such that

$$\Phi(2t) \leq k\Phi(t) \qquad \forall t \geq c.$$

If $c = 0$, we speak about the global Δ_2-condition. Note that we have for all $u \in E_\Phi(\Omega)$

$$\|u\|_\Phi \leq \int_\Omega \Phi(|u(x)|)\,\mathrm{d}x + 1,$$

while $E_\Phi(\Omega) = L_\Phi(\Omega)$ only if Φ fulfills the Δ_2-condition. For $\alpha \geq 0$ and $\beta \geq 1$ we denote by $L_{z^\beta \ln^\alpha(1+z)}(\Omega)$ the Orlicz spaces generated by $\Phi(z) = z^\beta \ln^\alpha(1+z)$. In the range mentioned above $z^\beta \ln^\alpha(1+z)$ fulfills the global Δ_2-condition. Recall that the complementary function to $z\ln^\alpha(1+z)$ behaves as $\mathrm{e}^{z^{1/\alpha}}$; however, this function does not satisfy the Δ_2-condition. We denote by $E_{\mathrm{e}(1/\alpha)}(\Omega)$ and $L_{\mathrm{e}(1/\alpha)}(\Omega)$ the corresponding sets of functions.

First note that $W^{1,2}(\Omega) \hookrightarrow L_{\mathrm{e}^{z^2}-1}(\Omega)$ and thus

$$\|u\|_{e(2)} \leq C(\|u\|_{1,2} + 1). \tag{2.2}$$

Further, the generalized Hölder inequality yields

$$\|uv\|_1 \leq \|u\|_{z\ln^\alpha(1+z)}\|v\|_{\mathrm{e}(1/\alpha)} \tag{2.3}$$

as well as

$$\|uv\|_{z\ln^\alpha(1+z)} \leq C\|u\|_{z^p\ln^\alpha(1+z)}\|v\|_{z^{p'}\ln^\alpha(1+z)}, \tag{2.4}$$

for any $\alpha \geq 0$ and $\frac{1}{p} + \frac{1}{p'} = 1$, $1 < p, p' < \infty$. The definition of the Luxemburg norm immediately yields for $\beta \geq 1$, $\alpha \geq 0$

$$\|u\|_{z^\beta\ln^\alpha(1+z)} \leq \Big(1 + \int_\Omega |u(x)|^\beta \ln^\alpha(1+|u(x)|)\,\mathrm{d}x\Big)^{\frac{1}{\beta}} \tag{2.5}$$

as well as for $\delta > 0$

$$\||u|^\delta\|_{\Phi(z)} = \|u\|^\delta_{\Phi(z^\delta)};$$

hence, especially for $\delta > 0$

$$\||u|^\delta\|_{\mathrm{e}(\alpha)} \le C\big(\|u\|^\delta_{\mathrm{e}(\delta\alpha)} + 1\big), \tag{2.6}$$

and for $\delta \ge 1$

$$\||u|^\delta\|_{z\ln^\alpha(1+z)} \le C(\delta)\big(\|u\|^\delta_{z^\delta\ln^\alpha(1+z)} + 1\big). \tag{2.7}$$

Finally, let us consider the problem

$$\begin{aligned} \operatorname{div}\boldsymbol{\varphi} &= f \quad \text{in } \Omega \in C^{0,1}, \\ \boldsymbol{\varphi} &= \mathbf{0} \quad \text{at } \partial\Omega. \end{aligned} \tag{2.8}$$

If $f \in L^p(\Omega), 1 < p < \infty$, and $\int_\Omega f \,\mathrm{d}x = 0$, then there exists a solution to (2.10) such that

$$\|\boldsymbol{\varphi}\|_{1,p} \le C\|f\|_p, \tag{2.9}$$

see e.g. [12]. A similar result holds also in Orlicz spaces such that the Young function Φ satisfies the global Δ_2-condition and for certain $\gamma \in (0,1)$ the function Φ^γ is quasiconvex, see [14]. Hence, especially for $\alpha \ge 0$ and $\beta > 1$ we have (provided $\int_\Omega f \,\mathrm{d}x = 0$) the existence of a solution such that

$$\||\nabla\boldsymbol{\varphi}|\|_{z^\beta\ln^\alpha(1+z)} \le C\|f\|_{z^\beta\ln^\alpha(1+z)}. \tag{2.10}$$

3 Approximation, a priori estimates

We can show that for any $\delta > 0$ and β, B sufficiently large, there exists a solution $\varrho_\delta \in L^{s\beta}(\Omega;\mathbb{R})$, $s < 2$ arbitrary, $\varrho_\delta \ge 0$ a.e. in Ω, $\int_\Omega \varrho_\delta \,\mathrm{d}x = M$, $\mathbf{u}_\delta \in W^{1,2}_0(\Omega;\mathbb{R}^2)$, $\vartheta_\delta \in W^{1,2}(\Omega;\mathbb{R})$, $\vartheta_\delta > 0$ a.e. in Ω such that

$$\int_\Omega \varrho_\delta\mathbf{u}_\delta\cdot\nabla\psi \,\mathrm{d}x = 0 \tag{3.1}$$

for all $\psi \in W^{1,r}(\Omega;\mathbb{R})$, $r > \frac{2\beta}{2\beta-1}$,

$$\int_\Omega \Big(-\varrho_\delta(\mathbf{u}_\delta\otimes\mathbf{u}_\delta) : \nabla\boldsymbol{\varphi} + \mathbb{S}(\vartheta_\delta,\mathbf{u}_\delta) : \nabla\boldsymbol{\varphi} - \big(p(\varrho_\delta,\vartheta_\delta) + \delta\varrho_\delta^\beta + \delta\varrho_\delta^2\big)\operatorname{div}\boldsymbol{\varphi}\Big)\,\mathrm{d}x = \int_\Omega \varrho_\delta\mathbf{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x \tag{3.2}$$

