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Algebraic classification of tensors in Lorentzian geometry and its applications

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Résumé

In this work, we describe an algebraic classification of tensors on Lorentzian manifolds and higher-dimensional generalizations of NP and GHP formalisms. We have developed these methods in collaborations with colleagues from Halifax, Canada and Cambridge, UK. As we point out in section 1.5, these methods have been already used by various authors in distinct general relativity research areas, such as numerical relativity, study of uniqueness, construction of new exact solutions, study of asymptotic behaviour of spacetimes, etc.

We defer technical aspects of these methods to sections 5 and 6 and first we focus on various applications of these methods developed in our own research. Namely in section 2, we discuss our results on universal spacetimes that are vacuum solutions to any geometric theory of gravity with the Lagrangian being a curvature invariant constructed from the curvature tensor and its covariant derivatives of arbitrary order.

In section 3, we discuss spacetimes with vanishing curvature invariants (VSI spacetimes) and we arrive at the necessary and sufficient conditions for the VSI property. We also discuss an extension of the VSI property to other non-curvature tensors, such as p-forms in generalized electromagnetism and we find that a VSI p-form necessarily "lives" in a degenerate Kundt spacetime.

In section 4, we present a generalization of the Goldberg-Sachs theorem to the case of five dimensions, which has been recently used to find new five-dimensional solutions to the Einstein equations [1].

Finally in the sections 5 and 6, we briefly summarize the algebraic classification of tensors and some aspects of the higher-

dimensional NP and GHP formalisms. For a more complete overview of these methods, we refer the reader to our topical review [2].

1 Introduction

It is notoriously difficult to solve the Einstein field equations using analytical methods without any assumptions leading to a considerable simplification of the field equations. An obvious simplifying assumption is to require an appropriate symmetry of the solution. In 1950s and 1960s, it turned out that instead of assuming symmetries of the metric, it is often efficient to make specific simplifying assumptions on the curvature tensor. This led to the development of the so called Petrov, Petrov-Penrose or algebraic classification of the Weyl tensor. The algebraic classification of the Weyl tensor in conjunction with the so-called Newman-Penrose formalism constitute powerful mathematical tools for studying various mathematical and physical aspects of the Einstein equations. Since 1970s, they have been considered standard methods of mathematical relativity. These methods have been also used to construct new exact solutions to the Einstein equations, for example, the famous Kerr solution describing a rotating black hole. In fact, as can be seen from the classical book on exact solutions of the Einstein equations [3], most exact solutions of the Einstein equations known today are indeed algebraically special.

1.1 Modified theories of gravity

In recent years, researchers have also started to consider various modifications of Einstein gravity which on the physics side are motivated e.g. by quantum corrections to the Einstein equations, dark matter, dark energy, string theory etc. Most of these modifications describe gravitational field in geometrical terms as in general relativity, i.e., in terms of curvature of a spacetime. However, the modified theories replace the Einstein-Hilbert Lagrangian with more general Lagrangians constructed from the metric, the Riemann tensor R_{abcd} and its covariant derivatives of an arbitrary order

$$L = L(g_{ab}, R_{abcd}, \nabla_{a_1} R_{bcde}, \ldots, \nabla_{a_1 \ldots a_p} R_{bcde}). \tag{1.1}$$

In general, the field equations of such theories are of the $(2p+4)^{\text{th}}$ order in derivatives of the metric and considerably more involved than the Einstein equations. Consequently, for most theories, very few exact solutions are known.

For example, in quadratic gravity, only linear and quadratic terms in curvature are considered in the action

$$S = \int \mathrm{d}^4 x \sqrt{-g} \left(\gamma' \left(R - 2\Lambda_0 \right) - \alpha' C_{abcd} C^{abcd} + \beta' R^2 \right). \tag{1.2}$$

The first non-trivial vacuum solution of the resulting field equations of quadratic gravity was found in 1990 in [4]. It represents a certain type of gravitational pp-waves with a non-trivial Ricci tensor. Recently, we have extended this class of known solutions in [5, 6]. Remarkably, in 2015 it was discovered that in quadratic gravity, static and spherically symmetric vacuum black holes more general than the Schwarzschild black holes exist [7].

1.2 Gravity in higher dimensions

In recent years, also the study of theories of gravity in more than four dimensions has attracted significant interest. It has become apparent that solutions of higher-dimensional gravity exhibit much richer behaviour, which is often qualitatively different from the four-dimensional case. As a striking example in five dimensions, we mention the five-dimensional rotating black ring [8] describing an asymptotically flat spacetime with a black hole with an event horizon of topology $S^1 \times S^2$. In the four-dimensional Einstein gravity, the no-hair theorem states that the only stationary vacuum black hole is the Kerr black hole. In five dimensions, a spherical black hole and a black ring with the same mass and angular momentum can exist. Thus the standard four-dimensional uniqueness theorem does not have a simple generalization to the case of five dimensions.

1.3 Algebraic classification of tensors in higher dimensions

Motivated by the success of the Petrov classification of the Weyl tensor and the Newman-Penrose formalism in four-dimensional mathematical relativity, we sought (in collaboration with A. Coley and R. Milson, Halifax, Canada) an appropriate generalization of these methods to higher dimensions.

