Navier–Stokes–Fourier system with general inflow/outflow boundary conditions

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Motto

Closed systems

Clausius:

> *The energy of the world is constant; its entropy tends to a maximum.*

Open systems

Ergodic hypothesis:

> *Time averages along trajectories of the flow converge, for large enough times, to an ensemble average given by a certain probability measure.*

Thermodynamically complete model in fluid mechanics

Mass conservation – equation of continuity

$$\partial_\varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Newton's second law – momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S} + \varrho \mathbf{f}$$

First law of thermodynamics – energy balance

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e \right) + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e + p \right) \mathbf{u} \right] + \mathrm{div}_x \mathbf{q} = \varrho \mathbf{f} \cdot \mathbf{u} + \mathrm{div}_x (\mathbb{S} \cdot \mathbf{u})$$

Second law of thermodynamics – entropy

Gibbs' equation

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right), \ s - -\text{entropy}$$

Second law of thermodynamics – entropy balance

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = (\geq)\frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right)$$

Thermodynamic stability hypothesis

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0, \ \varrho > 0, \ \vartheta > 0$$

$$\Leftrightarrow$$

$$S = \varrho s, \ (\varrho, S) \mapsto E_{\mathrm{int}} \equiv \varrho e(\varrho, S) \text{ is (strictly) convex}$$

Rheology: Navier–Stokes–Fourier system

Newton's rheological law

$$\mathbb{S} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}$$

$$\mu > 0, \ \eta \geq 0$$

Fourier's law

$$\mathbf{q} = -\kappa \nabla_x \vartheta, \ \kappa > 0$$

Boundary conditions

Physical space

$$\Omega \subset R^d, \; d = 1,2,3 \text{ a bounded domain with smooth boundary } \partial\Omega$$

Boundary velocity: inflow/ouflow boundary conditions

$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_B$$

$$\Gamma_{\rm in} = \left\{ x \in \partial\Omega \;\middle|\; \mathbf{u}_B(x)\cdot\mathbf{n} < 0 \right\}, \; \Gamma_{\rm out} = \left\{ x \in \partial\Omega \;\middle|\; \mathbf{u}_B(x)\cdot\mathbf{n} > 0 \right\}$$

Density influx

$$\varrho = \varrho_B \text{ on } \Gamma_{\rm in}$$

Internal energy flux

$$\Big[\varrho_B e \mathbf{u}_B + \mathbf{q} \Big] \cdot \mathbf{n} = F_{i,B} \text{ on } \Gamma_{\rm in}$$

$$\mathbf{q}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega \setminus \Gamma_{\rm in}$$

Conservative boundary conditions

No–slip

$$\mathbf{u}|_{\partial\Omega} = 0$$

No (heat) flux

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Long–time behavior EF, H. Petzeltová 2007

$$\mathbf{f} = \nabla_x F(x) \ \Rightarrow \ \text{any (weak) solution tends to equilibrium as} t \to \infty$$

$$\mathbf{f} = \mathbf{f}(x) \neq \nabla_x F(x) \Rightarrow \ E \equiv \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e \right] \ \mathrm{d}x \to \infty \text{ as } t \to \infty$$

Weak solutions I – mechanics

Equation of continuity

$$\left[\int_\Omega \varrho \varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} + \int_0^\tau \int_{\Gamma_{\mathrm{in}}} \varphi \varrho_B \mathbf{u}_B \cdot \mathbf{n} \ \mathrm{d}\sigma_x \mathrm{d}t + \boxed{\int_0^\tau \int_{\Gamma_{\mathrm{out}}} \varphi \varrho \mathbf{u}_B \cdot \mathbf{n} \ \mathrm{d}\sigma_x \mathrm{d}t}$$

$$= \int_0^\tau \int_\Omega \left[\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \right] \ \mathrm{d}x \mathrm{d}t$$

for any $0 \leq \tau \leq T$, and any $\varphi \in C^1([0,T] \times \overline{\Omega})$