for all $\boldsymbol{\varphi} \in W^{1,r}_0(\Omega;\mathbb{R}^2)$, $r > 2$,

$$\begin{aligned} &\int_\Omega \Big(\big(-\frac{1}{2}\varrho_\delta|\mathbf{u}_\delta|^2 - \varrho_\delta e(\varrho_\delta,\vartheta_\delta)\big)\mathbf{u}_\delta\cdot\nabla\psi + \big(\kappa(\vartheta_\delta) + \delta\vartheta_\delta^B + \delta\vartheta_\delta^{-1}\big)\nabla\vartheta_\delta\cdot\nabla\psi\Big)\,\mathrm{d}x \\ &\quad + \int_{\partial\Omega} \big(L + \delta\vartheta_\delta^{B-1}\big)(\vartheta_\delta - \Theta_0)\psi\,\mathrm{d}\sigma = \int_\Omega \varrho_\delta\mathbf{f}\cdot\mathbf{u}\psi\,\mathrm{d}x \\ &\quad + \int_\Omega \Big(\big(-\mathbb{S}(\vartheta_\delta,\mathbf{u}_\delta)\mathbf{u}_\delta + \big(p(\varrho_\delta,\vartheta_\delta) + \delta\varrho_\delta^\beta + \delta\varrho_\delta^2\big)\mathbf{u}_\delta\big)\cdot\nabla\psi + \delta\vartheta_\delta^{-1}\psi\Big)\,\mathrm{d}x \\ &\quad + \delta\int_\Omega \Big(\frac{1}{\beta-1}\varrho_\delta^\beta + \varrho_\delta^2\Big)\mathbf{u}_\delta\cdot\nabla\psi\,\mathrm{d}x \end{aligned} \tag{3.3}$$

for all $\psi \in C^1(\overline{\Omega};\mathbb{R})$; this solution also satisfies the entropy inequality

$$
\begin{aligned}
&\int_\Omega \Big(\vartheta_\delta^{-1}\mathbb{S}(\vartheta_\delta,\mathbf{u}_\delta):\nabla\mathbf{u}_\delta + \delta\vartheta_\delta^{-2} + \big(\kappa(\vartheta_\delta)+\delta\vartheta_\delta^B+\delta\vartheta_\delta^{-1}\big)\frac{|\nabla\vartheta_\delta|^2}{\vartheta_\delta^2}\Big)\psi \,\mathrm{d}x \\
(3.4)\qquad &\leq \int_\Omega \Big(\big(\kappa(\vartheta_\delta)+\delta\vartheta_\delta^B+\delta\vartheta_\delta^{-1}\big)\frac{\nabla\vartheta_\delta\cdot\nabla\psi}{\vartheta_\delta} - \varrho_\delta s(\varrho_\delta,\vartheta_\delta)\mathbf{u}_\delta\cdot\nabla\psi\Big)\,\mathrm{d}x \\
&\quad + \int_{\partial\Omega}\frac{L+\delta\vartheta_\delta^{B-1}}{\vartheta_\delta}(\vartheta_\delta-\Theta_0)\psi\,\mathrm{d}\sigma
\end{aligned}
$$

for all nonnegative $\psi \in C^1(\overline{\Omega};\mathbb{R})$.

More details can be found in [11], however, the detailed proof is performed in [9] in the case of three dimensional domains. The proof in two space dimensions is slightly easier, but contains all the main difficulties of the 3D problem and can be easily deduced from the proof in [9].

Moreover, using in (3.4) and (3.3) $\psi \equiv 1$ we can deduce (see again [9] and [11] for more details) the following a priori estimates

(3.5)

$$
\|\mathbf{u}_\delta\|_{1,2}+\|\nabla\vartheta_\delta^{\frac{m}{2}}\|_2+\|\nabla\ln\vartheta_\delta\|_2+\|\vartheta_\delta^{-1}\|_{1,\partial\Omega}+\delta\big(\|\nabla\vartheta_\delta^{\frac{B}{2}}\|_2^2+\|\nabla\vartheta_\delta^{-\frac12}\|_2^2+\|\vartheta_\delta\|_r^{B-2}+\|\vartheta_\delta^{-2}\|_1\big)\leq C,
$$

$r<\infty$, arbitrary, and

$$
(3.6)\qquad \|\vartheta_\delta^{\frac{m}{2}}\|_{1,2}^{\frac{2}{m}} \leq C\Big(1+\|\vartheta_\delta\|_{1,\partial\Omega}+\|\nabla\vartheta_\delta^{\frac{m}{2}}\|_2^{\frac{2}{m}}\Big)\leq C\Big(1+\Big|\int_\Omega \varrho_\delta\mathbf{f}\cdot\mathbf{u}_\delta\,\mathrm{d}x\Big|\Big)
$$

with C independent of δ. Note that the fundamental estimate (3.5), i.e. the control of $\|\mathbf{u}_\delta\|_{1,2}$ independently of the density, follows from the entropy inequality (3.4) and the form of the viscosity coefficients (1.8) due to the Korn inequality (2.1).