In [9, 10] (see also [2] for a recent topical review), we proposed an invariant[1] classification of an arbitrary tensor on a Lorentzian manifold of arbitrary dimension based on the existence and multiplicity of aligned null directions. In the case of the Weyl tensor in four dimensions, our classification is equivalent to the Petrov classification distinguishing algebraic types I, II, D, III, N, and O. However, it also allows us to classify the Weyl tensor in dimensions $n > 4$ as well and furthermore,

[1] Classification independent on the choice of the coordinates or the frame.

this classification extends to other tensors on Lorentzian manifolds. For instance, in the context of generalized gravities, it is often useful to classify not only the Weyl tensor but also the Riemann and Ricci tensors, their covariant derivatives, the Bach tensor etc.

Similarly as in four dimensions, most known exact solutions of higher-dimensional gravity, including e.g. Myers-Perry rotating black holes, are algebraically special[2], see [2].

1.4 NP and GHP formalisms in higher dimensions

In order to be able to perform frame calculations for higher-dimensional algebraically special spacetimes, we have developed a generalization of the Newman-Penrose (NP) formalism [11, 12] to the case of arbitrary dimension. Later in [13], we have also developed (in collaboration with H. Reall and M. Durkee, Cambridge, UK) a higher dimensional version of the Geroch-Held-Penrose (GHP) formalism which is often more efficient. These methods are briefly described in Sections 5 and 6, for a more thorough discussion, see [2, 13].

1.5 Applications and outline of this work

These higher-dimensional methods and, in particular the higher-dimensional generalization of the GHP formalism, have been employed by various authors to study a wide range of research problems, for example

- stability of certain black hole solutions [14, 15],

[2] Algebraically special spacetimes are spacetimes admitting a multiple Weyl aligned null direction (WAND) - see Sec. 5.

- uniqueness of selected exact solutions [16],
- construction of new exact solutions of the Einstein equations [1],
- numerical relativity [14, 17],
- study of near horizon geometries [15, 18],
- study of asymptotic properties of gravitational and electromagnetic fields [19–21].

However, in this thesis we focus only on applications studied by our group. We demonstrate that the algebraic classification and NP/GHP formalisms provide a powerful mathematical tool allowing to study various problems in theories of gravity.

As an illustration, let us briefly discuss the so called *universal spacetimes*. Already in late 80's and early 90's, it was shown in [22, 23] in the context of string theory that certain pp-waves are "immune" to all possible corrections of the field equations, i.e., they simultaneously solve vacuum equations of *all* gravitational theories with an arbitrary Lagrangian of the form (1.1) - hence universal spacetimes.

In [24] and [25], we have studied (in collaboration with S. Hervik, Stavanger, Norway) universal spacetimes and we have found much more general classes of these spacetimes in arbitrary dimension. Interestingly, for some classes the results strongly differ for prime number dimensions and composite number dimensions. Our results on universal spacetimes are discussed in section 2.

In section 3, we focus on spacetimes with vanishing scalar curvature invariants (VSI). In the Riemannian case (where the metric is positive definite), the only space with vanishing curvature invariant is the flat space. However, we show that in the Lorentzian case, one arrives at various non-trivial VSI metrics.

In four dimensions, the Goldberg-Sachs theorem is the key starting point for the construction of algebraically special solutions of the Einstein equations. In section 4, we present a generalization of the Goldberg-Sachs theorem to five dimensions found in collaboration with H. Reall (Cambridge, UK). In 2015-2016, this theorem was used by the Cambridge group to construct new exact solutions of the five-dimensional Einstein gravity [26].

Although the algebraic classification of tensors itself and the higher-dimensional generalization of the NP/GHP formalisms constitute one of the key parts of the results presented in this thesis, we have decided to postpone the brief overview of these methods to sections 5 and 6 in order to devote the first few sections to applications to demonstrate the effectiveness of these mathematical tools in proving various results and not to discourage the reader by a rather technical discussion from the start.

Finally, let us thank to all co-authors of the joint research papers presented and quoted in this thesis, namely J. Bičák, A. Coley, M. Durkee, S. Hervik, T. Málek, R. Milson, M. Ortaggio, A. Pravdová, J. Podolský, H. Reall and R. Švarc.

2 Universal spacetimes

This section is devoted to the study of *universal spacetimes* and it is based on our results published in [24] and [25].

Obviously in general, vacuum solutions to the Einstein gravity are not vacuum solutions to the more general theories of gravity. However, specific classes of spacetimes exist which are to some degree “immune” to certain corrections of the field equations. For example, in [5] we have shown that in arbitrary dimension all Weyl type N vacuum solutions to the Einstein gravity are also vacuum solutions to the quadratic gravity (1.2). Thus these vacuum solutions to the Einstein gravity are immune to the quadratic corrections to the Einstein-Hilbert action.

Here, we want to focus on spacetimes of arbitrary dimension that are immune to *all* possible corrections of the Einstein gravity that can be expressed in terms of curvature tensors and their derivatives - the so called universal spacetimes.

The formal definition of universal spacetimes proposed in [27] and generalized to k-universal metrics in [25] reads

Definition 2.1. A metric is called *k-universal* if all conserved symmetric rank-2 tensors constructed[3] from the metric, the Riemann tensor and its covariant derivatives up to the k^{th} order are multiples of the metric. If a metric is k-universal for all integers k then it is called *universal*.

We say that a rank-2 tensor $\boldsymbol{T}$ is conserved iff it obeys $T^{ab}{}_{;b} = 0$.

From the above definition it follows that universal spacetimes are vacuum solutions to any field equations following

[3] We consider only scalars constructed as contractions of *polynomials* from the metric, the Riemann tensor and its covariant derivatives of an arbitrary order or their analytic functions.

from the action

$$L = L(g_{ab}, R_{abcd}, \nabla_{a_1} R_{bcde}, \ldots, \nabla_{a_1 \ldots a_p} R_{bcde}), \qquad (2.1)$$

where L is an arbitrary scalar polynomial function of its arguments (or more generally, an analytic function of such polynomials).