Momentum balance

$$\left[\int_\Omega \varrho \mathbf{u} \cdot \boldsymbol{\varphi} \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \boldsymbol{\varphi} + p \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x \mathrm{d}t$$

$$- \int_0^\tau \int_\Omega \mathbb{S} : \nabla_x \boldsymbol{\varphi} \ \mathrm{d}x \mathrm{d}t + \int_0^\tau \int_\Omega \varrho \mathbf{f} \cdot \boldsymbol{\varphi} \ \mathrm{d}x \mathrm{d}t$$

for any $0 \leq \tau \leq T$, and any $\boldsymbol{\varphi} \in C^1_c([0,T] \times \Omega; R^d)$

$$\mathbf{u} - \mathbf{u}_B \in L^q(0,T; W^{1,q}_0(\Omega; R^d)), \ q > 1$$

Weak solutions II – total energy

Total energy balance

$$\left[\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}-\mathbf{u}_B|^2 + \varrho e\right)\psi \ \mathrm{d}x\right]_{t=0}^{t=\tau}$$
$$- \int_0^\tau \partial_t\psi \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}-\mathbf{u}_B|^2 + \varrho e\right) \ \mathrm{d}x\mathrm{d}t$$
$$+ \boxed{\int_0^\tau \psi \int_{\Gamma_{\rm out}} E_{\rm int}(\varrho, S)\mathbf{u}_B\cdot\mathbf{n} \ \mathrm{d}\sigma_x\mathrm{d}t}$$
$$\leq \int_0^\tau \psi \int_\Omega \mathbb{S} : \nabla_x \mathbf{u}_B \ \mathrm{d}x\mathrm{d}t + \int_0^\tau \psi \int_\Omega \varrho\mathbf{f}\cdot(\mathbf{u}-\mathbf{u}_B) \ \mathrm{d}x\mathrm{d}t$$
$$+ \frac{1}{2}\int_0^\tau \psi \int_\Omega \varrho\mathbf{u}\cdot\nabla_x|\mathbf{u}_B|^2 \ \mathrm{d}x\mathrm{d}t$$
$$- \int_0^\tau \psi \int_\Omega \left(\varrho\mathbf{u}\otimes\mathbf{u} + p\mathbb{I}\right) : \nabla_x\mathbf{u}_B \ \mathrm{d}x\mathrm{d}t - \int_0^\tau \psi \int_{\Gamma_{\rm in}} F_{i,B} \ \mathrm{d}\sigma_x\mathrm{d}t$$

for $0 \leq \tau \leq T$ and any $\psi \in C^1[0,T]$, $\psi \geq 0$

Weak solutions III – entropy

Entropy inequality

$$\left[\int_\Omega \varrho s \varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} - \int_0^\tau \int_\Omega \left[\varrho s \partial_t \varphi + \varrho s \mathbf{u} \cdot \nabla_x \varphi + \left(\frac{\mathbf{q}}{\vartheta} \right) \cdot \nabla_x \varphi \right] \ \mathrm{d}x\mathrm{d}t$$

$$+ \boxed{ \int_0^\tau \int_{\Gamma_{\mathrm{out}}} \varphi S \mathbf{u}_B \cdot \mathbf{n} \ \mathrm{d}\sigma_x \mathrm{d}t }$$

$$\geq \int_0^\tau \int_\Omega \frac{\varphi}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right) \ \mathrm{d}x\mathrm{d}t$$

$$- \int_0^\tau \int_{\Gamma_{\mathrm{in}}} \varphi \left(\frac{F_{i,B}}{\vartheta} + \left[s(\varrho_B, \vartheta) - \frac{e(\varrho_B, \vartheta)}{\vartheta} \right] \varrho_B \mathbf{u}_B \cdot \mathbf{n} \right) \mathrm{d}\sigma_x \mathrm{d}t$$

for a.a. $0 \leq \tau \leq T$, and any $\varphi \in C^1([0, T] \times \overline{\Omega})$, $\varphi \geq 0$

Relative energy

Relative energy in the standard variables

$$E\left(\varrho, \mathbf{u}, \vartheta \,\middle|\, \widetilde{\varrho}, \widetilde{\mathbf{u}}, \widetilde{\vartheta}\right) = \frac{1}{2}\varrho|\mathbf{u}-\widetilde{\mathbf{u}}|^2 + H_{\widetilde{\vartheta}}(\varrho,\vartheta) - \frac{\partial H_{\widetilde{\vartheta}}(\widetilde{\varrho},\widetilde{\vartheta})}{\partial \varrho}(\varrho-\widetilde{\varrho}) - H_{\widetilde{\vartheta}}(\widetilde{\varrho},\widetilde{\vartheta})$$

Ballistic free energy:

$$H_{\widetilde{\vartheta}}(\varrho,\vartheta) \equiv \varrho\Big(e(\varrho,\vartheta) - \widetilde{\vartheta}s(\varrho,\vartheta)\Big),$$