It remains to deduce estimates of density. To this aim, we take $0<s<1$, arbitrary, and consider the problem

$$
(3.7)\qquad
\begin{aligned}
\operatorname{div}\boldsymbol{\varphi} &= \varrho_\delta^s - \frac{1}{|\Omega|}\int_\Omega \varrho_\delta^s\,\mathrm{d}x \qquad \text{a.e. in } \Omega,\\
\boldsymbol{\varphi} &= \mathbf{0} \qquad \text{at } \partial\Omega,\\
\|\boldsymbol{\varphi}\|_{1,q} &\leq C\|\varrho_\delta\|_{qs}^s,
\end{aligned}
$$

and use this test function in (3.2). We get

$$
\begin{aligned}
&\int_\Omega\Big(\frac{\varrho_\delta^{2+s}}{1+\varrho_\delta}\ln^\alpha(1+\varrho_\delta)+\varrho_\delta^{1+s}\vartheta_\delta\Big)\,\mathrm{d}x+\delta\int_\Omega\big(\varrho_\delta^{\beta+s}+\varrho_\delta^{2+s}\big)\,\mathrm{d}x\\
(3.8)\qquad &=\frac{1}{|\Omega|}\int_\Omega\varrho_\delta^s\,\mathrm{d}x\int_\Omega\Big(\frac{\varrho_\delta^2}{\varrho_\delta+1}\ln^\alpha(1+\varrho_\delta)+\varrho_\delta\vartheta_\delta+\delta(\varrho_\delta^\beta+\varrho_\delta^2)\Big)\,\mathrm{d}x-\int_\Omega\varrho_\delta\mathbf{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\\
&\quad+\int_\Omega\mathbb{S}(\vartheta_\delta,\mathbf{u}_\delta):\nabla\boldsymbol{\varphi}\,\mathrm{d}x-\int_\Omega\varrho_\delta(\mathbf{u}_\delta\otimes\mathbf{u}_\delta):\nabla\boldsymbol{\varphi}\,\mathrm{d}x\\
&=J_1+J_2+J_3+J_4.
\end{aligned}
$$

The estimates of J_1 and J_2 are easy; hence we concentrate ourselves only on the remaining two terms. We have due to (2.2), (2.3) and (2.5) for $\alpha\geq 0$

$$
\begin{aligned}
|J_3| &\leq C\int_\Omega(1+\vartheta_\delta)|\nabla\mathbf{u}_\delta||\nabla\boldsymbol{\varphi}|\,\mathrm{d}x\leq C\|\nabla\mathbf{u}_\delta\|_2\|\nabla\boldsymbol{\varphi}\|_{\frac{1+s^2}{s}}\big(1+\|\vartheta_\delta\|_{\frac{2(1+s^2)}{(1-s)^2}}\big)\\
&\leq C\|\varrho_\delta\|_{1+s^2}^s\Big(1+\Big|\int_\Omega\varrho_\delta\mathbf{f}\cdot\mathbf{u}_\delta\,\mathrm{d}x\Big|\Big)\leq C+\frac14\int_\Omega\frac{\varrho_\delta^{2+s}}{1+\varrho_\delta}\ln^\alpha(1+\varrho_\delta)\,\mathrm{d}x
\end{aligned}
$$

and the last term can be shifted to the left-hand side. Note that we needed here $s < 1$. Finally, using (2.3), (2.6), (2.4), (2.10), (2.7) and (2.5), for $\alpha > 1$,

$$
\begin{aligned}
|J_4| &\leq \int_\Omega \varrho_\delta |\mathbf{u}_\delta|^2 |\nabla \boldsymbol{\varphi}| \, \mathrm{d}x \leq C \| |\mathbf{u}_\delta|^2 \|_{\mathrm{e}(1)} \| \varrho_\delta |\nabla \boldsymbol{\varphi}| \|_{z \ln(1+z)} \\
&\leq C \big(\| |\mathbf{u}_\delta| \|^2_{\mathrm{e}(2)} + 1 \big) \| \varrho_\delta \|_{z^{1+s} \ln(1+z)} \| |\nabla \boldsymbol{\varphi}| \|_{z^{\frac{1+s}{s}} \ln(1+z)} \leq C (1 + \| \varrho_\delta \|^{1+s}_{z^{1+s} \ln(1+z)}) \\
&\leq C \Big(1 + \int_\Omega \varrho_\delta^{1+s} \ln(1+\varrho_\delta) \, \mathrm{d}x \Big) \leq C + \frac{1}{4} \int_\Omega \frac{\varrho_\delta^{2+s}}{1+\varrho_\delta} \ln^\alpha(1+\varrho_\delta) \, \mathrm{d}x
\end{aligned}
$$

(hint: consider separately $\varrho_\delta \leq K$ and $\varrho_\delta \geq K$ for K sufficiently large). Thus we are left with the estimate

$$
\int_\Omega \varrho_\delta^{1+s} \ln^\alpha(1+\varrho_\delta) \, \mathrm{d}x \leq C(s) \tag{3.9}
$$

with $C(s) \to +\infty$ for $s \to 1^-$.

Remark 3.1 In [11], the authors used the same test function with $s = 1$. This leads to L^2-estimate of the density (and hence the limit $(\varrho, \mathbf{u})$ is immediately the renormalized solution to the continuity equation), however, also to additional bound of α on m. Note that in this paper we are able to get the estimate for any $m > 0$ only for $\alpha > 1$; nevertheless, a certain restriction on α in terms of m will appear later, when proving the strong convergence of density.