As we have mentioned in the introduction, first examples of universal spacetimes have been already found in [22, 23] in the context of string theory. We set out to systematically study universal spacetimes in arbitrary dimension and have found considerably more general classes of such spacetimes.

In [24], we have proven that

Theorem 2.2. *A universal spacetime is necessarily a CSI spacetime.*

This means that all curvature invariants for a universal spacetime are constant. However, note that CSI is not a sufficient condition for universality. Nevertheless, this result considerably narrows down possible candidates for universal spacetimes and allows us to systematically study universal spacetimes of various algebraic types introduced in the section 5.

For type N spacetime of arbitrary dimension, we have proven a theorem with a simple necessary and sufficient condition for universality [24]

Theorem 2.3. *A type N spacetime is universal if and only if it is an Einstein Kundt spacetime.*

Thus the problem of universality for type N is essentially solved since Kundt spacetimes and in particularly type N Kundt spacetimes are well studied in literature (see section 6 for the definition of the Kundt spacetimes and [2, 28] for further references).

Universal pp-waves of [22,23] are special examples of spacetimes belonging to the Kundt class. However, the type N Kundt metrics also allow for a non-vanishing cosmological constant Λ and in contrast with pp-waves, these spacetimes need not admit a covariantly constant null vector.

It turns out that universal spacetimes of more general algebraic types than the Weyl type N also exist. For type III, we have obtained [24]

Theorem 2.4. *Einstein type III, $\tau_i = 0$, Kundt spacetimes obeying $C_{acde}C_b{}^{cde} = 0$ are universal.*

Explicit examples of type III and N universal metrics can be found in [24].

The above results on universality of type III and N spacetimes hold for arbitrary dimension of the spacetime. In contrast, in [25] for type II, we have obtained results which critically depend on the dimensionality of a spacetime.

We have found explicit examples of type II universal spacetimes for all composite number dimensions. On the other hand, we have found no type II universal spacetimes for prime number dimensions. For five dimensions, we have proven

Theorem 2.5. *In five dimensions, genuine type II universal spacetimes do not exist.*

The problem of the existence of type II universal spacetimes for prime number dimensions $n > 5$ remains open.

3 VSI spacetimes

In the previous sections, we have stated that all universal spacetimes are CSI (i.e. with constant scalar invariants). An interesting subset of CSI spacetimes are the so called *VSI spacetimes* - spacetimes with vanishing curvature invariants[4]. First examples of VSI spacetimes beyond pp-waves were identified in our work [29] already in 1998.

Note that it is not possible to distinguish a VSI spacetime from the Minkowski spacetime using curvature invariants. In the Riemannian case (where the metric is positive-definite), the only space with vanishing curvature invariant is a flat space.

Employing the higher-dimensional NP formalism, we arrived at necessary and sufficient conditions for VSI spacetimes in four [30] and higher dimensions [31].

Theorem 3.1 (VSI theorem [30, 31]). *A Lorentzian manifold of arbitrary dimension is VSI if, and only if, the following two conditions are both satisfied:*

(A) The spacetime possesses a non-expanding, twistfree, shearfree, geodesic null vector field $\boldsymbol{\ell}$, i.e. it belongs to the Kundt class.

(B) Relative to $\boldsymbol{\ell}$, the Riemann tensor is of type III or more special.

In fact, the algebraic classification of tensors [9] and [10] and the first partial generalization of the Newman-Penrose for-

[4] Curvature invariants are polynomial invariants constructed from the metric, the Riemann tensor and its covariant derivatives of an arbitrary order.

malism to higher dimensions [11] have been developed in the context of the study of VSI spacetimes [30, 31].

Now, let us extend the definition of the VSI property to arbitrary tensors [32]

Definition 3.2. A tensor in a spacetime with a metric g_{ab} is VSI$_i$ if the scalar polynomial invariants constructed from the tensor itself and its covariant derivatives up to order i ($i = 0, 1, 2, 3, \dots$) vanish. It is VSI if all its scalar polynomial invariants of *arbitrary order* vanish. As in [31, 33], if the Riemann tensor of g_{ab} is VSI (or VSI$_i$), the spacetime itself is said to be VSI (or VSI$_i$).

The following result has been proven by S. Hervik for VSI$_0$ tensors in [34]

Theorem 3.3. *A tensor is VSI$_0$ iff it is of type III (or more special).*

Thus the algebraic classification of tensors gives a clear necessary and sufficient condition for the VSI$_0$ property, independently on the tensor studied, dimensionality of the background manifold and its metric.

In contrast with the VSI$_0$ condition, the VSI condition strongly constrains not only the tensor itself, but also the geometry of the background manifold.

In the case of a p-form we have arrived at the following theorem [32]

Theorem 3.4. *The following two conditions are equivalent:*

1. *a non-zero p-form field* $\boldsymbol{F}$ *is VSI in a spacetime with metric* g_{ab}
2. (a) $\boldsymbol{F}$ *possesses a multiple null direction* $\boldsymbol{\ell}$*, i.e., it is of type N*
   (b) $\pounds_{\boldsymbol{\ell}}\boldsymbol{F} = 0$
   (c) g_{ab} *is a* degenerate Kundt *metric, and* $\boldsymbol{\ell}$ *is the corresponding Kundt null direction.*

Thus VSI p-forms can exist only in degenerate Kundt spacetimes.