Relative energy in the conservative–entropy variables

$$E\left(\varrho, \mathbf{m}, S \,\middle|\, \widetilde{\varrho}, \widetilde{\mathbf{m}}, \widetilde{S}\right) = E\left(\varrho, \mathbf{u}, \vartheta \,\middle|\, \widetilde{\varrho}, \widetilde{\mathbf{u}}, \widetilde{\vartheta}\right)$$

$$\mathbf{m} = \varrho\mathbf{u},\ \widetilde{\mathbf{m}} = \widetilde{\varrho}\widetilde{\mathbf{u}},\ S = \varrho e(\varrho,\vartheta),\ \widetilde{S} = \widetilde{\varrho}e(\widetilde{\varrho},\widetilde{\vartheta})$$

Relative energy as a Bregman distance

Relative energy in the conservative–entropy variables

$$E\left(\varrho, \mathbf{m}, S \,\middle|\, \widetilde{\varrho}, \widetilde{\mathbf{m}}, \widetilde{S}\right)$$

$$= E(\varrho, \mathbf{m}, S) - \partial_{\varrho,\mathbf{m},S} E(\widetilde{\varrho}, \widetilde{\mathbf{m}}, \widetilde{S}) \cdot (\varrho - \widetilde{\varrho}, \mathbf{m} - \widetilde{\mathbf{m}}, S - \widetilde{S}) - E(\widetilde{\varrho}, \widetilde{\mathbf{m}}, \widetilde{S})$$

Bregman distance

thermodynamic stability

$$\Rightarrow$$

convexity of $(\varrho, \mathbf{m}, S) \mapsto E(\varrho, \mathbf{m}, S)$

$$\Rightarrow$$

$$E\left(\varrho, \mathbf{m}, S \,\middle|\, \widetilde{\varrho}, \widetilde{\mathbf{m}}, \widetilde{S}\right) \geq 0$$

$$E\left(\varrho, \mathbf{m}, S \,\middle|\, \widetilde{\varrho}, \widetilde{\mathbf{m}}, \widetilde{S}\right) = 0 \ \Leftrightarrow \ (\varrho, \mathbf{m}, S) = (\widetilde{\varrho}, \widetilde{\mathbf{m}}, \widetilde{S})$$

Technical hypotheses - EOS

Equation of state

$$p(\varrho,\vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4,\ \ a > 0,\ \ P \in C^1[0,\infty)$$

$$e(\varrho,\vartheta) = \frac{3}{2}\frac{\vartheta^{5/2}}{\varrho} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{\varrho}\vartheta^4,$$

$$s(\varrho,\vartheta) = \mathcal{S}\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4a}{3}\frac{\vartheta^3}{\varrho},$$

$$\mathcal{S}'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2}.$$

$$P'(Z) > 0 \text{ for any } Z \geq 0,\ \ \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} > 0 \text{ for any } Z > 0.$$

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0.$$

$$P(0) = 0,\ \ \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} < c \text{ for all } Z > 0.$$

Technical hypotheses - transport coefficients

Viscosity coefficients

$$\underline{\mu}(1+\vartheta^\Lambda) \leq \mu(\vartheta) \leq \overline{\mu}(1+\vartheta^\Lambda), \ |\mu'(\vartheta)| < c \text{ for all } \vartheta \in [0,\infty), \ \frac{1}{2} \leq \Lambda \leq 1$$

$$0 \leq \eta(\vartheta) \leq \overline{\eta}(1+\vartheta^\Lambda) \text{ for all } \vartheta \in [0,\infty)$$

Heat conductivity

$$\underline{\kappa}(1+\vartheta^3) \leq \kappa(\vartheta) \leq \overline{\kappa}(1+\vartheta^3) \text{ for all } \vartheta \in [0,\infty)$$

Technical hypotheses – data

Driving force

$$\mathbf{f} \in L^{\infty}(\Omega; R^d)$$

Inflow data

$$\varrho_B \in C^1, \ \varrho_B > 0, \ \mathbf{u}_B \in C^2$$

Heat (internal energy) flow

$$\sup_{x \in \Gamma_{\rm in}} \left(\frac{F_{i,B}}{|\mathbf{u}_B \cdot \mathbf{n}|}(x) + \frac{3}{2} p_\infty \varrho_B^{5/3}(x) \right) < 0$$

Results

Global existence – EF, A. Novotný 2020

There exists a global–in–time weak solution for any finite energy initial data

Weak–strong uniqueness – EF, A. Novotný 2020

A weak and strong solution solution of the problem with the same initial/boundary data coincide on the life span of the strong solution

Time periodic solutions – A. Abbatiello, EF 2020

Existence of time periodic solutions for the barotropic problem driven by time periodic boundary data in the case of a "hard sphere" pressure EOS