4 Limit passage

From a priori estimates proved above, i.e. using (3.5), (3.6) and (3.9), we deduce the existence of a subsequence denoted again by ϱ_δ, ϑ_δ and $\mathbf{u}_\delta$ such that

$$
\varrho_\delta \rightharpoonup \varrho \quad \text{in } L^s(\Omega; \mathbb{R}), \ \forall s < 2, \tag{4.1}
$$

$$
\mathbf{u}_\delta \rightharpoonup \mathbf{u} \quad \text{in } W^{1,2}_0(\Omega; \mathbb{R}^2), \qquad \mathbf{u}_\delta \to \mathbf{u} \quad \text{in } L^q(\Omega; \mathbb{R}^2), \ \forall q < \infty, \tag{4.2}
$$

$$
\vartheta_\delta \rightharpoonup \vartheta \quad \text{in } W^{1,r}(\Omega; \mathbb{R}), \ \forall r < 2, \quad \vartheta_\delta \to \vartheta \quad \text{in } L^q(\Omega; \mathbb{R}), \ \forall q < \infty. \tag{4.3}
$$

Hence, passing to the limit in (3.1)–(3.3) (we may also pass to the limit in (3.4) to get the entropy inequality for the limit problem, but we will not use this fact later on) we get (here and in the sequel, $\overline{g(\varrho, \vartheta, \mathbf{u})}$ denotes weak limit of the sequence $g(\varrho_\delta, \vartheta_\delta, \mathbf{u}_\delta)$)

$$
\int_\Omega \varrho \mathbf{u} \cdot \nabla \psi \, \mathrm{d}x = 0 \tag{4.4}
$$

for all $\psi \in W^{1,r}(\Omega; \mathbb{R})$, $r > 2$,
(4.5)

$$
\int_\Omega \Big(- \varrho (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\varphi} + \mathbb{S}(\vartheta, \mathbf{u}) : \nabla \boldsymbol{\varphi} - \Big(\overline{\frac{\varrho^2}{\varrho+1} \ln^\alpha(\varrho+1)} + c_v \varrho \vartheta \Big) \operatorname{div} \boldsymbol{\varphi} \Big) \, \mathrm{d}x = \int_\Omega \varrho \mathbf{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}x
$$

for all $\boldsymbol{\varphi} \in W_0^{1,r}(\Omega;\mathbb{R}^2)$, $r>2$,

$$
\begin{aligned}
\int_\Omega \Big(\big(-\frac{1}{2}\varrho|\mathbf{u}|^2 - \frac{1}{\alpha+1}\overline{\varrho\ln^{\alpha+1}(\varrho+1)} - c_v\varrho\vartheta\big)\mathbf{u}\cdot\nabla\psi + \kappa(\vartheta)\nabla\vartheta\cdot\nabla\psi\Big)\,\mathrm{d}x \\
+\int_{\partial\Omega} L(\vartheta-\Theta_0)\psi\,\mathrm{d}\sigma = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\psi\,\mathrm{d}x \\
+\int_\Omega \Big(-\mathbb{S}(\vartheta,\mathbf{u})\mathbf{u} + \Big(\overline{\frac{\varrho^2}{\varrho+1}\ln^\alpha(\varrho+1) + c_v\varrho\vartheta}\Big)\mathbf{u}\Big)\cdot\nabla\psi\,\mathrm{d}x
\end{aligned}
\tag{4.6}
$$

for all $\psi\in W^{1,r}(\Omega;\mathbb{R})$, $r>2$. Note that we have also used the fact that $\delta\int_\Omega \varrho_\delta^{\beta+s}\,\mathrm{d}x \le C$ for all $s<1$. Hence, the last step which remains to prove is the strong convergence of density in order to remove bars in (4.5) and (4.6). However, unlike [11], we do not know whether the limit function satisfies the continuity equation in the renormalized sense (cf. Definition 2) as we do not control density in $L^2(\Omega;\mathbb{R})$. Hence we have to proceed differently, similarly as in [9], [10] (cf. [1], [2] in the evolutionary case). We introduce the cut-off functions $T_k(z)=kT(\frac{z}{k})$, $k\in\mathbb{N}$, with

$$
T(z)=z,\ 0<z\le 1,\qquad T(z)=2,\ z\ge 3,\qquad T(z)\ \text{concave on}\ (0,\infty)
$$

and we aim to show the following version of the effective viscous flux identity

$$
\overline{p(\varrho,\vartheta)T_k(\varrho)} - \big(\mu(\vartheta)+\xi(\vartheta)\big)\overline{T_k(\varrho)\operatorname{div}\mathbf{u}} = \overline{p(\varrho,\vartheta)}\,\overline{T_k(\varrho)} - \big(\mu(\vartheta)+\xi(\vartheta)\big)\overline{T_k(\varrho)}\operatorname{div}\mathbf{u}.
\tag{4.7}
$$

First we recall several useful general results. We denote

$$
\begin{aligned}
\nabla\Delta^{-1}v \equiv \mathcal{F}^{-1}\Big[\frac{\mathrm{i}\xi}{|\xi|^2}\mathcal{F}(v)(\xi)\Big],\quad (\mathcal{R}[v])_{ij}\equiv(\nabla\otimes\nabla\Delta^{-1})_{ij}v = \mathcal{F}^{-1}\Big[\frac{\xi_i\xi_j}{|\xi|^2}\mathcal{F}(v)(\xi)\Big],\\
(\mathcal{R}[\mathbf{v}])_i = \mathcal{F}^{-1}\Big[\frac{\xi_i\xi_j}{|\xi|^2}\mathcal{F}(v_j)(\xi)\Big]
\end{aligned}
\tag{4.8}
$$

with $\mathcal{F}$ the Fourier transform.