4 A generalization of the Goldberg-Sachs theorem to higher dimensions

In this section, we focus on attempts to generalize the Goldberg-Sachs theorem to higher dimensions. In this theorem, geodeticity of multiple WANDs and number of independent components of the so called optical matrix ρ_{ij} (introduced in the section 6) for algebraically special Einstein spacetimes are studied. This theorem leads to a considerable simplification of the Einstein field equations for algebraically special spacetimes. In 2009, Durkee and Reall [35] proved that all algebraically special spacetimes admit a *geodetic* multiple WAND. Therefore, without loss of generality one can restrict the attention to geodetic multiple WANDs. In the case of five dimensions, we have proven [36]

Theorem 4.1. *In a five-dimensional algebraically special Einstein spacetime that is not conformally flat, there exists a geodetic multiple WAND* $\boldsymbol{\ell}$ *and one can choose the orthonormal basis vectors* $\boldsymbol{m}^{(i)}$ *so that the optical matrix of* $\boldsymbol{\ell}$ *takes one of the forms*

$$i) \qquad \boldsymbol{\rho} = b \begin{pmatrix} 1 & a & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 + a^2 \end{pmatrix}, \tag{4.1}$$

$$ii) \qquad \boldsymbol{\rho} = b \begin{pmatrix} 1 & a & 0 \\ -a & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.2}$$

$$iii) \qquad \boldsymbol{\rho} = b \begin{pmatrix} 1 & a & 0 \\ -a & -a^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.3}$$

where a*,* b *are functions that may vary in spacetime. If the spacetime is type III or type N, then the form must be* ii) [11].

A 3×3 orthogonal matrix has three parameters so one can eliminate at most three parameters from a 3×3 optical matrix by a change of orthonormal basis. Hence, the canonical form of a general optical matrix has 9 - 3 = 6 parameters. The theorem 4.1 shows that for algebraically special Einstein spacetimes, this is reduced to the two parameters a and b. Some partial results have been also obtained in dimensions $n > 5$. In particular, for type N we have obtained already in [11]

Proposition 4.2. *In a type N Einstein spacetime of dimension $n \geq 4$, one can choose the orthonormal basis vectors $\boldsymbol{m}^{(i)}$ so that the optical matrix of the unique multiple WAND $\boldsymbol{\ell}$ (which is geodesic) takes the form*

$$\boldsymbol{\rho} = b \begin{pmatrix} 1 & a & 0 \dots 0 \\ -a & 1 & 0 \dots 0 \\ 0 & 0 & 0 \dots 0 \\ & & \vdots \\ 0 & 0 & 0 \dots 0 \end{pmatrix}. \tag{4.4}$$

where a and b are functions that may vary in spacetime.

In fact, the same result holds also for "generic" type III spacetimes of arbitrary dimension. More results in the $n > 5$ case can be found in [37].

5 Algebraic classification of tensors

Let us briefly review the algebraic classification of tensors based on null alignment, introduced in [9, 10]. In this short summary, we follow our topical review [2], where a more complete description can be found.

5.1 General tensors

5.1.1 Null frames

In the tangent space of an n-dimensional Lorentzian manifold, we introduce a null frame $\boldsymbol{\ell}$, $\boldsymbol{n}$, $\boldsymbol{m}^{(i)}$ with two null vectors $\boldsymbol{\ell}$

and $\boldsymbol{n}$ and $n-2$ spacelike vectors $\boldsymbol{m}^{(i)} = \boldsymbol{m}_{(i)}$ obeying[5]

$$\ell^a \ell_a = n^a n_a = 0, \qquad \ell^a n_a = 1, \qquad m^{(i)a} m_a^{(j)} = \delta_{ij}. \quad (5.1)$$

Obviously, the metric has the form

$$g_{ab} = 2\ell_{(a}n_{b)} + \delta_{ij} m_a^{(i)} m_b^{(j)}, \tag{5.2}$$

which is preserved under Lorentz transformations. The group of (real) proper orthochronous Lorentz transformations is generated by *null rotations* of one of the null frame vectors about the other, i.e.,

$$\hat{\boldsymbol{\ell}} = \boldsymbol{\ell} + z_i \boldsymbol{m}^{(i)} - \frac{1}{2} z^i z_i \, \boldsymbol{n}, \quad \hat{\boldsymbol{n}} = \boldsymbol{n}, \hat{\boldsymbol{m}}^{(i)} = \boldsymbol{m}^{(i)} - z_i \boldsymbol{n}, \tag{5.3}$$

$$\hat{\boldsymbol{n}} = \boldsymbol{n} + z'_i \boldsymbol{m}^{(i)} - \tfrac{1}{2} z'^i z'_i \, \boldsymbol{\ell}, \quad \hat{\boldsymbol{\ell}} = \boldsymbol{\ell}, \; \hat{\boldsymbol{m}}^{(i)} = \boldsymbol{m}^{(i)} - z'_i \boldsymbol{\ell}, \tag{5.4}$$

with $2(n-2)$ real parameters z_i and z'_i, *spins* described by an $SO(n-2)$ matrix $X^i{}_j$

$$\hat{\boldsymbol{\ell}} = \boldsymbol{\ell}, \qquad \hat{\boldsymbol{n}} = \boldsymbol{n}, \qquad \hat{\boldsymbol{m}}^{(i)} = X^i{}_j \boldsymbol{m}^{(j)}, \tag{5.5}$$

and *boosts* with a parameter $\lambda > 0$

$$\hat{\boldsymbol{\ell}} = \lambda \boldsymbol{\ell}, \qquad \hat{\boldsymbol{n}} = \lambda^{-1} \boldsymbol{n}, \qquad \hat{\boldsymbol{m}}^{(i)} = \boldsymbol{m}^{(i)}. \tag{5.6}$$

In the following, we will be interested in how various tensor frame components transform under boosts. First, let us define a boost weight

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[5] Indices $i, j, \ldots$ take values from 2 to $n-1$, while frame indices denoted by $\hat{a}, \hat{b}, \ldots$ and coordinate indices denoted by $a, b, \ldots$ take values from 0 to $n-1$.

Definition 5.1. A quantity q has a *boost weight* (b.w.) b if it transforms under a boost (5.6) according to

$$\hat{q} = \lambda^{\mathrm{b}} q. \tag{5.7}$$

Various frame components of a tensor $\boldsymbol{T}$ transform according to (5.7) with in general distinct integer values of b, $|b| \leq \mathrm{rank}(\boldsymbol{T})$.

Definition 5.2. The *boost order* of a tensor $\boldsymbol{T}$ with respect to the null frame $\boldsymbol{\ell}$, $\boldsymbol{n}$, $\boldsymbol{m}^{(i)}$ is the maximum b.w. of its frame components

$$\mathrm{bo}(\boldsymbol{T}) = \max\left\{\mathrm{b}(T_{\hat{a}\ldots\hat{b}}) \mid T_{\hat{a}\ldots\hat{b}} \neq 0\right\}. \tag{5.8}$$

Looking at transformations (5.4)–(5.6), it is straightforward to show [10] that

Proposition 5.3. *Let $\boldsymbol{\ell}$, $\boldsymbol{n}$, $\boldsymbol{m}^{(i)}$ and $\hat{\boldsymbol{\ell}}$, $\hat{\boldsymbol{n}}$, $\hat{\boldsymbol{m}}^{(i)}$ be two null frames with $\boldsymbol{\ell}$ and $\hat{\boldsymbol{\ell}}$ being scalar multiples of each other. Then, the boost order of a given tensor is the same relative to both frames.*

Thus, the boost order of a tensor depends only on the choice of the null direction $\langle\boldsymbol{\ell}\rangle$ and will be denoted as $\mathrm{bo}_{\boldsymbol{\ell}}(\boldsymbol{T})$. This indicates that $\boldsymbol{T}$ may admit preferrend null directions defined as:

Definition 5.4. Let $\boldsymbol{T}$ be a tensor and let $\mathrm{bo}_{\max}(\boldsymbol{T})$ denote the maximum value of $\mathrm{bo}_{\boldsymbol{\ell}}(\boldsymbol{T})$ taken over all null vectors $\boldsymbol{\ell}$, i.e.,

$$\mathrm{bo}_{\max}(\boldsymbol{T}) = \max\{\mathrm{bo}_{\boldsymbol{\ell}}(\boldsymbol{T}) \mid \langle\boldsymbol{\ell}\rangle \text{ is null}\}. \tag{5.9}$$

We say that a vector $\boldsymbol{\ell}$ is an *aligned null direction* (AND) of a tensor $\boldsymbol{T}$ whenever $\mathrm{bo}_{\boldsymbol{\ell}}(\boldsymbol{T}) < \mathrm{bo}_{\max}(\boldsymbol{T})$, and we call the integer $\mathrm{bo}_{\max}(\boldsymbol{T}) - \mathrm{bo}_{\boldsymbol{\ell}}(\boldsymbol{T})$ its multiplicity.

Possible existence and multiplicity of such directions is a basis for alignment type classification of tensors.

Definition 5.5. We define the *principal alignment type* (PAT) of a tensor $\boldsymbol{T}$ as the integer

$$\mathrm{PAT} = \mathrm{bo}_{\max}(\boldsymbol{T}) - \mathrm{bo}_{\min}(\boldsymbol{T}), \tag{5.10}$$

where

$$\mathrm{bo}_{\min}(\boldsymbol{T}) = \min\{\mathrm{bo}_{\boldsymbol{\ell}}(\boldsymbol{T}) \mid \langle \boldsymbol{\ell} \rangle \text{ is null}\}. \tag{5.11}$$

Choosing $\boldsymbol{\ell}$ with the maximal multiplicity (which is equal to PAT), we define the *secondary alignment type*, SAT, to be the integer

$$\mathrm{SAT} = \mathrm{bo}_{\max}(\boldsymbol{T}) - \tilde{\mathrm{bo}}_{\min}(\boldsymbol{T}), \tag{5.12}$$

with

$$\tilde{\mathrm{bo}}_{\min}(\boldsymbol{T}) = \min\{\mathrm{bo}_{\boldsymbol{n}}(\boldsymbol{T}) \mid \langle \boldsymbol{n} \rangle \quad \text{is null}, \ \langle \boldsymbol{n} \rangle \neq \langle \boldsymbol{\ell} \rangle\}. \tag{5.13}$$

Definition 5.6. The *alignment type* of an arbitrary tensor consists of the pair of integers (PAT, SAT).

To determine the alignment type of a tensor, one has to project the tensor $\boldsymbol{T}$ on the null frame and sort its components by their b.w.

$$\boldsymbol{T} = \sum_{\mathrm{b}} (\boldsymbol{T})_{(\mathrm{b})}, \tag{5.14}$$

where

$$(\boldsymbol{T})_{(\mathrm{b})} = \sum T_{\hat{a}\ldots\hat{b}} \boldsymbol{m}^{(\hat{a})} \ldots \boldsymbol{m}^{(\hat{b})}, \qquad \mathrm{b}(T_{\hat{a}\ldots\hat{b}}) = \mathrm{b}. \tag{5.15}$$

Then using null rotations (5.3) and (5.4) about $\boldsymbol{n}$ and $\boldsymbol{\ell}$, one has to set as many leading and trailing terms in (5.14) as possible to zero.