We have

Lemma 2 (Continuity properties of $\nabla\otimes\nabla\Delta^{-1}$ and $\nabla\Delta^{-1}$) *Operator $\mathcal{R}$ is a continuous operator from $L^p(\mathbb{R}^2;\mathbb{R})$ to $L^p(\mathbb{R}^2;\mathbb{R}^{2\times 2})$ for any $1<p<\infty$.*

Operator $\nabla\Delta^{-1}$ is a continuous linear operator from the space $L^1(\mathbb{R}^2;\mathbb{R})\cap L^2(\mathbb{R}^2;\mathbb{R})$ to $L^2(\mathbb{R}^2;\mathbb{R}^2)+L^\infty(\mathbb{R}^2;\mathbb{R}^2)$ and from $L^p(\mathbb{R}^2;\mathbb{R})$ to $L^{\frac{2p}{2-p}}(\mathbb{R}^2;\mathbb{R}^2)$ for any $1<p<2$.

Lemma 3 (Commutators I) *Let $\mathbf{U}_\delta\rightharpoonup\mathbf{U}$ in $L^p(\mathbb{R}^2;\mathbb{R}^2)$, $v_\delta\rightharpoonup v$ in $L^q(\mathbb{R}^2;\mathbb{R})$, where*

$$
\frac{1}{p}+\frac{1}{q}=\frac{1}{r}<1.
$$

Then

$$
v_\delta\mathcal{R}[\mathbf{U}_\delta]-\mathcal{R}[v_\delta]\mathbf{U}_\delta \rightharpoonup v\mathcal{R}[\mathbf{U}]-\mathcal{R}[v]\mathbf{U}
$$

in $L^r(\mathbb{R}^2;\mathbb{R}^2)$.

Lemma 4 (Commutators II) *Let $w \in W^{1,r}(\mathbb{R}^2;\mathbb{R})$, $\mathbf{z} \in L^p(\mathbb{R}^2;\mathbb{R}^2)$, $1 < r < 2$, $1 < p < \infty$, $\frac{1}{r} + \frac{1}{p} - \frac{1}{2} < \frac{1}{s} < 1$. Then for all such s we have*

$$\|\mathcal{R}[w\mathbf{z}] - w\mathcal{R}[\mathbf{z}]\|_{a,s,\mathbb{R}^2} \leq C\|w\|_{1,r,\mathbb{R}^2}\|\mathbf{z}\|_{p,\mathbb{R}^2},$$

where $\frac{a}{2} = \frac{1}{s} + \frac{1}{2} - \frac{1}{p} - \frac{1}{r}$.

We are in position to prove

Lemma 5 *Under the assumptions used to deduce (4.4)–(4.6) we have the effective viscous flux identity (4.7) fulfilled for arbitrary $k \in \mathbb{N}$.*

Proof. We use as test function in (3.2) $\boldsymbol{\varphi} = \zeta(x)\nabla\Delta^{-1}(1_\Omega T_k(\varrho_\delta))$ and in (4.5) $\boldsymbol{\varphi} = \zeta(x)\nabla\Delta^{-1}(1_\Omega\overline{T_k(\varrho)})$ with $\zeta(x) \in C_0^\infty(\Omega;\mathbb{R})$. As $\nabla\Delta^{-1}(1_\Omega T_k(\varrho_\delta)) \to \nabla\Delta^{-1}(1_\Omega\overline{T_k(\varrho)})$ in $C(\overline{\Omega};\mathbb{R}^2)$, recalling (4.1)–(4.3), we deduce

$$\begin{aligned}
&\lim_{\delta\to 0^+}\int_\Omega \zeta(x)\Big(p(\varrho_\delta,\vartheta_\delta)T_k(\varrho_\delta) - \mathbb{S}(\vartheta_\delta,\mathbf{u}_\delta) : \mathcal{R}[1_\Omega T_k(\varrho_\delta)]\Big)\,\mathrm{d}x \\
&= \int_\Omega \zeta(x)\Big(\overline{p(\varrho,\vartheta)}\;\overline{T_k(\varrho)} - \mathbb{S}(\vartheta,\mathbf{u}) : \mathcal{R}[1_\Omega\overline{T_k(\varrho)}]\Big)\,\mathrm{d}x \\
&+ \lim_{\delta\to 0^+}\int_\Omega \zeta(x)\Big(T_k(\varrho_\delta)\mathbf{u}_\delta\cdot\mathcal{R}[1_\Omega\varrho_\delta\mathbf{u}_\delta] - \varrho_\delta(\mathbf{u}_\delta\otimes\mathbf{u}_\delta) : \mathcal{R}[1_\Omega T_k(\varrho_\delta)]\Big)\,\mathrm{d}x \\
&- \int_\Omega \zeta(x)\Big(\overline{T_k(\varrho)}\mathbf{u}\cdot\mathcal{R}[1_\Omega\varrho\mathbf{u}] - \varrho(\mathbf{u}\otimes\mathbf{u}) : \mathcal{R}[1_\Omega\overline{T_k(\varrho)}]\Big)\,\mathrm{d}x.
\end{aligned}$$