For a tensor $\boldsymbol{T}$, we define the following algebraic types in terms of its PAT (and SAT) [9, 10, 34]:

Definition 5.7 (Algebraic types)**.** A non-vanishing tensor $\boldsymbol{T}$ is of

- type G if PAT$= 0$, i.e. for all frames $(\boldsymbol{T})_{(\mathrm{bo}_{\mathrm{max}}(\boldsymbol{T}))} \neq 0$,
- type I if PAT≥ 1, i.e. there exists a frame such that $(\boldsymbol{T})_{(\mathrm{bo}_{\mathrm{max}}(\boldsymbol{T}))} = 0$,[6]
- type II if PAT$\geq \mathrm{bo}_{\mathrm{max}}(\boldsymbol{T})$, i.e. there exists a frame such that $\boldsymbol{T} = \sum_{\mathrm{b}\leq 0} (\boldsymbol{T})_{(\mathrm{b})}$,
- type D if PAT$= \mathrm{bo}_{\mathrm{max}}(\boldsymbol{T})$ =SAT, i.e. there exists a frame such that $\boldsymbol{T} = (\boldsymbol{T})_{(0)}$,
- type III if PAT$\geq \mathrm{bo}_{\mathrm{max}}(\boldsymbol{T}) + 1$, i.e. there exists a frame such that $\boldsymbol{T} = \sum_{\mathrm{b}<0} (\boldsymbol{T})_{(\mathrm{b})}$,
- type N if PAT$= 2\mathrm{bo}_{\mathrm{max}}(\boldsymbol{T})$, i.e. there exists a frame such that $\boldsymbol{T} = (\boldsymbol{T})_{(-\mathrm{bo}_{\mathrm{max}}(\boldsymbol{T}))}$.

Note that the above definitions of algebraic types are frame independent and thus invariant (this follows from proposition 5.3). Type N is regarded as a special case of type III and type III which is not type N is sometimes denoted as *genuine* type III, etc. (see [2]).

Let us summarize algebraic types of tensors in Table 1 [2].

[6] Sometimes a different definition of type I is used, e.g. in [38]. However, in the case of the Weyl tensor, this definition becomes in fact equivalent to ours.

Genuine type	PAT	$\boldsymbol{T}$
G	0	$\sum_{\rm b}(\boldsymbol{T})_{(\rm b)}$
I	$\geq 1, < \mathrm{bo}_{\max}$	$\sum_{\mathrm{b}<\mathrm{bo}_{max}}(\boldsymbol{T})_{(\rm b)}$
II	$\mathrm{bo}_{\max}(\boldsymbol{T})$	$\sum_{\mathrm{b}\leq 0}(\boldsymbol{T})_{(\rm b)}$
III	$\geq \mathrm{bo}_{\max}(\boldsymbol{T})+1, < 2\mathrm{bo}_{\max}$	$\sum_{\mathrm{b}<0}(\boldsymbol{T})_{(\rm b)}$
N	$2\mathrm{bo}_{\max}(\boldsymbol{T})$	$\sum_{\mathrm{b}=-\mathrm{bo}_{\max}}(\boldsymbol{T})_{(\rm b)}$

Table 1: (Genuine) principal alignment types for a tensor $\boldsymbol{T}$. Recall that the (secondary) type D is a subtype of type II such that PAT= $\mathrm{bo}_{\max}(\boldsymbol{T})$ =SAT, i.e. $\boldsymbol{T}=\sum_{\mathrm{b}=0}(\boldsymbol{T})_{(\rm b)}$.

Now, let us discuss two examples of simple tensors. We start with a trivial case of a vector

$$\boldsymbol{v} = v_0\boldsymbol{n} + v_i\boldsymbol{m^{(i)}} + v_1\boldsymbol{\ell}. \tag{5.16}$$

In this case, $\mathrm{bo}_{\max}(\boldsymbol{v}) = 1$ and the three algebraic classes of vectors are (i) A timelike vector ($v^a v_a < 0$) is of type G (alignment type $(0,0)$), i.e. there are no ANDs. (ii) A spacelike vector ($v^a v_a > 0$) is of type D (alignment type $(1,1)$). (iii) A null vector ($v^a v_a = 0$) is of type N (alignment type $(2,0)$).

A symmetric rank-2 tensor $\boldsymbol{R}$ (such as the Ricci tensor), with $\mathrm{bo}_{\max}(\boldsymbol{R}) = 2$,

$$\begin{aligned} R_{ab} = \; & \overbrace{R_{00}\, n_a n_b}^{+2} + \overbrace{2R_{0i}\, n_{(a} {m^{(i)}}_{b)}}^{+1} + \overbrace{2R_{01}\, n_{(a}\ell_{b)} + R_{ij}\, {m^{(i)}}_{(a} {m^{(j)}}_{b)}}^{0} \\ & + \overbrace{2R_{1i}\, \ell_{(a} {m^{(i)}}_{b)}}^{-1} + \overbrace{R_{11}\, \ell_a \ell_b}^{-2} \end{aligned} \tag{5.17}$$

admits the following possible types: G - $(0,0)$, I$_i$ - $(1,1)$, II - $(2,0)$, II$_i$ - $(2,1)$, D - $(2,2)$, III - $(3,0)$, III$_i$ - $(3,1)$, N - $(4,0)$. In the case of four dimensions, the relation with the Segre types is discussed in [10].

5.2 Classification of the Weyl tensor

Let us apply the above classification scheme to the Weyl tensor ($\mathrm{bo}_{\max}(\boldsymbol{C}) = 2$), which obeys the identities

$$C_{abcd} = C_{\{abcd\}} \equiv \tfrac{1}{2}(C_{[ab][cd]} + C_{[cd][ab]}), \tag{5.18}$$

$$C^{c}{}_{acb} = 0, \qquad C_{a[bcd]} = 0 \tag{5.19}$$

and in terms of frame components it can be written as

$$\begin{aligned}
C_{abcd} \;=\; & \overbrace{4C_{0i0j}\, n_{\{a}m^{(i)}_{\ b}n_{c}m^{(j)}_{\ d\}}}^{\text{boost weight } +2} \\
& + \overbrace{8C_{010i}\, n_{\{a}l_{b}n_{c}m^{(i)}_{\ d\}} + 4C_{0ijk}\, n_{\{a}m^{(i)}_{\ b}m^{(j)}_{\ c}m^{(k)}_{\ d\}}}^{+1} \\
& \left.\begin{aligned} &+4C_{0101}\, n_{\{a}l_{b}n_{c}l_{d\}} + 4C_{01ij}\, n_{\{a}l_{b}m^{(i)}_{\ c}m^{(j)}_{\ d\}} \\ &+8C_{0i1j}\, n_{\{a}m^{(i)}_{\ b}l_{c}m^{(j)}_{\ d\}} + C_{ijkl}\, m^{(i)}_{\ \{a}m^{(j)}_{\ b}m^{(k)}_{\ c}m^{(l)}_{\ d\}} \end{aligned}\right\}0 \\
& + \overbrace{8C_{101i}\, l_{\{a}n_{b}l_{c}m^{(i)}_{\ d\}} + 4C_{1ijk}\, l_{\{a}m^{(i)}_{\ b}m^{(j)}_{\ c}m^{(k)}_{\ d\}}}^{-1} \\
& + \overbrace{4C_{1i1j}\, l_{\{a}m^{(i)}_{\ b}l_{c}m^{(j)}_{\ d\}}}^{-2},
\end{aligned} \tag{5.20}$$

where the various components have been ordered by boost weight.

In the case of the Weyl tensor, it is useful to specialize definition 5.4 to

Definition 5.8. A null vector field $\boldsymbol{\ell}$ is a *Weyl aligned null direction* (WAND) if it is an aligned null direction of the Weyl tensor. A WAND $\boldsymbol{\ell}$ is a multiple WAND (mWAND) if its multiplicity is greater than 1.

We are now in a position to introduce a term algebraically special in arbitrary dimension (in four dimensions this is equivalent to the standard use of this term):

Definition 5.9. A Weyl tensor is said to be *algebraically special* if it admits a multiple WAND.[7]

If the Weyl tensor is of the same algebraic type at all points of the spacetime, then the spacetime is said to be of the corresponding algebraic (Weyl) type (and similarly for an open region of the spacetime).

Genuine algebraic types of the Weyl tensor ("Weyl types", see the text following definition 5.7) and the comparison with Petrov types in four dimensions are given in table 2. Note that in four dimensions, this algebraic classification is equivalent to the Petrov classification, and the notion of WAND coincides with that of a principal null direction (PND). However, the case $n = 4$ is somewhat special since there always exist exactly four PNDs (possibly repeated), so that the type G does not exist, and there are even fewer possible types since I=I$_i$, II=II$_i$, III=III$_i$ [10]. By contrast, an $n > 4$ spacetime may admit no WANDs (type G, which is the generic situation [10]), a finite

[7]Note that this is different from the definition originally used in [9,10] and later in some other papers. There, type I was also considered "special" when $n > 4$. However, in analogy with the $n = 4$ case (where algebraically special means admitting a *multiple* PND), and for other reasons [34,39], we use the terminology of definition 5.9.

number of WANDs, or infinitely many. In any dimension, there is a *unique* multiple WAND for the genuine types II (double), III (triple) and N (quadruple), whereas there exist at least two (both double) for type D. However, in the latter case, there may exist also an infinity of mWANDs. For a given type, further subtypes can be defined when some (but not all) of the components of the Weyl tensor having maximal b.w. vanish [9, 40, 41] (see also [2]).

n >4 dimensions		4 dimensions
Weyl type	alignment type	Petrov type
G	(0,0)	
I	(1,0)	
I_i	(1,1)	I
II	(2,0)	
II_i	(2,1)	II
D	(2,2)	D
III	(3,0)	
III_i	(3,1)	III
N	(4,0)	N

Table 2: Possible Weyl/alignment (genuine) types in higher dimensions (definitions 5.7 and 5.6) compared to the four-dimensional case [9].

It is of interest to study constraints on the Weyl types following from symmetries. For example [42],

Proposition 5.10. *An $n \geq 4$ static spacetime can be only of type G, I_i, D(d) or O.*

Proposition 5.11. *An $n \geq 4$ spherically symmetric spacetime is of type D or O.*

As in four dimensions, it turns out that most higher-dimensional vacuum solutions to the Einstein equations are algebraically special (see Table 7 in [2]).

6 Comments on NP and GHP formalisms

Newman-Penrose and Geroch-Held-Penrose formalisms are the basic tools of the four-dimensional mathematical relativity. In [11–13], we have developed higher-dimensional generalizations of these methods. Since the resulting equations are very complicated in the generic case, here we limit ourselves to a description of (some of) the mathematical objects appearing in the formalism. A complete presentation of the formalism can be found in [13].