Now we apply Lemma 3 with

$$\begin{aligned}
v_\delta = T_k(\varrho_\delta) \rightharpoonup \overline{T_k(\varrho)} \text{ in } L^q(\Omega;\mathbb{R}), \quad \forall q < \infty, \\
\mathbf{U}_\delta = \varrho_\delta\mathbf{u}_\delta \rightharpoonup \varrho\mathbf{u} \text{ in } L^p(\Omega;\mathbb{R}^2), \quad \forall p < 2.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_\Omega \zeta(x)\Big(\mathbf{u}_\delta\cdot\Big(\mathcal{R}[1_\Omega\varrho_\delta\mathbf{u}_\delta]T_k(\varrho_\delta) - \mathcal{R}[1_\Omega T_k(\varrho_\delta)]\varrho_\delta\mathbf{u}_\delta\Big)\Big)\,\mathrm{d}x \\
&\to \int_\Omega \zeta(x)\Big(\mathbf{u}\cdot\Big(\mathcal{R}[1_\Omega\varrho\mathbf{u}]\overline{T_k(\varrho)} - \mathcal{R}[1_\Omega\overline{T_k(\varrho)}]\varrho\mathbf{u}\Big)\Big)\,\mathrm{d}x.
\end{aligned}$$

Next we write

$$\begin{aligned}
&\lim_{\delta\to 0^+}\int_\Omega \zeta(x)\mathbb{S}(\vartheta_\delta,\mathbf{u}_\delta) : \mathcal{R}[1_\Omega T_k(\varrho_\delta)]\,\mathrm{d}x = \lim_{\delta\to 0^+}\int_\Omega \zeta(x)\big(\mu(\vartheta_\delta) + \xi(\vartheta_\delta)\big)\operatorname{div}\mathbf{u}_\delta T_k(\varrho_\delta)\,\mathrm{d}x \\
&+ \lim_{\delta\to 0^+}\int_\Omega T_k(\vartheta_\delta)\Big(\mathcal{R} : \Big[\zeta(x)\mu(\vartheta_\delta)\big(\nabla\mathbf{u}_\delta + (\nabla\mathbf{u}_\delta)^T\big)\Big] - \\
&\qquad \zeta(x)\mu(\vartheta_\delta)\mathcal{R} : \big[\nabla\mathbf{u}_\delta + (\nabla\mathbf{u}_\delta)^T\big]\Big)\,\mathrm{d}x \\
&+ \lim_{\delta\to 0^+}\int_\Omega T_k(\vartheta_\delta)\Big(\mathcal{R} : \Big[\zeta(x)(\xi(\vartheta_\delta) - \mu(\vartheta_\delta))\operatorname{div}\mathbf{u}_\delta\mathbb{I}\Big] - \\
&\qquad \zeta(x)(\xi(\vartheta_\delta) - \mu(\vartheta_\delta))\mathcal{R} : \big[\operatorname{div}\mathbf{u}_\delta\mathbb{I}\big]\Big)\,\mathrm{d}x,
\end{aligned}$$

similarly for the limit term, and apply Lemma 4 with $w_\delta = \zeta(x)\mu(\vartheta_\delta) \sim (1+\vartheta_\delta)$ bounded in $W^{1,r}(\Omega;\mathbb{R})$ with $r < 2$ arbitrary (recall that $\mu(\cdot)$ is Lipschitz continuous) and $z_i =$

$\partial_i(u_\delta)_j + \partial_j(u_\delta)_i$, $j = 1, 2, 3$, bounded in $L^2(\Omega;\mathbb{R}^2)$. Note that the last term on the right hand-side is identically zero. Hence we get (4.7). $\square$

The next and in fact the most important change with respect to [11] appears at this moment. As we do not know whether the pair $(\varrho, \mathbf{u})$ is a renormalized solution to the continuity equation, we have to show it by means of another technique. To this aim, we introduce the oscillation defect measure defined in a more general context of the Orlicz spaces

$$\mathrm{osc}_\Phi[T_k(\varrho_\delta) - T_k(\varrho)] = \sup_{k\in\mathbb{N}} \limsup_{\delta\to 0^+} \|T_k(\varrho_\delta) - T_k(\varrho)\|_\Phi.$$

In what follows, we show that there exists $\sigma > 0$ such that

$$\mathrm{osc}_{z^2 \ln^\sigma(1+z)}[T_k(\varrho_\delta) - T_k(\varrho)] < +\infty; \tag{4.9}$$

further we verify that this fact implies the renormalized continuity equation to be satisfied. Note that to show the latter we cannot use the approach from [2] (or [9]) as there it is required that $\Phi = z^{2+\sigma}$ for $\sigma > 0$ which are not able to show.

We recall one lemma concerning properties of weakly convergent sequences. For the proof see e.g. [2, Appendix].

Lemma 6 (Weak convergence, monotone functions) *Let $(P, G) \in C(\mathbb{R}) \times C(\mathbb{R})$ be a couple of nondecreasing functions. Assume that $\varrho_n \in L^1(\Omega;\mathbb{R})$ is a sequence such that*

$$\left.\begin{array}{r} P(\varrho_n) \rightharpoonup \overline{P(\varrho)}, \\ G(\varrho_n) \rightharpoonup \overline{G(\varrho)}, \\ P(\varrho_n)G(\varrho_n) \rightharpoonup \overline{P(\varrho)G(\varrho)} \end{array}\right\} \text{ in } L^1(\Omega;\mathbb{R}).$$

i) *Then*

$$\overline{P(\varrho)}\;\overline{G(\varrho)} \le \overline{P(\varrho)G(\varrho)}$$

a.e. in Ω.