Let us denote the covariant derivatives of the frame vectors as

$$L_{ab} = \nabla_b \ell_a, \qquad N_{ab} = \nabla_b n_a, \qquad \overset{i}{M}_{ab} = \nabla_b m_{(i)a}. \tag{6.1}$$

The projections onto the basis are the Ricci rotation coefficients $L_{\hat{a}\hat{b}}$, $N_{\hat{a}\hat{b}}$, $\overset{i}{M}_{\hat{a}\hat{b}}$. Orthogonality properties of the null frame give the following constraints

$$\begin{aligned} &N_{0\hat{a}} + L_{1\hat{a}} = 0, \quad \overset{i}{M}_{0\hat{a}} + L_{i\hat{a}} = 0, \quad \overset{i}{M}_{1\hat{a}} + N_{i\hat{a}} = 0, \\ &\overset{i}{M}_{j\hat{a}} + \overset{j}{M}_{i\hat{a}} = 0, \quad L_{0\hat{a}} = N_{1\hat{a}} = \overset{i}{M}_{i\hat{a}} = 0. \end{aligned} \tag{6.2}$$

Geometric interpretation of selected spin coefficients is given in Table 3 [13].

NP	GHP	b	s	Interpretation
L_{ij}	ρ_{ij}	1	2	expansion, shear and twist of $\boldsymbol{\ell}$
L_{ii}	$\rho = \rho_{ii}$	1	0	expansion of $\boldsymbol{\ell}$
L_{i0}	κ_i	2	1	non-geodesity of $\boldsymbol{\ell}$
L_{i1}	τ_i	0	1	transport of $\boldsymbol{\ell}$ along $\boldsymbol{n}$
N_{ij}	ρ'_{ij}	-1	2	expansion, shear and twist of $\boldsymbol{n}$
N_{ii}	$\rho' = \rho'_{ii}$	-1	0	expansion of $\boldsymbol{n}$
N_{i1}	κ'_i	-2	1	non-geodesity of $\boldsymbol{n}$
N_{i0}	τ'_i	0	1	transport of $\boldsymbol{n}$ along $\boldsymbol{\ell}$

Table 3: List of the Ricci coefficients appearing in the higher-dimensional GHP formalism. The first and second columns give the coefficients in the NP and GHP notation, respectively. Boost weight b and spin s are also indicated.

The *optical matrix* ρ_{ij}

$$\rho_{ij} \equiv L_{ij} \equiv L_{ab} m^{(i)a} m^{(j)b} \tag{6.3}$$

describes expansion, shear and twist of the null congruence $\boldsymbol{\ell}$ corresponding to its trace, traceless symmetric and antisymmetric parts, respectively. Note that the optical matrix appears e.g. in the formulation of the five-dimensional Goldberg-Sachs theorem in section 4. A special class of spacetimes can be defined by vanishing of the optical matrix. These so called Kundt spacetimes admit a geodetic null congruence with vanishing shear, expansion and twist. These spacetimes have particular geometrical properties and have been widely studied in the literature (see [2] for an overview and references). Kundt spacetimes also appear in sections 2 and 3 in the context of universal and VSI spacetimes, respectively.

The equations of the NP/GHP formalisms then consist of contractions of the Ricci identity $v_{a;bc} - v_{a;cb} = R_{sabc}v^s$, the Bianchi identity $R_{ab[cd;e]} = 0$ and commutators with various combinations of the frame vectors. The resulting equations are in general very complex but they simplify considerably for algebraically special spacetimes. Thus the main advantage of this approach is in the study of algebraically spacetimes. See [13] for the full set of the equations in algebraically general and algebraically special cases.

Here, let us just demonstrate this simplification on one component[8] of the Ricci identity which in full generality reads

$$\begin{aligned} DL_{ij} - \ \delta_j L_{i0} &= L_{10}L_{ij} - L_{i0}(2L_{1j} + N_{j0}) - L_{i1}L_{j0} \\ &+ 2L_{k[0|}\overset{k}{M}_{i|j]} - L_{ik}(L_{kj} + \overset{k}{M}_{j0}) - C_{0i0j} - \tfrac{1}{n-2}R_{00}\delta_{ij} \,, \end{aligned} \tag{6.4}$$

where $D \equiv \ell^a \nabla_a$ is the covariant derivative along $\boldsymbol{\ell}$.

In a frame parallelly transported along $\boldsymbol{\ell}$, this equation reduces to

$$D\rho_{ij} = -\rho_{ik}\rho_{kj} - C_{0i0j} - \frac{1}{n-2}R_{00}\delta_{ij} \tag{6.5}$$

and for algebraically special vacuum spacetimes, it further simplifies to

$$D\rho_{ij} = -\rho_{ik}\rho_{kj}. \tag{6.6}$$

The above equation (in four dimensions called the Sachs equation) is a starting point for the studies of asymptotic properties

[8] Note that in this case, the simplification is more impressive than for a generic NP equation.

of vacuum solutions of the Einstein equations and for partial or full integration of the Einstein equations in higher dimensions (see e.g. [1, 43, 44]).

7 List of publications of the dissertation

1. V. Pravda, A. Pravdová, A. Coley, and R. Milson. All spacetimes with vanishing curvature invariants. *Class. Quantum Grav.*, 19:6213–6236, 2002.

2. A. Coley, R. Milson, V. Pravda, and A. Pravdová. Vanishing scalar invariant spacetimes in higher dimensions. *Class. Quantum Grav.*, 21:5519–5542, 2004.

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