ii) *If, in addition,*

$$G(z) = z, \quad P \in C(\mathbb{R}), P \text{ non-decreasing}$$

and

$$\overline{P(\varrho)}\;\varrho = \overline{P(\varrho)\varrho}$$

(where we have denoted by $\varrho = \overline{G(\varrho)}$), then

$$\overline{P(\varrho)} = P(\varrho).$$

First, we have

Lemma 7 *Under the assumptions of Theorem 1 we have (4.9).*

Proof. As $g(t) = \frac{t^2}{t+1}\ln^\alpha(t+1)$ is for $\alpha > 1$ convex, we get similarly as in [1, Lemma 3.9] that for $z > y \ge 0$

$$\begin{aligned} &\frac{z^2}{1+z}\ln^\alpha(1+z) - \frac{y^2}{1+y}\ln^\alpha(1+y) = \int_y^z g'(t)\,\mathrm{d}t \ge \int_y^z g'(t-y)\,\mathrm{d}t \\ &= \frac{(z-y)^2}{1+z-y}\ln^\alpha(1+z-y) \ge \frac{1}{2}(z-y)\ln^\alpha(1+z-y) - \ln^\alpha 2\, 1_{\{z-y\le 1\}}. \end{aligned} \tag{4.10}$$

Moreover,

$$
\begin{aligned}
&\limsup_{\delta\to 0^+}\int_\Omega \Big[p(\varrho_\delta,\vartheta_\delta)T_k(\varrho_\delta) - \overline{p(\varrho,\vartheta)}\,\overline{T_k(\varrho)}\Big]\ \mathrm{d}x \\
&= \limsup_{\delta\to 0^+}\Big[\int_\Omega (g(\varrho_\delta)-g(\varrho))\big(T_k(\varrho_\delta)-T_k(\varrho)\big)\ \mathrm{d}x + \int_\Omega \big(\varrho_\delta\vartheta_\delta T_k(\varrho_\delta) - \varrho\vartheta\overline{T_k(\varrho)}\big)\ \mathrm{d}x\Big] \\
&\quad + \int_\Omega \big(\overline{g(\varrho)}-g(\varrho)\big)\big(T_k(\varrho)-\overline{T_k(\varrho)}\big)\ \mathrm{d}x.
\end{aligned}
$$

As $z\mapsto T_k(z)$ is concave and $z\mapsto g(z)$ is convex, we have (using also Lemma 6 in the second integral)

$$
\begin{aligned}
&\limsup_{\delta\to 0^+}\int_\Omega (g(\varrho_\delta)-g(\varrho))\big(T_k(\varrho_\delta)-T_k(\varrho)\big)\ \mathrm{d}x \\
&\le \limsup_{\delta\to 0^+}\int_\Omega \Big[p(\varrho_\delta,\vartheta_\delta)T_k(\varrho_\delta) - \overline{p(\varrho,\vartheta)}\,\overline{T_k(\varrho)}\Big]\ \mathrm{d}x.
\end{aligned}
\tag{4.11}
$$

Thus, due to (4.10) and Lipschitz continuity of $T_k(\cdot)$ with Lipschitz constant 1, together with (2.5), we arrive at

$$
\begin{aligned}
&\limsup_{\delta\to 0^+}\|T_k(\varrho_\delta)-T_k(\varrho)\|^2_{z^2\ln^\alpha(z+1)} \\
&\le 1+\limsup_{\delta\to 0^+}\int_\Omega |T_k(\varrho_\delta)-T_k(\varrho)|^2\ln^\alpha(1+|T_k(\varrho_\delta)-T_k(\varrho|)\ \mathrm{d}x \\
&\le C\Big(1+\limsup_{\delta\to 0^+}\int_\Omega \Big[p(\varrho_\delta,\vartheta_\delta)T_k(\varrho_\delta) - \overline{p(\varrho,\vartheta)}\,\overline{T_k(\varrho)}\Big]\ \mathrm{d}x\Big).
\end{aligned}
\tag{4.12}
$$

Denote now $G_k^\delta(x,z) = |T_k(z)-T_k(\varrho(x))|^2\ln^\alpha(1+|T_k(z)-T_k(\varrho(x))|)$. Hence, due to (4.7)

$$
\overline{G_k(\cdot,\varrho)} \le C\Big((\mu(\vartheta)+\xi(\vartheta))\big(\overline{T_k(\varrho)\operatorname{div}\mathbf{u}} - \overline{T_k(\mathbf{u})}\operatorname{div}\mathbf{u}\big)+1\Big)
$$

for all $k\ge 1$. Then

$$
\begin{aligned}
\int_\Omega (1+\vartheta)^{-1}\overline{G_k(\cdot,\varrho)}\ \mathrm{d}x &\le C\Big(\sup_{\delta>0}\|\operatorname{div}\mathbf{u}_\delta\|_2\limsup_{\delta\to 0^+}\|T_k(\varrho_\delta)-T_k(\varrho)\|_2+1\Big) \\
&\le C\Big(\limsup_{\delta\to 0^+}\|T_k(\varrho_\delta)-T_k(\varrho)\|_2+1\Big).
\end{aligned}
\tag{4.13}
$$

On the other hand, take $\sigma>0$, $s>1$ and compute

$$
\begin{aligned}
&\int_\Omega |T_k(\varrho_\delta)-T_k(\varrho)|^2\ln^\sigma(1+|T_k(\varrho_\delta)-T_k(\varrho)|)\ \mathrm{d}x \\
&\le \int_\Omega |T_k(\varrho_\delta)-T_k(\varrho)|^2\ln^\sigma(1+|T_k(\varrho_\delta)-T_k(\varrho)|)(1+\vartheta)^{-s}(1+\vartheta)^s\ \mathrm{d}x \\
&\le C\|1+\vartheta^s\|_{e(\frac{m}{s})}\||T_k(\varrho_\delta)-T_k(\varrho)|^2\ln^\sigma(1+|T_k(\varrho_\delta)-T_k(\varrho)|)(1+\vartheta)^{-s}\|_{z\ln^{\frac{s}{m}}(1+z)} \\
&\le C\Big(1+\int_\Omega |T_k(\varrho_\delta)-T_k(\varrho)|^2\ln^\alpha(1+|T_k(\varrho_\delta)-T_k(\varrho)|)(1+\vartheta)^{-1}\ \mathrm{d}x\Big)
\end{aligned}
$$

provided $\alpha>\frac{1}{m}$ and $\sigma>0$, $s-1>0$ are sufficiently small, with respect to $\alpha-\frac{1}{m}$. Thus we have shown that for a certain $\sigma>0$ it holds, due to (4.13),

$$
\begin{aligned}
&\limsup_{\delta\to 0^+}\int_\Omega |T_k(\varrho_\delta)-T_k(\varrho)|^2\ln^\sigma(1+|T_k(\varrho_\delta)-T_k(\varrho)|)\ \mathrm{d}x \\
&\le C\Big(1+\limsup_{\delta\to 0^+}\Big(\int_\Omega |T_k(\varrho_\delta)-T_k(\varrho)|^2\ \mathrm{d}x\Big)^{\frac{1}{2}}\Big),
\end{aligned}
$$

and hence

$$\limsup_{\delta\to0^+}\int_\Omega |T_k(\varrho_\delta)-T_k(\varrho)|^2\ln^\sigma(1+|T_k(\varrho_\delta)-T_k(\varrho)|)\ \mathrm{d}x \le C<+\infty \tag{4.14}$$

with C independent of k. □

Next we have to show that (4.9) is sufficient to guarantee that $(\varrho,\mathbf{u})$ verifies the renormalized continuity equation. As the proof is the same (even slightly easier) than in the evolutionary case, we recall that Lemma 4.5 in [1] gives us this result. Recall that the main idea is to show that

$$\limsup_{k\to\infty}\int_\Omega |b'(\overline{T_k(\varrho)})|\overline{(T_k(\varrho)-T_k'(\varrho)\varrho)\operatorname{div}\mathbf{u}}\ \mathrm{d}x=0$$

for $b(\cdot)$ sufficiently smooth such that $b'(z)\equiv 0$ for z sufficiently large. In order to get it, we need to control $T_k(\varrho_\delta)-T_k(\varrho)$ in a slightly better space than $L^2(\Omega)$ and (4.9) is sufficient to this aim.

Finally, let us show the strong convergence of the density. We use the renormalized continuity equation with

$$b(\varrho)=\varrho\int_1^\varrho \frac{T_k(t)}{t^2}\,\mathrm{d}t.$$

Thus

$$\int_\Omega T_k(\varrho)\operatorname{div}\mathbf{u}\ \mathrm{d}x=0; \tag{4.15}$$

recall also that

$$\int_\Omega T_k(\varrho_\delta)\operatorname{div}\mathbf{u}_\delta\ \mathrm{d}x=0,$$

whence

$$\int_\Omega \overline{T_k(\varrho)\operatorname{div}\mathbf{u}}\ \mathrm{d}x=0. \tag{4.16}$$

Coming back to the effective viscous flux identity (4.7) and using (4.15) with (4.16),

$$\int_\Omega \frac{1}{\mu(\vartheta)+\xi(\vartheta)}\Big(\overline{p(\varrho,\vartheta)T_k(\varrho)}-\overline{p(\varrho,\vartheta)}\,\overline{T_k(\varrho)}\Big)\ \mathrm{d}x=\int_\Omega \big(T_k(\varrho)-\overline{T_k(\varrho)}\big)\operatorname{div}\mathbf{u}\ \mathrm{d}x. \tag{4.17}$$

We also know that

$$\lim_{k\to\infty}\|T_k(\varrho)-\varrho\|_1=\lim_{k\to\infty}\|\overline{T_k(\varrho)}-\varrho\|_1=0;$$

it yields

$$\lim_{k\to\infty}\|T_k(\varrho)-\overline{T_k(\varrho)}\|_1=0. \tag{4.18}$$

Recalling (4.14) and (4.18) it is easy to see (consider the sets where $|T_k(\varrho)-\overline{T_k(\varrho)}|$ is larger and smaller than a suitably chosen number)

$$\lim_{k\to\infty}\|T_k(\varrho)-\overline{T_k(\varrho)}\|_2=0;$$

thus

$$\lim_{k\to\infty} \int_\Omega \left(\overline{p(\varrho,\vartheta)T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\,\overline{T_k(\varrho)} \right) \,\mathrm{d}x = 0 \tag{4.19}$$

and by (4.10) and (4.11)

$$\lim_{k\to\infty} \limsup_{\delta\to 0^+} \int_\Omega \frac{|T_k(\varrho_\delta) - T_k(\varrho)|^3}{1 + |T_k(\varrho_\delta) - T_k(\varrho)|} \ln^\alpha(1 + |T_k(\varrho_\delta) - T_k(\varrho)|) \,\mathrm{d}x = 0. \tag{4.20}$$

Now we write

$$\|\varrho_\delta - \varrho\|_1 \le \|\varrho_\delta - T_k(\varrho_\delta)\|_1 + \|T_k(\varrho_\delta) - T_k(\varrho)\|_1 + \|T_k(\varrho) - \varrho\|_1,$$

and we immediately see

$$\varrho_\delta \to \varrho \quad \text{in } L^1(\Omega;\mathbb{R})$$

as well as by interpolation

$$\varrho_\delta \to \varrho \quad \text{in } L^p(\Omega;\mathbb{R}) \quad \forall p < 2.$$

Theorem 1 is proved.

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